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The Grothendieck group of polytopes and norms

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Abstract. Polytopes in $\mathbb{R}^n$ with integral vertices form a monoid under the Minkowski sum, and the Grothendieck construction gives rise to an abelian group. Symmetric polytopes generate a subgroup. Similarly, difference bodies (which we refer to as norms) also generate a subgroup. We show that for every n the two subgroups agree.

1. Introduction

In this paper we define a *polytope* in $\mathbb{R}^n$ to be the convex hull of a finite subset of $\mathbb{R}^n$. If the finite subset lies in the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$, then we say that the polytope is *integral*.

We denote by $\mathfrak{P}(n)$ the set of all integral polytopes in $\mathbb{R}^n$. Given two polytopes $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^n$, the *Minkowski sum of* $\mathcal{P}$ *and* $\mathcal{Q}$ is defined to be the polytope

$$\mathcal{P} + \mathcal{Q} := \big\{p + q \mid p \in \mathcal{P} \text{ and } q \in \mathcal{Q}\big\}.$$

Under the Minkowski sum $\mathfrak{P}(n)$ becomes an abelian monoid, where the identity element is the polytope consisting of the origin. We denote by $\mathfrak{G}(n)$ the Grothendieck group of the monoid $\mathfrak{P}(n)$. (See Section 2.1 for details.)

We introduce a few more definitions:

- The *reflection in the origin* of a polytope $\mathcal{P}$ is $\overline{\mathcal{P}} := \{-x \mid x \in \mathcal{P}\}$.
- A polytope $\mathcal{P}$ is *symmetric* if $\mathcal{P} = \overline{\mathcal{P}}$. Symmetric polytopes form a submonoid $\mathfrak{P}^{\mathrm{sym}}(n) \subset \mathfrak{P}(n)$ and a subgroup $\mathfrak{G}^{\mathrm{sym}}(n) \subset \mathfrak{G}(n)$.
- An (integral) polytope $\mathcal{P}$ is an (*integral*) *norm* if there exists an (integral) polytope $\mathcal{Q}$ such that $\mathcal{P} = \mathcal{Q} + \overline{\mathcal{Q}}$. (What we call norms are often referred

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to as difference bodies, but in light of Section 2.2 we prefer the non-standard name of a norm.) Integral norms form a submonoid $\mathfrak{P}^{\text{norm}}(n) \subset \mathfrak{P}(n)$ and they generate a subgroup $\mathfrak{G}^{\text{norm}}(n) \subset \mathfrak{G}(n)$.

Clearly a polytope that is a norm is also symmetric. In the real setting the converse holds. More precisely, any symmetric polytope $\mathcal{P}$ can be written as

$$\mathcal{P} = \tfrac{1}{2}\mathcal{P} + \tfrac{1}{2}\mathcal{P} = \tfrac{1}{2}\mathcal{P} + \overline{\tfrac{1}{2}\mathcal{P}}.$$

This shows that symmetric polytopes are also norms.

In the remainder of the paper we study only integral polytopes and integral norms. Since any integral norm is symmetric, it follows that $\mathfrak{P}^{\text{norm}}(n) \subset \mathfrak{P}^{\text{sym}}(n)$ and $\mathfrak{G}^{\text{norm}}(n) \subset \mathfrak{G}^{\text{sym}}(n)$ for any n. We address the question whether all symmetric integral polytopes are integral norms. The question arises naturally on its own, and in addition, there is a motivation from the study of group rings. See Section 2.2 for a related discussion.

Every one-dimensional symmetric integral polytope $\mathcal{P}$ is of the form $\mathcal{P} = [-x, x]$ for some $x \in \mathbb{Z}_{\geq 0}$. It can be written as $\mathcal{P} = \mathcal{Q} + \overline{\mathcal{Q}}$ where $\mathcal{Q} = [0, x]$. This shows that every one-dimensional symmetric integral polytope is in fact an integral norm. Thus $\mathfrak{P}^{\text{norm}}(1) = \mathfrak{P}^{\text{sym}}(1)$ and $\mathfrak{G}^{\text{norm}}(1) = \mathfrak{G}^{\text{sym}}(1)$.

