

# Equivariant correspondences and the Borel–Bott–Weil theorem

Heath Emerson and Robert Yuncken

(Communicated by Siegfried Echterhoff)

**Abstract.** We prove an analog of the Borel–Bott–Weil theorem in equivariant KK-theory by constructing certain canonical equivariant correspondences between minimal flag varieties  $G/B$ , with  $G$  a complex semisimple Lie group.

## 1. INTRODUCTION

Let  $G$  be a complex semisimple Lie group and  $B \subset G$  a minimal parabolic subgroup. Let  $\mu$  be a weight for  $G$  and  $E_\mu$  the corresponding induced holomorphic line bundle on the flag manifold  $X = G/B$ . The Dolbeault cohomology group  $H^*(X, E_\mu)$  with its canonical action of  $G$ , is a graded-finite-dimensional representation of  $G$ , and, more relevantly for us, of its maximal compact subgroup  $K \subset G$ . The Borel–Bott–Weil theorem computes this representation [4].

Bott’s key observation was that there is a Weyl-group symmetry in the solution to the problem: if the weights  $\mu$  and  $\mu'$  are in the same orbit of the shifted Weyl group action, then  $H^*(X, E_\mu)$  and  $H^*(X, E_{\mu'})$  are equal, up to a shift in degree. In this paper, we will look at this symmetry from the point of view of correspondences in geometric equivariant K-theory.

The bridge between Dolbeault cohomology and K-theory is provided by index theory of elliptic operators:  $H^*(G/K, E_\mu)$ , as a virtual  $K$ -representation, is the same as the  $K$ -index  $\text{Index}_K[\bar{\partial}]_\mu \in R(K)$ , in the sense of Atiyah and Singer [1], of the Dolbeault operator twisted by  $E_\mu$ . From the point of view of Kasparov theory, the class  $[\bar{\partial}]_\mu$  is an element of the  $K$ -equivariant K-homology  $\text{KK}^K(G/B, \mathbb{C})$  of  $G/B$ . This  $R(K)$ -module is acted on by the bivariant group  $\text{KK}^K(G/B, G/B)$ , for which a topological model was developed in [6] using

---

Heath Emerson was supported by a Natural Science and Engineering Council of Canada (NSERC) Discovery grant. Part of this research was carried out while Robert Yuncken was supported by a PIMS post-doctoral fellowship at the University of Victoria, and part was supported by the project SINGSTAR of the Agence Nationale de la Recherche, ANR-14-CE25-0012-01.

the theory of equivariant correspondences. The correspondence theory is the main tool used in this work. We consider certain canonical correspondences  $\Lambda(w)$ , parameterized by the elements of the Weyl group  $W$ , compute how these correspondences act on equivariant K-homology, and relate it to the Borel–Bott–Weil theorem.

Let  $\mathfrak{h}$  be a Cartan subalgebra of the Lie algebra  $\mathfrak{g}$  of  $G$  and  $\Gamma_W \subset \mathfrak{h}^*$  the lattice of weights. Let  $\Delta^+$  be a set of positive roots for  $G$ , which brings with it a generating set of simple reflections for the Weyl group  $W$  and a corresponding word length function  $l : W \rightarrow \mathbb{N}$ . Up to conjugacy, the minimal parabolic subgroup  $B \subset G$  is the subgroup with Lie algebra  $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ , where  $\mathfrak{g}_\alpha$  is the  $\alpha$ -root space of the Lie algebra of  $G$ .

Let

$$[G/B]_\mu \in \mathrm{KK}^K(C(G/B), \mathbb{C}) =: \mathrm{K}_0^K(G/B)$$

be the class of the Dolbeault operator on  $G/B$  twisted by the  $K$ -equivariant line bundle  $E_\mu$ . Let  $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$  be half the sum of the positive roots. The following theorem is essentially due to Bott.

**Theorem 1.1.** *In the above notation, for any weight  $\mu$  of  $G$  and any  $w \in W$ , the identity*

$$\mathrm{Index}_K[G/B]_\mu = (-1)^{l(w)} \mathrm{Index}_K[G/B]_{w(\mu+\rho)-\rho}$$

holds in  $\mathrm{R}(K) = \mathrm{KK}^K(\mathbb{C}, \mathbb{C})$ .

