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Non-commutative twisted Euler characteristic

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Abstract. It is well known that given a finitely generated torsion module M over the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$, where $\Gamma \cong \mathbb{Z}_p$, there exists a continuous p-adic character ρ of Γ such that, for the twist $M(\rho)$ of M, the $\Gamma_n := \Gamma^{p^n}$ Euler characteristic, i.e. $\chi(\Gamma_n, M(\rho))$, is finite for every n. We prove a generalization of this result by considering modules over the Iwasawa algebra of a general p-adic Lie group G, instead of Γ. We relate this twisted Euler characteristic to the evaluation of the *Akashi series* at the twist and in turn use it to indicate some application to the Iwasawa theory of elliptic curves. This article is a natural generalization of the result established in [8].

1. Introduction

In this article, we study certain topics in non-commutative Iwasawa theory following the set-up of [4]. Fix an odd integer prime p. Let L be a finite extension of $\mathbb{Q}_p$ and let O denote the ring of integers of L. Let G be a compact p-adic Lie group without any element of order p and let H be a closed subgroup of G such that $\Gamma := G/H \cong \mathbb{Z}_p$. We fix a topological generator γ of Γ and a lift $\tilde{\gamma}$ of γ in G. For a general profinite group $\mathcal{G}$, let $\Lambda_O(\mathcal{G})$ denote the Iwasawa algebra defined by

$$\Lambda_O(\mathcal{G}) := \varprojlim_U O[\mathcal{G}/U],$$

where U varies over open normal subgroups of $\mathcal{G}$ and the inverse limit is taken with respect to natural projection maps. Given any ring R and a left R-module M, we denote by $M[p^r]$ the set of p^r-torsion elements of M and we define

$$M(p) := \bigcup_{r\geq 1} M[p^r].$$

Let $\mathfrak{M}_H(G)$ denote the category of finitely generated $\Lambda_O(G)$-modules M such that $M/M(p)$ is a finitely generated $\Lambda_O(H)$-module. For a $\Lambda_O(G)$-module M and a continuous character $\rho : \Gamma \to \mathbb{Z}_p^\times$, we denote by $M(\rho)$ the $\Lambda_O(G)$-module $M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\rho)$ with diagonal G-action. Recall that, for a compact p-adic Lie group G which has no element of order p, the Iwasawa algebra $\Lambda_O(G)$ has finite global dimension [2, 9].

Definition 1.1. Let G be a compact p-adic Lie group without any element of order p. For a finitely generated $\Lambda_O(G)$-module M, we say the G-Euler characteristic of M exists if the homology groups $H_i(G, M)$ are finite for every $i \geq 0$. When the G-Euler characteristic of M exists, then it is defined by

$$\chi(G, M) := \prod_i (\# H_i(G, M))^{(-1)^i}.$$

Given a $\Lambda_O(G)$-module M, $\chi(G, M)$ is an invariant attached to M, and for an appropriate motive $\mathcal{M}$ and G, under suitable hypotheses, the G-Euler characteristic of certain Selmer group $S(\mathcal{M})$ attached to $\mathcal{M}$, i.e. $\chi(G, S(\mathcal{M}))$, often encodes important arithmetical properties of $\mathcal{M}$. We discuss a concrete example.

Let $E/\mathbb{Q}$ be an elliptic curve with good, ordinary reduction at $p \geq 11$ and E has no prime of additive reduction. Let $\mathbb{Q}_{\mathrm{cyc}}$ denote the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$ and $S_p(E/\mathbb{Q})$ the p^∞-Selmer group of E over $\mathbb{Q}$ (defined in Section 3.8). We assume that $L_E(s)$, the Hasse–Weil (complex) L-function of E over $\mathbb{Q}$, satisfies $L_E(1) \neq 0$. Then the Birch and Swinnerton–Dyer conjecture describing the leading term of $L_E(s)$ at 1 together with [6, Thm. 4.1] imply that

$$\chi(\Gamma, S_p(E/\mathbb{Q}_{\mathrm{cyc}})^\vee) =_p \frac{L_E(1)}{\Omega_E} (\# \tilde{E}(\mathbb{F}_p)(p))^2, \tag{1}$$

where $=_p$ denotes the equality up to a p-adic unit and Ω_E denotes the smallest positive real period of E. Note that due to works of Gross–Zagier, Kolyvagin and others, it is established that $L_E(1) \neq 0$ implies that $E(\mathbb{Q})$ is finite. Then for such an E and p as above, the leading term formula for $L_E(s)$ in BSD is derived from the proof of the Iwasawa main conjecture for E at p due to works of Kato, Skinner–Urban and others (see [13, Thm. 3.35 (a)] for details). As a consequence of this, formula (1) is established using [6, Conj. 1.13 and §4].