The situation is more subtle in dimension two and higher. First of all we have the following elementary lemma.

Lemma 1.1. *For any $n \geq 2$ we have $\mathfrak{P}^{\text{sym}}(n) \neq \mathfrak{P}^{\text{norm}}(n)$.*

Our main result is that, to our surprise, the situation is very different if one considers the Grothendieck group. More precisely, we have the following theorem.

Theorem 1.2. *For any n we have $\mathfrak{G}^{\text{sym}}(n) = \mathfrak{G}^{\text{norm}}(n)$.*

The polytope group $\mathfrak{G}(n)$ has garnered a lot of interest over the last few years. For example the second author, Wolfgang Lück and Stephan Tillmann [2, 3, 4] associate to an L^2-acyclic group π that satisfies the Atiyah conjecture an element in $\mathfrak{G}(n)$, where $n = b_1(\pi)$. This invariant has been examined by the third author and Dawid Kielak for free-by-cyclic groups in [6]. Furthermore the third author showed in [5, Thm. 4.4] that $\mathfrak{G}(n)$ is a free abelian group by constructing an explicit basis and proved a statement dual to that of Theorem 1.2, see [5, Thm. 6.3].

2. Preliminaries

2.1. **The polytope group.** Let $n \in \mathbb{N}$. It is straight-forward to show that the monoid $\mathfrak{P}(n)$ of integral polytopes has the cancellation property, i.e. for polytopes $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathfrak{P}(n)$ with $\mathcal{P} + \mathcal{Q} = \mathcal{P} + \mathcal{R}$ we have $\mathcal{Q} = \mathcal{R}$. (For instance see [9, Lem. 3.1.8].)

On $\mathfrak{P}(n) \times \mathfrak{P}(n)$, define $(\mathcal{P}, \mathcal{Q}) \sim (\mathcal{P}', \mathcal{Q}')$ if $\mathcal{P} + \mathcal{Q}' = \mathcal{P}' + \mathcal{Q}$. This is an equivalence relation since $\mathfrak{P}(n)$ has the cancellation property. Let $\mathfrak{G}(n)$ be the set of equivalence classes. It is straight-forward to see that $\mathfrak{G}(n)$ is an abelian group under

$$(\mathcal{P},\mathcal{Q})+(\mathcal{P}',\mathcal{Q}') := (\mathcal{P}+\mathcal{P}',\mathcal{Q}+\mathcal{Q}').$$

It is referred to as the *Grothendieck group* of $\mathfrak{P}(n)$. It is also straight-forward to see that the map

$$\mathfrak{P}(n) \to \mathfrak{G}(n), \quad \mathcal{P} \mapsto (\mathcal{P}, 0)$$

is a monomorphism. We will use this monomorphism to identify $\mathfrak{P}(n)$ with its image in $\mathfrak{G}(n)$. As usual, given $\mathcal{P}$ and $\mathcal{Q} \in \mathfrak{P}(n)$, we write $\mathcal{P}-\mathcal{Q} = (\mathcal{P},\mathcal{Q})$.

2.2. **Motivation: The marked polytope for a group ring element.** Here we discuss a motivation which leads us to consider *integral* polytopes. Let G be a finitely generated group. An *integral polytope in* $H_1(G;\mathbb{R})$ is the convex hull of a finite number of points in $\operatorname{Im}\{H_1(G;\mathbb{Z}) \to H_1(G;\mathbb{R})\}$. All the concepts and definitions for polytopes in $\mathbb{R}^n$ generalize in an obvious way to polytopes in $H_1(G;\mathbb{R})$. In particular, we can consider the monoid $\mathfrak{P}(H)$ of integral polytopes in $H := H_1(G;\mathbb{R})$ and we can consider the corresponding group $\mathfrak{G}(H)$.