The focus of this article is the KK-theory which lies behind Theorem 1.1. We show how to prove Theorem 1.1 using the theory [6] of equivariant correspondences. For a verification of the Weyl character formula using similar techniques, see the paper [3].

The Weyl group element  $w$  conjugates the subgroup  $B$  to another minimal parabolic subgroup  $B_w$ . The homogeneous space  $G/B \cap B_w$  admits a pair of natural  $K$ -equivariant holomorphic fibrations to  $G/B$  and  $G/B_w$ . Since the latter space is  $K$ -equivariantly biholomorphic to  $G/B$ , we have realized  $G/B \cap B_w$  as a holomorphic fibered space over  $G/B$  in two different ways. In fact, in each case  $G/B \cap B_w$  is  $K$ -equivariantly biholomorphic to the total space of a complex vector bundle over  $G/B$ . Using the Thom class  $\tau(q_w)$  associated to the latter of these fibrations, we get a  $K$ -equivariant holomorphic correspondence

$$G/B \xleftarrow{p_w} (G/B \cap B_w, \tau(q_w)) \xrightarrow{q_w} G/B_w \cong G/B$$

from  $G/B$  to itself. This yields an element of  $\widehat{\mathrm{KK}}^K(G/B, G/B)$  which we denote by  $\Lambda(w)$  and call the *Borel–Bott–Weil morphism with parameter  $w \in W$* .

The main result of the paper is the following.

**Theorem 1.2** (Borel–Bott–Weil product formula). *For any weight  $\mu$  and  $w \in W$ , the identity*

$$\Lambda(w) \otimes_{G/B} [G/B]_\mu = (-1)^{l(w)} [G/B]_{w(\mu+\rho)-\rho} \in \mathrm{KK}^K(G/B, \star)$$

holds, where  $\Lambda(w)$  is the Borel–Bott–Weil morphism with parameter  $w$ .

This easily implies the analog Theorem 1.1 of the Borel–Bott–Weil theorem above.

**Remark 1.3.** The ring  $\text{KK}^K(G/B, G/B)$  is computed explicitly in [3], see also [9]. We will see that the class  $\Lambda(w)$  above corresponds to the class which is referred to as the “intertwiner”  $I_w$  in [3].

We close by noting that we can replace  $K$ -equivariance by  $G$ -equivariance in Theorem 1.1, using the Baum–Connes conjecture. Classically, the Borel–Bott–Weil theorem is a statement about holomorphic (non-unitary) representations of noncompact groups. Kasparov theory does not admit such representations. Instead, equivariant Kasparov theory for noncompact groups uses unitary, but possibly infinite-dimensional representations, and almost-equivariant Fredholm operators; these are the cycles for the Kasparov representation ring  $\text{KK}^G(\mathbb{C}, \mathbb{C})$ . There is a restriction map

$$\text{KK}^G(A, B) \rightarrow \text{KK}^K(A, B)$$

when  $K \subset G$  is a maximal compact subgroup as above, by forgetting  $G$ -equivariance to  $K$ -equivariance on cycles. The Baum–Connes apparatus shows that this map is an isomorphism when  $A$  has the form  $A = \mathcal{C}(G/B) \otimes A'$  for some  $G$ - $C^*$ -algebra  $A'$ ; this follows from a theorem of Tu [10]. Since all the analytic Kasparov classes defined by us have this form, Theorems 1.1 and 1.2 have their counterparts with  $K$  replaced by  $G$ .

## 2. PRELIMINARIES

**2.1. Equivariant correspondences.** The environment in which the calculations of this paper will take place is the topological model for equivariant Kasparov theory developed in [6]. We refer the reader to this article for details on the framework. All correspondences used in this paper will be smooth, which simplifies the definitions. Let  $K$  be a compact Lie group and let  $X$  and  $Y$  be smooth  $K$ -manifolds, i.e., smooth manifolds with smooth actions of  $K$ . A smooth correspondence is given by a quadruple  $(M, f, b, \xi)$  where

- $M$  is a smooth  $K$ -manifold,
- $f : M \rightarrow Y$  is a smooth  $K$ -equivariantly  $K$ -oriented map,
- $b : M \rightarrow X$  is a smooth  $K$ -equivariant map, and
- $\xi \in \text{RK}_{K,X}^*(M)$  is a smooth  $K$ -equivariant  $K$ -theory class with compact support along the fibers of  $b$  (in the terminology of [6], a  $K$ -theory class with  $M$ -compact support).