On the other hand, for a general p-adic Lie extension K_∞ of a number field K with Galois group G as above and for an elliptic curve $E/\mathbb{Q}$, the existence of a p-adic L-function of E over K_∞ is not known and even the algebraicity of the L-values has not been established in general. However, a formulation of the Iwasawa main conjecture is given in [4] for dual Selmer groups in $\mathfrak{M}_H(G)$ and in this context the twisted Euler characteristic of modules in $\mathfrak{M}_H(G)$ via Akashi series has been extensively studied in [4]. In particular, for a general $\rho : G \to \mathrm{GL}_n(\mathbb{Z}_p)$, the conjectural relation between $\chi(G, S_p(E/K_\infty)^\vee(\rho))$ and special values of twisted L-function is discussed in detail [4, Thm. 3.6].

The main result of this article is the following theorem.

Theorem 1.2. *Let M be a $\Lambda_O(G)$-module which is in $\mathfrak{M}_H(G)$, i.e., $M/M(p)$ is finitely generated over $\Lambda_O(H)$. Then there exists a continuous character $\rho : \Gamma \to \mathbb{Z}_p^\times$ such that $\chi(U, M(\rho))$ exists for every open normal subgroup U of G.*

Remark 1.3. Actually, we will establish a slightly stronger result; we will prove that there is a countable subset S_M of the set

$$S := \{\text{all continuous characters from } \Gamma \text{ to } \mathbb{Z}_p^\times\}$$

such that if we choose and fix any $\rho \in S \setminus S_M$, then $\chi(U, M(\rho))$ exists for every open normal subgroup U of G.

Theorem 1.2 is a generalization and uses the proof of the following result proved in [8], coauthored by the first-named author.

Theorem 1.4 ([8])**.** *Let M be a $\Lambda_O(G)$-module which is finitely generated over $\Lambda_O(H)$. Then there exists a continuous character $\rho : \Gamma \to \mathbb{Z}_p^\times$, such that $M(\rho)_U := H_0(U, M)$ is finite for every open normal subgroup U of G.*

Remark 1.5. (i) For $G = \Gamma$, $H = 1$, Theorem 1.4 and Theorem 1.2 are equivalent. Indeed it is the well-known "twisting lemma" which can be found in [5, 11].

(ii) More generally, if G is a 'finite by nilpotent' group, then Theorem 1.4 and Theorem 1.2 are equivalent [14]. Examples of 'finite by nilpotent' groups include $G \cong \mathbb{Z}_p^d$, $d \in \mathbb{N}$.

(iii) However, for a general compact p-adic Lie group G without any element of order p and for a finitely generated $\Lambda_O(G)$-module M, M_G being finite does not necessarily imply that $\chi(G, M)$ exists [1]. Thus indeed Theorem 1.2 is more general than Theorem 1.4.

(iv) The assumption that G has no element of order p is assumed in Theorem 1.2 but it is not needed in Theorem 1.4. However, in our case this assumption is clearly necessary; otherwise $\Lambda_O(G)$ may not have finite global dimension and hence the Euler characteristic may not be defined.

(v) It is necessary to assume $M \in \mathfrak{M}_H(G)$ in Theorem 1.2. See the example on [4, p. 194], where for certain M and G with M not in $\mathfrak{M}_H(G)$, even $\chi(G, M(\rho))$ does not exist for any ρ.

In Section 2, we prove Theorem 1.2. In Section 3, we first discuss the relation between Theorem 1.2 and the Akashi series and apply the theorem to deduce Corollary 3.7. We then apply Theorem 1.2 and Corollary 3.7 to p^∞-Selmer groups of elliptic curves to deduce Corollary 3.11.

2. Proof of Theorem 1.2

We start with the following remark.

Remark 2.1. By our assumption on G, $\Lambda_O(G)$ is (left and right) noetherian. It is then well known that the U-Euler characteristic of a finitely generated, p-primary torsion $\Lambda_O(G)$-module always exists [7, Prop. 1.6]. Thus to establish Theorem 1.2, without any loss of generality, we may assume that M is a finitely generated $\Lambda_O(H)$-module.

We next prove the following lemma and use it to make Remark 2.3.