We denote by $\epsilon : G \to H = H_1(G;\mathbb{R})$ the canonical map. Given a nonzero element

$$f = \sum_{g\in G} f_g g \in \mathbb{Z}[G] \quad (f_g \in \mathbb{Z})$$

we refer to

$$\mathcal{P}(f) := \text{convex hull of } \big\{\epsilon(g) \mid g \in G \text{ with } f_g \neq 0\big\} \subset H = H_1(G;\mathbb{R})$$

as the *polytope of* f. Now suppose that the ring $\mathbb{Z}[G]$ is a domain, i.e. it has no nonzero element which is a left or right zero-divisor. Conjecturally this is precisely the case when G is torsion-free. It is straight-forward to see that in this case the map

$$\mathcal{P} : \mathbb{Z}[G] \setminus \{0\} \to \mathfrak{P}(H), \quad f \mapsto \mathcal{P}(f)$$

is a monoid homomorphism. We refer to [4, Lem. 3.2] for details.

Now let G be a group that is torsion-free elementary amenable. It follows from [7, Thm. 1.4] that the group ring $\mathbb{Z}[G]$ is a domain. Furthermore by [1, Cor. 6.3] the ring $\mathbb{Z}[G]$ satisfies the Ore condition, that is, for any two nonzero elements $x, y \in \mathbb{Z}[G]$ there exist nonzero elements $p, q \in \mathbb{Z}[G]$ such that $xp = yq$. This implies that $\mathbb{Z}[G]$ admits a 'naive' ring of fractions $\mathbb{K}(G)$, which usually is referred to as the Ore localization of $\mathbb{Z}[G]$. We refer to [8, §4.4] for details. In the following we denote by $\mathbb{K}(G)^\times_{\mathrm{ab}}$ the abelianization of the multiplicative group $\mathbb{K}(G)^\times = \mathbb{K}(G) \setminus \{0\}$. It is straight-forward to see that the above map $\mathcal{P} : \mathbb{Z}[G] \setminus \{0\} \to \mathcal{P}(H)$ extends to a group homomorphism

$$\mathcal{P} : \mathbb{K}(G)^\times_{\mathrm{ab}} \to \mathfrak{G}(H).$$

This group homomorphism, and a generalization thereof to groups that satisfy the Atiyah conjecture, are used in [2, 3, 4] to study elements in $\mathbb{K}(G)^\times_{\mathrm{ab}}$.

The inversion $g \mapsto g^{-1}$ on G extends linearly to the standard involution $f \mapsto \overline{f}$ on $\mathbb{Z}[G]$. It extends naturally to an involution on $\mathbb{K}(G)$ and on $\mathbb{K}(G)^\times_{\mathrm{ab}}$. A *norm* in $\mathbb{K}(G)^\times_{\mathrm{ab}}$ is an element that can be written as $f \cdot \overline{f}$ for some $f \in \mathbb{K}(G)$. The above group homomorphism $\mathcal{P} : \mathbb{K}(G)^\times_{\mathrm{ab}} \to \mathfrak{G}(H)$ sends norms to norms. Our paper grew out of an attempt to detect elements in $\mathbb{K}(K)^\times_{\mathrm{ab}}$ that are not norms.

3. Proofs

3.1. **Proof of Lemma 1.1.** Let $\mathcal{P}$ be a polytope in $\mathbb{R}^n$. A *face of* $\mathcal{P}$ is any subset of $\mathcal{P}$ of the form

$$F = \{p \in \mathcal{P} \mid \phi(p) = \max\{\phi(p') \mid p' \in \mathcal{P}\}\},$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a homomorphism. Recall that a face of $\mathcal{P}$ is a polytope in its own right. In fact, a face of $\mathcal{P}$ is the convex hull of a proper subset of the vertex set of $\mathcal{P}$. We call a polytope contained in a face a *subface of* $\mathcal{P}$. Then we have the following elementary lemma.