We usually use the notation

$$X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y,$$

as in [5] (the origin of the theory) to denote the quadruple above.

Note that if  $X$  is compact (the case throughout in this article), then

$$\text{RK}_{K,X}^*(M) = \text{K}_K^*(M),$$

the ordinary, compactly supported,  $K$ -equivariant  $K$ -theory of  $M$ .

The *degree* of the correspondence is the sum of the degrees of  $\xi$  and of  $f$ .

Equivalence classes of equivariant correspondences make up the morphisms in the additive category  $\widehat{\text{KK}}^K$  explained in [6]; there is a natural transformation  $\widehat{\text{KK}}^K \rightarrow \text{KK}^K$  to the usual analytic equivariant Kasparov category, inducing an *isomorphism*  $\widehat{\text{KK}}^K(X, Y) \rightarrow \text{KK}^K(X, Y)$  if  $X$  is a *normally non-singular*  $K$ -manifold, that is, if  $X$  admits a smooth,  $K$ -equivariant embedding into a finite-dimensional representation of  $K$ . A smooth  $K$ -manifold of finite orbit type is automatically normally non-singular, and in particular, all smooth, compact  $K$ -manifolds are normally non-singular. All concrete  $K$ -manifolds we meet in this paper are normally non-singular.

We generally operate in the category  $\widehat{\text{KK}}^K$  in this paper.

For any pair of  $K$ -spaces  $X$  and  $Y$ ,  $\widehat{\text{KK}}_*^K(X, Y)$  denotes the abelian group of equivalence classes of equivariant correspondences from  $X$  to  $Y$ , graded by parity of degree.

Two standard examples of  $\widehat{\text{KK}}^K$ -classes are important; to fix notation, we recall them.

**Example 2.2.** If  $b : Y \rightarrow X$  is a *proper*  $K$ -equivariant map, we define

$$b^* := [X \xleftarrow{b} (Y, \mathbf{1}_Y) \xrightarrow{\text{id}} Y],$$

where  $\mathbf{1}_Y$  is the class of the trivial line bundle  $Y \times \mathbb{C}$ , the unit in  $\text{RK}_{K,X}^*(Y) = \text{RK}_K^*(Y)$ .

**Example 2.3.** If  $\Phi$  is an equivariantly  $K$ -oriented smooth map from  $X$  to  $Y$ , where  $X$  and  $Y$  are smooth  $K$ -manifolds, we define the *wrong-way class* of  $\Phi$  as

$$\Phi_! := [X \xleftarrow{\text{id}} (X, \mathbf{1}_X) \xrightarrow{\Phi} Y],$$

where  $\mathbf{1}_X$  is the class of the trivial line bundle  $E \times \mathbb{C}$  in  $\text{RK}_K^*(X)$ .

By a *complex  $K$ -manifold* we shall mean a smooth complex manifold  $X$  equipped with a holomorphic action of  $K$ . The tangent bundle  $\text{TX}$  has a canonical  $K$ -equivariant complex structure and a corresponding  $K$ -equivariant  $K$ -orientation. This supplies an equivariant  $K$ -orientation on the map from  $X$  to a point. The corresponding wrong-way class is called the (*topological*) *fundamental class* of  $X$ , and denoted by  $[X]$ . Its image in  $\text{KK}_0^K(C_0(X), \mathbb{C})$  is the class of the  $K$ -equivariant Dolbeault operator on  $X$ .

Next, let  $M_1, M_2, Y$  be complex  $K$ -manifolds. Assume that both  $M_1$  and  $M_2$  are normally non-singular  $K$ -manifolds.