Lemma 2.2. *Let G be a compact p-adic Lie group without any element of order p and let W be any normal subgroup of G. Let M be a finitely generated $\Lambda_O(G)$-module. If the W-Euler characteristic of M exists, then the G-Euler characteristic also exists.*

Proof. Since the W-Euler characteristic of M exists, the groups $H_j(W, M)$ are by definition finite for every integer $j \geq 0$. Thus $H_i(G/W, H_j(W, M))$ is finite for every integer $i, j \geq 0$ (see [9]). Now it follows from the Hochschild–Serre spectral sequence that $H_r(G, M)$ is finite for every integer $r \geq 0$. Since G is a compact p-adic Lie group without any element of order p, the ring $\Lambda_O(G)$ has finite global dimension. Consequently, there exists an integer $m \geq 0$ such that $H_r(G, M) = 0$ for all $r \geq m$. This shows that the G-Euler characteristic exists for M. □

Remark 2.3. By a result of Lazard [9, Chap. 3, §3.1], any compact p-adic Lie group G has an open normal uniform subgroup W. In particular, W is a pro-p, p-adic Lie group without any element of order p. We choose and fix such a W. Now let M be a finitely generated $\Lambda_O(G)$-module. If for every open normal subgroup V of W, the V-Euler characteristic of M exists, then it follows from Lemma 2.2 that for any open normal subgroup U of G, the U-Euler characteristic of M also exists. Hence from this discussion, we further reduce Theorem 1.2 to the special case where G is a compact, pro-p, p-adic Lie group without any element of order p.

Lemma 2.4. *Let G be a pro-p, p-adic Lie group without any element of order p and let M be a finitely generated $\Lambda_O(G)$-module which is also a finitely generated $\Lambda_O(H)$-module. Then there exist an open subgroup G^{00} of G with $H \subset G^{00}$ and a resolution*

$$0 \to N_k \to N_{k-1} \to \cdots \to N_1 \to M \to 0$$

of M by finitely generated $\Lambda_O(G^{00})$-modules N_i, $i = 1, \ldots, k$, such that each N_i is a free $\Lambda_O(H)$-modules.

Proof. We make two observations which are used in the proof of this lemma.

First, as H does not have any element of order p either, the global dimension of $\Lambda_O(H)$ is also finite. Moreover, H is also a pro-p group, $\Lambda_O(H)$ is a local ring and hence any finitely generated projective $\Lambda_O(H)$-module is a finitely generated free $\Lambda_O(H)$-module.

Second, we recall a basic fact from homological algebra (see for example [10, p. 37, (5.6)]). Given an exact sequence of $\Lambda_O(H)$-modules,

$$0 \to K \to M \to L \to 0,$$

let k, m, l be the $\Lambda_O(H)$ projective dimensions of K, M, L, respectively. We have $m \leq \max\{k, l\}$, and if this is a strict inequality, then $k = l - 1$.

If the M given in the lemma is a projective $\Lambda_O(H)$-module, then $N_1 = M$ works. Let $t > 0$ be the $\Lambda_O(H)$-projective dimension of M (which is finite by first observation). By [8, Key Lemma], there exist an open subgroup G_1^0 of G, a finitely generated $\Lambda_O(G^0)$-module N_1 which is a free $\Lambda_O(H)$-module, and an exact sequence of $\Lambda_O(G^0)$-modules

$$0 \to K_1 \to N_1 \to M \to 0.$$

If M is not a projective $\Lambda_O(H)$-module, then the $\Lambda_O(H)$-projective dimension of K_1 is equal to $d-1$, by our second observation. If K_1 is projective over $\Lambda_O(H)$, then we are done. Otherwise, again we apply [8, Key Lemma] to get an open subgroup G_2^0 of G_1^0 containing H and an exact sequence

$$0 \to K_2 \to N_2 \to K_1 \to 0$$

of $\Lambda_O(G_2^0)$-modules such that N_2 is a projective $\Lambda_O(H)$-module. The $\Lambda_O(H)$-projective dimension of K_2 is equal to $d-2$. By repeating this process, at most d times, we get an open subgroup G_0^k of G containing H and an exact sequence

$$0 \to N_k \to N_{k-1} \to \cdots \to N_1 \to M \to 0$$

of $\Lambda_O(G_0^k)$-module such that each N_i is a finitely generated free $\Lambda_O(H)$-module. This proves the lemma.

Note that in the special case of the false Tate curve extension (see [15] for its definition) where $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$, Lemma 2.4 was proved in [15, Lem. 4.3]. $\square$

Let M be a $\Lambda_O(G)$-module which is also a finitely generated free $\Lambda_O(H)$-module. Fix a $\Lambda_O(H)$ isomorphism $\phi : M \cong \Lambda_O(H)^d$. Let $\{e_1, e_2, \ldots, e_d\}$ be the standard basis of $\Lambda_O(H)^d$, namely, e_i denotes the row vector with 1 at ith entry and 0 elsewhere. Since M is a G-module, via the isomorphism ϕ, G acts on $\Lambda_O(H)^d$. Let A be the matrix of multiplication of $\tilde{\gamma}$ defined by

$$\tilde{\gamma} * e_i := \sum_j a_{ij} e_j$$

for $a_{ij} \in \Lambda_O(H)$. Thus we have, $\tilde{\gamma} * e_i = e_i A$. Let $x = \sum_k a_k e_k \in \Lambda_O(H)^d$. We get

$$\tilde{\gamma} * x = \sum_k (\tilde{\gamma} a_k \tilde{\gamma}^{-1}) \tilde{\gamma} * e_k = \Big(\sum_k \tilde{\gamma} a_k \tilde{\gamma}^{-1} e_k\Big) A. \tag{2}$$