Lemma 3.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be polytopes in $\mathbb{R}^n$. Then any face of $\mathcal{P}$ is, up to translation, a subface of $\mathcal{P} + \mathcal{Q}$.*

Proof. Given a face F of $\mathcal{P}$, there exists $\phi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ such that

$$F = \{p \in \mathcal{P} \mid \phi(p) = \max\{\phi(p') \mid p' \in \mathcal{P}\}\}.$$

If we let

$$G = \{q \in \mathcal{Q} \mid \phi(q) = \max\{\phi(q') \mid q' \in \mathcal{Q}\}\},$$

then $F + G$ is a face of $\mathcal{P} + \mathcal{Q}$ which contains a translate of F. □

Now we are in a position to prove Lemma 1.1.

Proof of Lemma 1.1. We first show that $\mathfrak{P}^{\mathrm{sym}}(2) \neq \mathfrak{P}^{\mathrm{norm}}(2)$. Let $k \in \mathbb{N}$. We denote by $\mathcal{P}$ the integral two-dimensional symmetric polytope spanned by $(k, 0), (k, 1), (-k, 0)$ and $(-k, -1)$. We want to show that $\mathcal{P}$ is not an integral norm. We denote by $\mathcal{X}$ the integral polytope spanned by $(0, 0)$ and $(2k, 1)$, and by $\mathcal{Y}$ the integral polytope spanned by $(0, 0)$ and $(0, 1)$.

Suppose $\mathcal{P}$ is an integral norm. Thus we can write $\mathcal{P} = \mathcal{Q} + \overline{\mathcal{Q}}$, where $\mathcal{Q}$ is an integral polytope. By Lemma 3.2, each face of $\mathcal{Q}$ is, up to translation, a subface of $\mathcal{P}$. This implies that, up to translation, each face of $\mathcal{Q}$ is a subface of $\mathcal{X}$ or of $\mathcal{Y}$. Since neither $\mathcal{X}$ or $\mathcal{Y}$ admits one-dimensional integral subpolytopes, we see that, up to translation, each face of $\mathcal{Q}$ is either $\mathcal{X}$, $\mathcal{Y}$ or a point. In particular, up to translation, $\mathcal{Q}$ equals either $\{0\}$, $\mathcal{X}$, $\mathcal{Y}$ or $\mathcal{X} + \mathcal{Y}$. But it is straight-forward to verify that in each case $\mathcal{Q} + \overline{\mathcal{Q}} \neq \mathcal{P}$.

This shows that $\mathfrak{P}^{\mathrm{sym}}(2) \neq \mathfrak{P}^{\mathrm{norm}}(2)$. Now let $n \geq 3$. We consider the two maps

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^n, \quad (x, y) \mapsto (x, y, 0),$$
$$\Psi : \mathbb{R}^n \to \mathbb{R}^2, \quad (x_1, \ldots, x_n) \mapsto (x_1, x_2).$$

Both maps induce homomorphisms on the polytope monoids that map symmetric polytopes to symmetric polytopes and integral norms to integral norms. Clearly Ψ is a splitting of Φ. Now it follows that if $\mathcal{P}$ is an integral symmetric polytope in $\mathbb{R}^2$ that is not a norm, then $\Phi(\mathcal{P}) \subset \mathbb{R}^n$ is also a symmetric polytope that is not an integral norm. □

3.3. **Proof of Theorem 1.2.** In this section we will prove Theorem 1.2. So given any n we want to show that $\mathfrak{G}^{\mathrm{sym}}(n) = \mathfrak{G}^{\mathrm{norm}}(n)$. This is equivalent to showing that given any integral polytope $\mathcal{P}$ in $\mathbb{R}^n$ there exist integral polytopes $\mathcal{Q}$ and $\mathcal{R}$ such that

$$\mathcal{P} + \mathcal{Q} + \overline{\mathcal{Q}} = \mathcal{R} + \overline{\mathcal{R}}.$$

The key idea is to prove this statement by induction on n where we perform the induction step by cutting along a hyperplane.