Two smooth maps  $f_1 : M_1 \rightarrow Y$  and  $b_2 : M_2 \rightarrow Y$  are *transverse* if for every pair of points  $m_1 \in M_1$  and  $m_2 \in M_2$  with  $f_1(m_1) = b_2(m_2)$ , the map

$$T_{m_1}M_1 \oplus T_{m_2}M_2 \rightarrow T_{f_1(m_1)}Y, \quad (\xi_1, \xi_2) \mapsto D_{m_1}f_1(\xi_1) + D_{m_2}b_2(\xi_2)$$

is surjective. It is shown in [6] that when transversality holds, the fiber product

$$M_1 \times_Y M_2 := \{(m_1, m_2) \mid f_1(m_1) = b_2(m_2)\}$$

is itself a smooth  $K$ -manifold (of finite orbit type) and the projection

$$\text{pr}_2 : M_1 \times_Y M_2 \rightarrow M_2$$

inherits a canonical equivariant  $K$ -orientation from the  $K$ -orientation on  $f_1$ .

If  $f_1$  and  $b_1$  are *holomorphic* maps, the fiber product  $M_1 \times_Y M_2$  will be a complex manifold, and the projection  $\text{pr}_2$  will be a holomorphic map; the corresponding  $K$ -orientation agrees with the one described in the previous paragraph.

**2.4. Complex semisimple Lie groups.** Here we review some standard structure theory for semisimple groups and fix notation for the remainder of the paper. For details, see, for example, [8].

Let  $G$  be a complex connected semisimple Lie group and  $\mathfrak{g}$  its Lie algebra. Denote by  $B(\cdot)$  its Killing form. Let  $\theta$  be a Cartan involution on  $\mathfrak{g}$ , so that

$$\langle v, w \rangle := -B(\theta(v), w), \quad v, w \in \mathfrak{g}$$

is a positive definite inner product on  $\mathfrak{g}$ ; the archetypal example is the operation of negative-conjugate-transpose on  $\mathfrak{sl}_n(\mathbb{C})$ . The  $+1$ -eigenspace of  $\theta$  is the Lie algebra  $\mathfrak{k}$  of a maximal compact subgroup  $K$  of  $G$ .

Fix  $\mathfrak{h}$ , a  $\theta$ -stable Cartan subalgebra. Let  $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ , which is the Lie algebra of a maximal torus  $T$  in  $K$ . We have  $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ , where  $\mathfrak{a} = i\mathfrak{t}$ , and we let  $A$  denote the subgroup of  $G$  with Lie algebra  $\mathfrak{a}$ .

The set of roots will be denoted by  $\Delta$ , with  $\mathfrak{g}_\alpha$  denoting the root space of  $\alpha \in \Delta$ . We fix a choice of positive roots  $\Delta^+$ , and recall that every positive root is a nonnegative integral combination of simple roots. The lattice of weights will be denoted by  $\Gamma_W$ , and the dominant weights are those  $\lambda \in \Gamma_W$  for which  $\langle \lambda, \alpha \rangle \geq 0$  for every positive root  $\alpha$ . We will frequently abuse notation by blurring the distinction between a weight  $\mu \in \Gamma_W$ , the corresponding representation of  $T$ , and the corresponding holomorphic representation of  $H = T \cdot A$ .

The Weyl group is  $W = N_G(H)/Z_G(H)$ . We will frequently identify elements  $w \in W$  with a lift to an element of  $N_G(H) \subseteq G$ , at least when the choice of lift makes no difference. The usual action of the Weyl group on weights will be denoted by  $\mu \mapsto w(\mu)$ . Let  $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$  be the half-sum of positive roots. We will often refer to the *shifted action* of the Weyl group, which is the action

$$w : \lambda \mapsto w(\lambda + \rho) - \rho.$$

We fix the standard Borel subalgebra  $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$ , where  $\mathfrak{n}$  is the nilpotent subalgebra  $\mathfrak{n} := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ . The associated subgroups are denoted by  $B$  and  $N$ . For each element  $w$  of the Weyl group, there are conjugate subgroups

$$B_w := wBw^{-1}, \quad N_w := wNw^{-1}$$

with corresponding Lie algebras  $\mathfrak{b}_w$  and  $\mathfrak{n}_w$ . We also define the Lie algebra  $\bar{\mathfrak{n}} := \theta\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$ , as well as its conjugates  $\bar{\mathfrak{n}}_w := \text{Ad}(w)\bar{\mathfrak{n}}$  for each  $w \in W$ .