Let us denote the left $\Lambda_O(G)$-module

$$\frac{\Lambda_O(G)^d}{\Lambda_O(G)^d(\tilde{\gamma} I_d - A)}$$

by $\tilde{M}$. Here I_d denotes the $d \times d$ identity matrix. Consider the natural map

$$\psi : \Lambda_O(H)^d \to \frac{\Lambda_O(G)^d}{\Lambda_O(G)^d(\tilde{\gamma} I_d - A)}$$

induced by the natural inclusion map $\Lambda_O(H) \to \Lambda_O(G)$. It can be easily shown that ψ is an isomorphism of left $\Lambda_O(H)$-modules. We shall next show that it is an isomorphism of $\Lambda_O(G)$-modules. Recall that $\Lambda_O(H)^d$ is a $\Lambda_O(G)$-module via the isomorphism ϕ. To show that ψ is a $\Lambda(G)$-linear it is enough to show that $\psi(\tilde{\gamma} * x) = \tilde{\gamma}\psi(x)$ for all x. Let $x = \sum_k a_k e_k \in \Lambda_O(H)^d$. From (2), we have

$$\psi(\tilde{\gamma} * x) = \psi\Big(\Big(\sum_k \tilde{\gamma} a_k \tilde{\gamma}^{-1} e_k\Big)A\Big) = (\sum_k \tilde{\gamma} a_k \tilde{\gamma}^{-1} e_k)A \in \tilde{M}.$$

Now the right multiplication by A in $\tilde{M}$ is the same as the right multiplication by $\tilde{\gamma} I_d$ in $\tilde{M}$. Thus we have

$$\Big(\sum_k \tilde{\gamma} a_k \tilde{\gamma}^{-1} e_k\Big)A = \Big(\sum_k \tilde{\gamma} a_k \tilde{\gamma}^{-1} e_k\Big)\tilde{\gamma} I_d \in \tilde{M}.$$

Using the scalar multiplication by $\tilde{\gamma}$ in $\Lambda_O(G)^d$, $\tilde{\gamma} e_i = e_i \tilde{\gamma} I_d$, we deduce that

$$\psi(\tilde{\gamma} * x) = \Big(\sum_k \tilde{\gamma} a_k \tilde{\gamma}^{-1} e_k\Big)\tilde{\gamma} I_d = \sum_k \tilde{\gamma} a_k e_k = \tilde{\gamma}\Big(\sum_k a_k e_k\Big) \in \tilde{M}.$$

This proves that $\psi(\tilde{\gamma} * x) = \tilde{\gamma}\psi(x)$ in $\tilde{M}$. Thus we have proved the following lemma which generalizes [8, Lem. 1].

Lemma 2.5. *Let M be a $\Lambda_O(G)$-module which is a finitely generated free $\Lambda_O(H)$-module. Then there is a $\Lambda_O(G)$-module isomorphism*

$$M \cong \frac{\Lambda_O(G)^d}{\Lambda_O(G)^d(\tilde{\gamma} I_d - A)}.$$

Proposition 2.6. *Let G be a compact, pro-p, p-adic Lie group without any element of order p and let M be a finitely generated $\Lambda_O(G)$-module which is also a finitely generated free $\Lambda_O(H)$-module of rank d. Then M has $\Lambda_O(G)$-projective dimension equal to 1.*

Proof. Since M is a finitely generated $\Lambda_O(H)$-module, it is a torsion $\Lambda_O(G)$-module. In particular, it is not a free $\Lambda_O(G)$-module. Thus the $\Lambda_O(G)$ projective dimension of M is greater than or equal to 1. From Lemma 2.5, there is a $\Lambda_O(G)$-isomorphism

$$M \cong \frac{\Lambda(G)^d}{\Lambda(G)^d(\tilde{\gamma} I_d - A)}$$

for a $d \times d$ matrix A with entries in $\Lambda_O(H)$. Thus we have a resolution

$$\Lambda_O(G)^d \xrightarrow{f} \Lambda_O(G)^d \xrightarrow{q} M \to 0 \tag{3}$$

of the left $\Lambda_O(G)$-module where the second map f is the multiplication by $\tilde{\gamma} I_d - A$ from right. Set $K = \mathrm{Ker}(f)$. We claim that K is a (finitely generated) $\Lambda_O(G)$ torsion module. Note that as G is a compact p-adic Lie group, $\Lambda_O(G)$ is a (left and right) noetherian ring and as G has no element of order p, $\Lambda_O(G)$ has no nonzero left or right zero divisors. It follows that the total skew field of fraction $Q_O(G)$ of $\Lambda_O(G)$ exists [7, §1.1]. As M is a torsion $\Lambda_O(G)$-module,