In the proof of Theorem 1.2 we will use the following definitions and notations. A hyperplane H in $\mathbb{R}^n$ can be written as $H = \{x \in \mathbb{R}^n \mid x \cdot v = 0\}$ for some $v \in \mathbb{R}^n$. We define the *halves* of a real polytope $\mathcal{P} \subset \mathbb{R}^n$ with respect H to be

$$\mathcal{P}_+ := \big\{x \in \mathcal{P} \mid x \cdot v \geq 0\big\}, \quad \mathcal{P}_- := \big\{x \in \mathcal{P} \mid x \cdot v \leq 0\big\}.$$

Informally speaking, when H meets $\mathcal{P}$ in a proper subset, $\mathcal{P}_+$ and $\mathcal{P}_-$ are obtained by cutting $\mathcal{P}$ along H. We remark that $\mathcal{P}_+$ and $\mathcal{P}_-$ may be exchanged depending on the choice of v, but it will not cause any issue for our purpose. It is known that each of $\mathcal{P}_+$, $\mathcal{P}_-$ and $\mathcal{P} \cap H$ is a real polytope whenever it is nonempty.

Lemma 3.4 (Normalization by a hyperplane)**.** *Suppose $\mathcal{P} \in \mathbb{R}^n$ is a symmetric polytope and $H \subset \mathbb{R}^n$ is a hyperplane. Let $\mathcal{P}_+$ and $\mathcal{P}_-$ be the halves of $\mathcal{P}$ with respect to H. Then*

$$\mathcal{P}_+ + \overline{\mathcal{P}_+} = \mathcal{P}_- + \overline{\mathcal{P}_-} = \mathcal{P}_+ + \mathcal{P}_- = \mathcal{P} + (\mathcal{P} \cap H).$$

Proof. Since $\mathcal{P}$ is symmetric, $\overline{\mathcal{P}_\pm} = \mathcal{P}_\mp$. Therefore it suffices to show that $\mathcal{P}_+ + \mathcal{P}_- = \mathcal{P} + (\mathcal{P} \cap H)$.

Each $x \in \mathcal{P}$ lies in either $\mathcal{P}_+$ or $\mathcal{P}_-$. If $x \in \mathcal{P}_+$, then since

$$\mathcal{P} \cap H = \mathcal{P}_+ \cap \mathcal{P}_- \subset \mathcal{P}_-,$$

we have $\{x\} + (\mathcal{P} \cap H) \subset \mathcal{P}_+ + \mathcal{P}_-$. By symmetry, $\{x\} + (\mathcal{P} \cap H) \subset \mathcal{P}_+ + \mathcal{P}_-$ when $x \in \mathcal{P}_-$. It follows that $\mathcal{P} + (\mathcal{P} \cap H) \subset \mathcal{P}_+ + \mathcal{P}_-$.

For the reverse inclusion, suppose $x \in \mathcal{P}_+$ and $y \in \mathcal{P}_-$. Since $\mathcal{P}$ is convex, there is $t \in [0, 1]$ such that the point $z := tx + (1-t)y$ lies on $\mathcal{P} \cap H$. Consider $p := (1-t)x + ty$. Since $\mathcal{P}$ is convex, $p \in \mathcal{P}$. Therefore $x + y = p + z$ lies in $\mathcal{P} + (\mathcal{P} \cap H)$. This shows $\mathcal{P}_+ + \mathcal{P}_- \subset \mathcal{P} + (\mathcal{P} \cap H)$. □

We can not directly apply Lemma 3.4 to an integral polytope, since in general, given an integral polytope $\mathcal{P}$ there does not exist a hyperplane, such that $\mathcal{P} \cap H$ is again integral. To overcome this, the following is useful:

Lemma 3.5 (Vertical stretching)**.** *Let $\mathcal{P}$ be an integral polytope in $\mathbb{R}^n$. Denote by $d\mathcal{Z}$ the line segment in $\mathbb{R}^n$ joining the origin and the point $(0,\dots,0,d)$. As usual, identify $\mathbb{R}^{n-1}$ with the hyperplane of points with last coordinate zero in $\mathbb{R}^n$. Then for all sufficiently large $d>0$,*

$$(\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}})\cap\mathbb{R}^{n-1}$$

is integral.