The flag variety of  $G$  is the complex homogeneous space  $G/B$ . It is  $K$ -equivariantly diffeomorphic to  $K/T$  via the map

$$\iota : K/T \xrightarrow{\cong} G/B, \quad kT \mapsto kB.$$

However, we shall try to distinguish the two spaces as much as possible. The difference is technical but important:  $G/B$ , having a natural complex structure, is canonically  $K$ -oriented, while  $K/T$  only inherits a  $K$ -orientation once it is identified with  $G/B$ . Moreover, for any  $w \in W$ , there is a  $K$ -equivariant diffeomorphism

$$\iota_w : K/T \xrightarrow{\cong} G/B_w, \quad kT \mapsto kB_w,$$

each inducing a different  $K$ -orientation on  $K/T$ . This technicality is of course absolutely central to what follows.

### 3. THE BOREL–BOTT–WEIL THEOREM

**3.1. Twisted fundamental classes.** Let  $\mu$  be a weight of  $G$ . As mentioned above, it corresponds to a holomorphic representation of  $H$ , and one can extend it to a holomorphic character of  $B$  which is trivial on  $N$ . We denote the one-dimensional representation space by  $\mathbb{C}_\mu$ .

We shall use the notation  $E_\mu$  throughout to denote the induced  $G$ -equivariant line bundle

$$E_\mu := G \times_B \mathbb{C}_\mu.$$

We also have  $E_\mu \cong K \times_T \mathbb{C}_\mu$  by restriction.

Recall (see, e.g., Antony Wassermann's Frobenius reciprocity theorem [2, Thm. 20.5.5]) that  $K_K^*(K/T)$  is isomorphic to  $K_T^*(\mathbb{C}) = R(T)$ , the representation ring of  $T$ , as a  $\mathbb{Z}$ -module. The representation ring is just  $\mathbb{Z}[\Gamma_W]$ , the group ring of the weight lattice, and the isomorphism is given by induction:

$$\text{Ind}_T^K : R(T) \xrightarrow{\cong} K_K^*(K/T), \quad [\mu] \mapsto [E_\mu].$$

**Definition 3.2.** Given  $\mu \in \Gamma_W$ , we define the  $\mu$ -twisting class to be the element  $[[\mu]] \in \widehat{KK}^K(G/B, G/B)$  given by the following correspondence:

$$G/B \xleftarrow{\text{id}} (G/B, [E_\mu]) \xrightarrow{\text{id}} G/B.$$

The  $\mu$ -twisted fundamental class of  $G/B$ , denoted by  $[G/B]_\mu$ , is the class of the  $K$ -equivariant correspondence

$$G/B \xleftarrow{\text{id}} (G/B, [E_\mu]) \rightarrow \star$$

in  $\widehat{KK}^K(G/B, \star)$ .

Thus,  $[G/B]_\mu = [[\mu]] \otimes_{G/B} [G/B]$ , where  $[G/B] := [G/B]_0$  is the (untwisted) fundamental class of  $G/B$ . The reason for the terminology is that  $[G/B]$  institutes a duality isomorphism (see [6])

$$\widehat{KK}_*^K(G/B \times X, Y) \cong \widehat{KK}_*^K(X, G/B \times Y)$$

valid for arbitrary  $K$ -spaces  $X$  and  $Y$ . For example if  $X = Y = \star$  then duality gives an isomorphism

$$R(T) = \mathbb{Z}[\Gamma_W] \cong \widehat{KK}^K(G/B, \star).$$

This duality can easily be verified to send the point mass at a weight  $\mu \in \Gamma_W$  to the class  $[G/B]_\mu$ .

**3.3. Borel–Bott–Weil correspondences.** Let  $w$  be an element of the Weyl group  $W = N_G(H)/Z_G(H)$ . Recall that the subgroup  $B_w := wBw^{-1}$  is independent of the choice of lift of  $w$  to  $N_G(H) \subseteq G$ . It is another minimal parabolic subgroup of  $G$ .