$$M \otimes_{\Lambda_O(G)} Q_O(G) = 0.$$

Hence applying $-\otimes_{\Lambda_O(G)} Q_O(G)$ on (3) and considering the additive property of the dimension of finite-dimensional $Q_O(G)$ vector spaces in a short exact sequence, we deduce that

$$K \otimes_{\Lambda_O(G)} Q_O(G) = 0.$$

This in turn proves the claim that K is a $\Lambda_O(G)$ torsion module. As $\Lambda_O(G)^d$ is a free $\Lambda_O(G)$-module, it has no nonzero $\Lambda_O(G)$ torsion submodule. Hence, we deduce that $K = 0$. Consequently, f in (3) is injective and thus we conclude that

$$0 \to \Lambda_O(G)^d \xrightarrow{f} \Lambda_O(G)^d \xrightarrow{q} M \to 0$$

is a short exact sequence and therefore the projective dimensions of $M(\rho)$ as a $\Lambda_O(G)$-module is less than or equal to 1. This completes the proof of the proposition. □

Proposition 2.7. *Let G be a pro-p, p-adic Lie group without any element of order p and let M be a finitely generated $\Lambda_O(G)$-module which is also a finitely generated free $\Lambda_O(H)$-module of rank d. Then there is a countable subset S_M of the set $S := \{$all continuous characters from $\Gamma \to \mathbb{Z}_p^\times\}$ such that if we choose and fix any $\rho \in S \setminus S_M$, the U-Euler characteristic of $M(\rho)$ exists for every open normal subgroup U of G and we have $\chi(U, M(\rho)) = \#H_0(U, M(\rho))$.*

Proof. From Lemma 2.6, we have a $\Lambda_O(G)$ projective resolution of M of the form

$$0 \to \Lambda_O(G)^d \to \Lambda_O(G)^d \to M \to 0.$$

Twisting by a character ρ of Γ and taking co-invariance by an open normal subgroup U, we get an exact sequence

$$0 \to H_1(U, M(\rho)) \to O^{d[G:U]} \to O^{d[G:U]} \to H_0(U, M(\rho)) \to 0.$$

From Theorem 1.4, we get that $H_0(U, M(\rho))$ is finite for every U once we have chosen ρ as in the statement of this proposition. This implies that $H_1(U, M(\rho)) = 0$, proving the proposition. □

We are now ready to our main theorem.

Proof of Theorem 1.2. By Lemma 2.4, there exist an open normal subgroup G^{00} of G containing H, and a resolution

$$0 \to N_k \to N_{k-1} \to \cdots \to N_1 \to M \to 0 \tag{4}$$

of M by finitely generated $\Lambda_O(G^{00})$-modules N_i, $i = 1, \dots, k$, such that each N_i is a free $\Lambda_O(H)$-modules. Let $\Gamma^{00} := G^{00}/H$.

Since Γ^{00} is a finite p-power index subgroup of Γ and $\mathbb{Z}_p^\times$ has no element of order p, the following natural restriction map in (5) is injective:

$$\mathrm{res} : \mathrm{Hom}(\Gamma, \mathbb{Z}_p^\times) \to \mathrm{Hom}(\Gamma^{00}, \mathbb{Z}_p^\times). \tag{5}$$

For each N_i there is a countable subset $S_i \subset \mathrm{Hom}(\Gamma^{00}, \mathbb{Z}_p^\times)$ such that if we choose and fix any $\rho \in \mathrm{Hom}(\Gamma^{00}, \mathbb{Z}_p^\times) \setminus S_i$, then $\chi(U, N_i(\rho))$ exists for every open normal subgroup U of G^{00}. Hence by choosing a $\rho \in \mathrm{Hom}(\Gamma^{00}, \mathbb{Z}_p^\times) \setminus \bigcup_{1 \le i \le n} S_i$, simultaneously for every i, $\chi(U, N_i(\rho))$ exists for each open normal subgroup U of G^{00}. This in turn from (4) implies that for any such chosen $\rho : \Gamma^{00} \to \mathbb{Z}_p^\times$, $\chi(U, M(\rho))$ exists for each open normal subgroup U of G^{00}.