Proof. Denote by $\pi:\mathbb{R}^n\to\mathbb{R}^{n-1}$ the projection map which forgets the last coordinate. Denote by $v_1,\dots,v_k\in\mathbb{Z}^n$ the vertices of $\mathcal{P}$. Let $w_i=\pi(v_i)\in\mathbb{R}^{n-1}$ and write $v_i=(w_i,a_i)$ with $a_i\in\mathbb{Z}$. Let $\mathcal{Y}$ be the convex hull of $\{w_1,\dots,w_k\}$. Suppose d satisfies $d>|a_i|$ for all i. Now it suffices to prove the following:

Claim. $(\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}})\cap\mathbb{R}^{n-1}=\mathcal{Y}$.

Obviously we have

$$(\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}})\cap\mathbb{R}^{n-1}=\pi\big((\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}})\cap\mathbb{R}^{n-1}\big)\subset\pi(\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}}).$$

Since $\pi(d\mathcal{Z})=\{0\}$, we deduce $\pi(\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}})=\pi(\mathcal{P})$. Since the projection of the convex hull of a set is the convex hull of the projection of the set, we have $\pi(\mathcal{P})=\mathcal{Y}$. It follows that $(\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}})\cap\mathbb{R}^{n-1}\subset\mathcal{Y}$.

For the reverse inclusion, observe that $(w_i,a_i\pm d)\in\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}}$ for each i. Note that one of $a_i\pm d$ is negative and the other is positive. By convexity we have $(w_i,0)\in\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}}$. Once again by convexity, it follows that

$$\mathcal{Y}\subset(\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}})\cap\mathbb{R}^{n-1}. \qquad \square$$

Now we are ready for the proof of the main result of this paper.

Proof of Theorem 1.2. We prove the theorem by induction on n. For $n=0$, the statement is trivial. Suppose the conclusion holds for $n-1$, and suppose $\mathcal{P}$ is a symmetric integral polytope in $\mathbb{R}^n$. As above we identify $\mathbb{R}^{n-1}$ with the hyperplane of points with last coordinate zero in $\mathbb{R}^n$. By Lemma 3.5, there is $d\in\mathbb{N}$ such that $(\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}})\cap\mathbb{R}^{n-1}$ is an integral polytope in $\mathbb{R}^{n-1}$. Write $\mathcal{Y}=\mathcal{P}+d\mathcal{Z}+\overline{d\mathcal{Z}}$ for brevity. Since $\mathcal{Y}$ is symmetric, $\mathcal{Y}\cap\mathbb{R}^{n-1}$ is symmetric too. Therefore, by the induction hypothesis, there are integral polytopes $\mathcal{Q}$ and $\mathcal{R}$ in $\mathbb{R}^{n-1}$ such that

$$(\mathcal{Y}\cap\mathbb{R}^{n-1})+\mathcal{Q}+\overline{\mathcal{Q}}=\mathcal{R}+\overline{\mathcal{R}}.$$

By Lemma 3.4, we have

$$\mathcal{Y}+(\mathcal{Y}\cap\mathbb{R}^{n-1})=\mathcal{Y}_++\overline{\mathcal{Y}_+},$$

where $\mathcal{Y}_+$ denotes a half of $\mathcal{Y}$ with respect to the hyperplane $\mathbb{R}^{n-1}$. Since $\mathcal{Y}\cap\mathbb{R}^{n-1}$ is integral, we also deduce that $\mathcal{Y}_+$ is an integral polytope. From the above equations, it follows that

$$\mathcal{P}+(d\mathcal{Z}+\mathcal{Q})+\overline{(d\mathcal{Z}+\mathcal{Q})}=(\mathcal{Y}_++\mathcal{R})+\overline{(\mathcal{Y}_++\mathcal{R})}. \qquad \square$$

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