Consider the homogeneous space  $G/(B \cap B_w)$ . This admits two  $G$ -equivariant fibrations, given by the natural maps

$$p_w : (G/B \cap B_w) \rightarrow G/B, \quad q_w : (G/B \cap B_w) \rightarrow G/B_w.$$

Viewing  $G/(B \cap B_w)$  as a  $K$ -space by restriction, both of these fibrations can be realized as  $K$ -equivariant vector bundle projections, as we now describe.

Recall that we define  $N_w := wNw^{-1}$ ,  $\bar{N}_w := w\bar{N}w^{-1}$ , with Lie algebras  $\mathfrak{n}_w$  and  $\bar{\mathfrak{n}}_w$  respectively. Then  $\mathfrak{n} = (\mathfrak{n} \cap \bar{\mathfrak{n}}_w) \oplus (\mathfrak{n} \cap \mathfrak{n}_w)$  is a decomposition of  $\mathfrak{n}$  into Lie subalgebras. Since  $N$  is a connected simply-connected nilpotent Lie group, there is a corresponding factorization  $N = (N \cap \bar{N}_w)(N \cap N_w)$ .

**Lemma 3.4.** *Let  $w \in W$ . One can define a  $K$ -equivariant diffeomorphism*

$$\varphi_w : |K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)| \xrightarrow{\cong} G/(B \cap B_w)$$

by the formula  $\varphi_w : [k, X] \mapsto k \exp(X).(B \cap B_w)$  such that the diagram

$$(1) \quad \begin{array}{ccc} K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w) & \xrightarrow[\cong]{\varphi_w} & G/(B \cap B_w) \\ \pi_w \downarrow & & \downarrow p_w \\ K/T & \xrightarrow[\cong]{} & G/B \end{array}$$

commutes. Moreover,  $\varphi_w$  is fiberwise holomorphic (with respect to the fibrations  $\pi_w$  and  $p_w$ ).

In other words, the  $K$ -equivariant fibrations  $K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w) \rightarrow K/T$  and  $G/(B \cap B_w) \rightarrow G/B$  are equivalent in the category of  $K$ -equivariant fibrations with holomorphic fibers.

*Proof.* To see that the map  $\varphi_w$  is well-defined we compute, for any  $k \in K$ ,  $t \in T$ ,  $X \in \mathfrak{n} \cap \bar{\mathfrak{n}}_w$ :

$$\begin{aligned} \varphi_w([kt, \text{Ad}(t^{-1})X]) &= kt.t^{-1} \exp(X)t(B \cap B_w) \\ &= k \exp(X).(B \cap B_w) = \varphi_w([k, X]). \end{aligned}$$

Next we show surjectivity. Let  $g \in G$  be arbitrary. There is a decomposition

$$G = KNA = K(N \cap \bar{N}_w)(N \cap N_w)A,$$

and we decompose  $g$  as  $g = kn_1n_2a$  accordingly. Since  $(N \cap N_w)A \subseteq B \cap B_w$ , we have  $\varphi_w([k, \log(n_1)]) = g(B \cap B_w)$ .

Next suppose  $[k, X]$  and  $[k', X'] \in K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)$  have the same image under  $\varphi_w$ . Since  $B \cap B_w = T \cdot (N \cap N_w)A$ , there exist  $t \in T$  and  $n_2a \in (N \cap N_w)A$  such that

$$k' \exp(X') = k \exp(X)tn_2a = kt \exp(\text{Ad}(t^{-1})X)n_2a.$$

By the uniqueness of the  $K(N \cap \bar{N}_w)(N \cap N_w)A$ -decomposition, we have  $k' = kt$  and  $X' = \text{Ad}(t^{-1})X$ , which is to say  $[k', X'] = [k, X]$ .

Next we show that  $\varphi_w$  is a diffeomorphism. By  $K$ -equivariance it suffices to show that it is a local diffeomorphism at each  $[e, X]$  where  $X \in \mathfrak{n} \cap \bar{\mathfrak{n}}_w$  and  $e \in K$  is the identity. The derivative of the diagram (1) at  $[e, X]$  is

$$\begin{array}{ccc} T_{[e, X]}(K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)) & \xrightarrow{D\varphi_w} & \mathfrak{g}/(\mathfrak{b} \cap \mathfrak{b}_w) \\ D\pi_w \downarrow & & \downarrow D\rho_w \\ \mathfrak{k}/\mathfrak{t} & \xrightarrow{\cong} & \mathfrak{g}/\mathfrak{b}. \end{array}$$