Let U be an open normal subgroup of G. Choose and fix $\rho : \Gamma \to \mathbb{Z}_p^\times$ such that $\rho|_{\Gamma^{00}} \notin \bigcup_{1\leq i\leq n} S_i$. Then $\chi(U \cap G^{00}, M(\rho|_{\Gamma^{00}}))$ exists by the preceding discussion. Moreover, applying Lemma 2.2 and (5), we see that $\chi(U, M(\rho))$ also exists. This completes the proof. $\square$

3. APPLICATION

3.1. **Akashi series and Euler characteristic.** Let G be as before and set $\Lambda(G) := \Lambda_{\mathbb{Z}_p}(G)$. Moreover, for $G = \Gamma$, we will sometime write $\Lambda := \Lambda(\Gamma)$. Fix an isomorphism $\Lambda \cong \mathbb{Z}_p[[X]]$ by sending γ to $1 + X$. For a finitely generated torsion $\Lambda(\Gamma)$-module N, let $\mathrm{char}_{\Lambda(\Gamma)}(N)$ denote its usual characteristic element $\in (\Lambda(\Gamma)\backslash\{0\})/(\Lambda(\Gamma))^\times$. Write $Q(\Lambda(\Gamma))$ for the quotient field of $\Lambda(\Gamma)$. We recall the following definition of Akashi series.

Definition 3.2 (see [4])**.** Let $M \in \mathfrak{M}_H(G)$. We say the *Akashi series* of M with respect to H exists, if the Galois homology groups $H_i(H, M)$ are finitely generated torsion $\Lambda(\Gamma)$-modules for every $i \geq 0$. If the Akashi series of M exists, then it is given by

$$Ak_H^G(M) = \prod_i \big(\mathrm{char}_{\Lambda(\Gamma)} H_i(H, M)\big)^{(-1)^i} \in Q(\Lambda(\Gamma))^\times/\Lambda(\Gamma)^\times.$$

Lemma 3.3. *Let $M \in \mathfrak{M}_H(G)$. Then $Ak_{H\cap U}^U(M)$ exists for every open normal subgroup U of G.*

Proof. Put $T = H \cap U$. Then T is a finite index subgroup of H and therefore $\Lambda(H)$ is a finite module over $\Lambda(T)$. In particular, $M/M(p)$ is a finitely generated $\Lambda(T)$-module. Therefore, by [4, Lem. 3.1], $Ak_T^U(M)$ exists. $\square$

For a character $\rho : \Gamma \to \mathbb{Z}_p^\times$, let $\tilde{\rho} : \Lambda(\Gamma) \to \mathbb{Z}_p$ denote the corresponding ring homomorphism induced by the evaluation map. We extend this homomorphism to

$$\tilde{\rho} : Q(\Lambda(\Gamma))^\times \to \mathbb{Q}_p \cup \{\infty\}$$

as $\tilde{\rho}(f/g) = \tilde{\rho}(f)/\tilde{\rho}(g)$ if f and g have no common divisor and $\tilde{\rho}(g) \neq 0$. If f and g have no common divisor and $\tilde{\rho}(g) = 0$, then put $\tilde{\rho}(f/g) = \infty$. To simplify the notation we denote $\widetilde{\rho^{-1}}$ by $\tilde{\rho}^{-1}$.

Definition 3.4. We say $\tilde{\rho}(f/g)$ exists if $\tilde{\rho}(f/g) \neq 0, \infty$.

Let M be a finitely generated torsion $\Lambda(\Gamma)$-module. It can be shown that if $\chi(\Gamma, M)$ exists, then $\chi(\Gamma, M) = \mathrm{char}_{\Lambda(\Gamma)}(M)|_{X=0} = \tilde{\rho}(\mathrm{char}_{\Lambda(\Gamma)}(M))$, where ρ denotes the trivial character of Γ (see [6, Lem. 4.2]). More generally we have the following lemma.

Lemma 3.5. *If ρ is an arbitrary character of Γ and $\chi(\Gamma, M(\rho))$ exists, then*

$$\chi(\Gamma, M(\rho)) = \tilde{\rho}^{-1}(\mathrm{char}_{\Lambda(\Gamma)}(M)).$$

Proof. For a character $\tau : \Gamma \to \mathbb{Z}_p^\times$ define $Tw_\tau : \Lambda \to \Lambda$ be the $\mathbb{Z}_p$-linear isomorphism induced by

$$Tw_\tau(\gamma) = \tau(\gamma)\gamma.$$

For a finitely generated torsion Λ-module B from [12, Chap. VI, Lem. 1.2] we have

$$Tw_\tau(\mathrm{char}_\Lambda(B \otimes \tau)) = \mathrm{char}_\Lambda(B).$$

Taking $B = M \otimes \rho$ and $\tau = \rho^{-1}$, we get

$$Tw_{\rho^{-1}}(\mathrm{char}_\Lambda(M)) = \mathrm{char}_\Lambda(M \otimes \rho). \tag{6}$$

We get

$$\begin{aligned}\tilde{\rho}^{-1}(\mathrm{char}_\Lambda(M)) &= Tw_{\rho^{-1}}(\mathrm{char}_\Lambda(M))|_{X=0} \\ &= \mathrm{char}_\Lambda(M \otimes \rho)|_{X=0} \\ &= \chi(\Gamma, M(\rho)).\end{aligned}$$