The left and bottom maps are surjective. But also, since the exponential on  $\mathfrak{n} \cap \bar{\mathfrak{n}}_w$  is a diffeomorphism onto its image,  $D\varphi_w$  maps the vertical tangent space  $\mathfrak{n} \cap \bar{\mathfrak{n}}_w \subset T_{[e, X]}(K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w))$  onto  $(\mathfrak{n} \cap \bar{\mathfrak{n}}_w)/(\mathfrak{b} \cap \mathfrak{b}_w) = \ker(D\rho_w)$ . Therefore  $D\varphi_w$  is surjective.

That the map is  $K$ -equivariant is straight-forward, as is the commutativity of the diagram of bundle maps. Fiberwise holomorphicity follows from the holomorphicity of the exponential map.  $\square$

**Remark 3.5.** There is an alternative realization of the space  $G/(B \cap B_w)$  as a  $K$ -equivariant vector bundle via the diagram

$$(2) \quad \begin{array}{ccc} K \times_T (\mathfrak{n}_w \cap \bar{\mathfrak{n}}) & \xrightarrow[\cong]{\varphi'_w} & G/(B \cap B_w) \\ \pi'_w \downarrow & & \downarrow \mathfrak{q}_w \\ K/T & \xrightarrow[\cong]{} & G/B_w, \end{array}$$

where the top map has essentially the same defining formula:

$$\varphi'_w : [k, X'] \mapsto k \exp(X') \cdot (B \cap B_w).$$

The proof is basically identical. Thus, the holomorphic manifold  $G/(B \cap B_w)$  admits two distinct structures as a complex  $K$ -equivariant vector bundle over  $K/T$ , via the maps  $\pi_w$  and  $\pi'_w$ . This point will be of crucial importance later.

**Definition 3.6.** Using the diagrams (1) and (2), we may consider the zero sections of the two complex vector bundles  $K \times_T (\mathfrak{n}_w \cap \bar{\mathfrak{n}})$  and  $K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)$  as  $K$ -equivariant maps  $\zeta_w : G/B \rightarrow G/(B \cap B_w)$  and  $\zeta'_w : G/B_w \rightarrow G/(B \cap B_w)$ . They are given simply by

$$\zeta_w : kB \mapsto k(B \cap B_w), \quad \zeta'_w : kB_w \mapsto k(B \cap B_w)$$

for  $k \in K$ , where we stress that in applying these formulas, we are obliged to choose coset representatives  $k$  belonging to the compact subgroup  $K$ .

The importance of realizing  $G/(B \cap B_w)$  as a complex  $K$ -vector bundle over  $G/B$  is that there is a Thom class

$$\tau(p_w) \in K_K^*(G/(B \cap B_w)),$$

obtained by pushing forward the Thom class from  $K^*(|K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)|)$ .

Note that this Thom class is dependent upon the fibration map

$$p_w : G/(B \cap B_w) \rightarrow G/B.$$

The alternative fibration  $q_w : G/(B \cap B_w) \rightarrow G/B_w$  defines a different class  $\tau(q_w)$ , pushed forward from  $K_K^*(|K \times_T (\mathfrak{n}_w \cap \bar{\mathfrak{n}})|)$ .

Let  $w \in W$ . The spaces  $G/B_w$  and  $G/B$  are  $G$ -equivariantly diffeomorphic, even biholomorphic, via the right multiplication map  $R_w : g \cdot (wBw^{-1}) \mapsto gw \cdot B$ . We can now define one of our main objects of study.

**Definition 3.7.** The *Borel–Bott–Weil morphism*  $\Lambda(w) \in \widehat{KK}^K(G/B, G/B)$  with parameter  $w \in W$  is the class of the  $K$ -equivariant holomorphic correspondence

$$G/B \xleftarrow{p_w} (G/B \cap B_w, \tau(q_w)) \xrightarrow{q_w} G/B_w \xrightarrow[\simeq]{R_w} G/B.$$

**Example 3.8.** If  $w = e$  is the identity element, then  $B \cap B_e = B$ ,  $\mathfrak{n} \cap \bar{\mathfrak{n}}_e$  is the zero Lie subalgebra, inducing to the zero vector bundle on  $K/T$ , and  $\tau(p_e)$  is the Thom class [1] of the zero vector bundle. Thus  $\Lambda(e) = 1$  is represented by the correspondence

$$G/B \xleftarrow{\text{id}} (G/B, [1]) \xrightarrow{\text{id}} G/B$$

which is the identity correspondence. Thus  $\Lambda(e) = 1 \in \widehat{KK}^K(G/B, G/B)$ .