□

Lemma 3.6. *Let $M \in \mathfrak{M}_H(G)$ and let U be an open normal subgroup of G. For a character $\rho : \Gamma \to \mathbb{Z}_p^\times$ if $\chi(U, M(\rho))$ exists, then $\tilde{\rho}^{-1}(Ak^U_{U\cap H}(M))$ exists, too, and*

$$\chi(U, M(\rho)) = \tilde{\rho}^{-1}(Ak^U_{U\cap H}(M)).$$

Proof. As $M/M(p)$ is a finitely generated $\Lambda(H)$-module, $(M(\rho))/((M(\rho))(p))$ is also a finitely generated $\Lambda(H)$-module. Therefore $Ak^U_{U\cap H}(M(\rho))$ exists. Since $\chi(U, M(\rho))$ exists, it follows from [3, Lem. 4.2] that $Ak^U_{U\cap H}(M \otimes \rho)|_{X=0}$ is finite, nonzero and

$$\chi(U, M(\rho)) = Ak^U_{U\cap H}(M(\rho))|_{X=0}. \tag{7}$$

We have $H_i(U \cap H, M(\rho)) \cong H_i(U \cap H, M)(\rho)$ for each $i \geq 0$. Therefore from (6) we get that

$$Ak^U_{U\cap H}(M(\rho)) = Tw_{\rho^{-1}}(Ak^U_{U\cap H}(M)). \tag{8}$$

Since $Ak^U_{U\cap H}(M \otimes \rho)|_{X=0}$ is finite and nonzero,

$$Tw_{\rho^{-1}}(Ak^U_{U\cap H}(M))|_{X=0} = \tilde{\rho}^{-1}(Ak^U_{U\cap H}(M))$$

also exists. From (7) and (8) we get that $\chi(U, M(\rho)) = \tilde{\rho}^{-1}(Ak^U_{U\cap H}(M))$. This proves the lemma. □

The next result is a consequence of Theorem 1.2 and Lemma 3.6.

Corollary 3.7. *Let $M \in \mathfrak{M}_H(G)$. Then there is a countable subset S_M of the set all continuous characters from Γ to $\mathbb{Z}_p^\times$ such that if we choose and fix any character ρ outside S_M, then $\tilde{\rho}^{-1}(Ak^U_{U\cap H}(M))$ exists for all open normal subgroups U of G and satisfies*

$$\chi(U, M(\rho)) = \tilde{\rho}^{-1}(Ak^U_{U\cap H}(M)).$$

3.8. **Relation to the Iwasawa theory of elliptic curves.** Throughout this subsection, E will be an elliptic curve defined over a number field K with good and ordinary reduction at all primes of K dividing p. The p^∞-Selmer group $S_p(E/L)$ of E over a finite extension L/K is defined by

$$0 \to S_p(E/L) \to H^1(L, E(p)) \to \prod_v H^1(L_v, E),$$

where $E(p) := \bigcup_{n\geq 1} E(\bar{\mathbb{Q}})[p^n]$ and v varies over the set of finite primes of L. If L_∞/K is an infinite algebraic extension of K, we define

$$S_p(E/L_\infty) := \varinjlim_L S_p(E/L),$$

where L varies over number fields $L \subset L_\infty$ and the direct limit is taken over the natural corestriction maps. The natural action of $\mathrm{Gal}(L_\infty/K)$ on $S_p(E/L_\infty)$ (endowed with discrete topology) makes it a discrete $\Lambda(\mathrm{Gal}(L_\infty/K))$-module. For a p-adic Lie extension L_∞, the Pontryagin dual

$$S_p(E/L_\infty)^\vee := \mathrm{Hom}_{\mathrm{cont}}\big(S_p(E/L_\infty), \mathbb{Q}_p/\mathbb{Z}_p\big)$$

is a finitely generated $\Lambda(\mathrm{Gal}(L_\infty/K))$-module (see for example [10]).

Definition 3.9. A Galois extension K_∞ of K is called *admissible* if

(i) K_∞ is unramified outside finitely many primes of K,
(ii) $G := \mathrm{Gal}(K_\infty/K)$ is a p-adic Lie group without any element of order p,
(iii) $K_{\mathrm{cyc}} := K\mathbb{Q}_{\mathrm{cyc}} \subset K_\infty$.

We write $\Gamma := \mathrm{Gal}(K_{\mathrm{cyc}}/K)$, $H := \mathrm{Gal}(K_\infty/K_{\mathrm{cyc}})$.

An admissible extension K_∞/K is said to be *strongly admissible* if for every prime $v|p$ of K and a prime $w|v$ of K_∞, the completion at w, denoted by $K_{\infty,w}$, contains the unramified $\mathbb{Z}_p$-extension of K_v.

Conjecture 3.10 ($\mathfrak{M}_H(G)$ conjecture, see [4, Conj. 5.1])**.** *Let E and p be as above. Let K_∞/K be admissible. Then the $\Lambda(G)$-module $S_p(E/K_\infty)^\vee$ lies in $\mathfrak{M}_H(G)$. (Note that for $K_\infty = K_{\mathrm{cyc}}$ this is precisely Mazur's conjecture.)*

If $S_p(E/K_{\mathrm{cyc}})^\vee$ is a finitely generated $\mathbb{Z}_p$-module, then it can be shown that $S_p(E/K_\infty)^\vee \in \mathfrak{M}_H(G)$ (see [4, Prop. 5.6]). In particular, if $S_p(E/K_{\mathrm{cyc}})^\vee$ is a finitely generated $\mathbb{Z}_p$-module, then $Ak^U_{U\cap H}(S_p(E/K_\infty)^\vee)$ exists for every open normal subgroup U.

For an open normal subgroup U of $G = \mathrm{Gal}(K_\infty/K)$, $K_U := K_\infty^U$ is the fixed field of K_∞ by U. We write $K_{U,\mathrm{cyc}} := K_U K_{\mathrm{cyc}}$. Note that $H \cap U = \mathrm{Gal}(K_\infty/K_{U,\mathrm{cyc}})$. Put $\Gamma_U = \mathrm{Gal}(K_{U,\mathrm{cyc}}/K_U) \cong \mathbb{Z}_p$. For a number field L and a prime v of L, set

$$J_v(E/L_{\mathrm{cyc}}) := \prod_{w|v} H^1(L_{\mathrm{cyc},w}, E(p)),$$

where w varies over the set of primes of L_{cyc} lying over v. The Pontryagin dual of $J_v(E/L_{\mathrm{cyc}})$, denoted by $X_v(E/L_{\mathrm{cyc}})$, is a finitely generated $\mathbb{Z}_p$-module and hence a torsion $\Lambda(\mathrm{Gal}(L_{\mathrm{cyc}}/L))$-module.

Let E and p be as before and take K_∞/K to be a strongly admissible extension. Assume

$$\frac{S_p(E/K_\infty)^\vee}{S_p(E/K_\infty)^\vee(p)}$$

is finitely generated over $\Lambda(H)$. As a consequence of [16, Thm. 1.3], we have for every open normal subgroup U of G,

$$\begin{aligned}(9)\qquad & Ak^U_{U\cap H}(S_p(E/K_\infty)^\vee) \\ &\qquad = \mathrm{char}_{\Lambda(\Gamma_U)}(S_p(E/K_{U,\mathrm{cyc}})^\vee)\prod_{v\in\Sigma_U}\mathrm{char}_{\Lambda(\Gamma_U)}(X_v(E/K_{U,\mathrm{cyc}})),\end{aligned}$$

where

$$(10)\qquad \Sigma_U = \big\{v \text{ prime in } K_U \mid I_v(K_\infty/K_U) \text{ is infinite}\big\}$$

and $I_v(K_\infty/K_U)$ denotes the inertia subgroup of $\mathrm{Gal}(K_\infty/K_U)$ at v.

Using Corollary 3.7 and Lemma 3.5 in (9), we deduce an application of Theorem 1.2.

Corollary 3.11. *Let E and p be as before and take K_∞/K to be a strongly admissible extension with $G = \mathrm{Gal}(K_\infty/K)$. Assume*

$$\frac{S_p(E/K_\infty)^\vee}{S_p(E/K_\infty)^\vee(p)}$$

is finitely generated over $\Lambda(H)$. Then there is a countable subset S_M of the set all continuous characters from Γ to $\mathbb{Z}_p^\times$, such that if we choose and fix any character ρ outside S_M, then for every open normal subgroup U of G, $\chi(U, S_p(E/K_\infty)^\vee(\rho))$ exists and

$$\begin{aligned}&\chi\big(U, S_p(E/K_\infty)^\vee(\rho)\big)\\ &\quad= \tilde\rho^{-1}\big(Ak^U_{U\cap H}(S_p(E/K_\infty)^\vee)\big)\\ &\quad= \tilde\rho^{-1}\big(\mathrm{char}_{\Lambda(\Gamma_U)}(S_p(E/K_{U,\mathrm{cyc}})^\vee)\big)\prod_{v\in\Sigma_U}\tilde\rho^{-1}\big(\mathrm{char}_{\Lambda(\Gamma_U)}(X_v(E/K_{U,\mathrm{cyc}}))\big)\end{aligned}$$

with Σ_U given in (10).

We refer to the discussions in [16, §1.2] where the strong admissibility condition in Corollary 3.11 can be replaced by a different set of hypotheses.

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