**3.9. Product structure.** For  $w \in W$ , we denote by  $\iota_w : K/T \xrightarrow{\simeq} G/B_w$  the  $K$ -equivariant diffeomorphism defined by  $kT \mapsto kB_w$  for  $k \in K$ .

Each of these maps identifies  $K/T$  with a complex manifold, with  $K$  acting by a holomorphic action, and this complex structure induces a corresponding  $K$ -equivariant  $\text{Spin}^c$ -structure on  $K/T$ . All of these  $\text{Spin}^c$ -structures will be different. To keep track of them, we use the complex picture whenever possible.

**Definition 3.10.** We denote by  $I_w : G/B \rightarrow G/B_w$  the (non-holomorphic)  $K$ -equivariant diffeomorphism defined by the commuting diagram

$$\begin{array}{ccc} G/B & \xrightarrow{I_w} & G/B_w \\ \iota_e \uparrow \simeq & & \uparrow \simeq \iota_w \\ K/T & \xrightarrow{\text{id}} & K/T. \end{array}$$

Thus,  $I_w$  corresponds to the identity map on  $K/T$  but with an unusual  $K$ -orientation.

If  $w \in W$ , then right translation  $R_w : G/B_w \rightarrow G/B$  is a  $K$ -equivariant map yielding an element  $R_w^* \in \widehat{KK}^K(G/B, G/B_w)$ . The following proposition asserts, roughly, that after twisting  $R_w$  by the change of equivariant  $K$ -orientation induced by  $I_w$ , we get exactly the Borel–Bott–Weil correspondence  $\Lambda(w)$ .

**Proposition 3.11.** *The identity*

$$\Lambda(w) = (I_w^{-1} \circ R_w^{-1})^*$$

holds in  $\widehat{KK}^K(G/B, G/B)$ .

*Proof.* The map  $R_w$  is biholomorphic, so  $(R_w^{-1})^* = R_{w!}$ . Using the realization of  $G/(B \cap B_w)$  as a  $K$ -equivariant vector bundle over  $G/B_w$ , we can perform a Thom modification to get

$$\begin{aligned} (3) \quad (I_w^{-1} \circ R_w^{-1})^* &= [G/B \xleftarrow{I_w^{-1}} G/B_w \xrightarrow{R_w} G/B] \\ &= [G/B \xleftarrow{I_w^{-1} \circ q_w} (G/(B \cap B_w), \tau(q_w)) \xrightarrow{R_w \circ q_w} G/B]. \end{aligned}$$

We claim that this is equivalent, via a bordism, to the correspondence

$$(4) \quad \Lambda(w) = [G/B \xleftarrow{p_w} (G/(B \cap B_w), \tau(q_w)) \xrightarrow{R_w \circ q_w} G/B].$$

To see this, consider first the linear retraction  $\gamma_t$  of

$$G/(B \cap B_w) \cong K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)$$

onto its zero section:

$$\begin{aligned} \gamma_t : G/(B \cap B_w) &\xrightarrow{\varphi_w^{-1}} K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w) \rightarrow K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w) \xrightarrow{\varphi_w} G/(B \cap B_w), \\ &(k, X) \mapsto (k, tX). \end{aligned}$$

We use this to define the smooth  $K$ -equivariant homotopy

$$h_t := I_w^{-1} \circ q_w \circ \gamma_t : G/(B \cap B_w) \rightarrow G/B$$

between  $h_0 = p_w$  and  $h_1 = I_w^{-1} \circ q_w$ .

We want to show that this homotopy yields a bordism of correspondences

$$(5) \quad [G/B \xleftarrow{h} (G/(B \cap B_w) \times [0, 1], \text{pr}_1^* \tau(q_w)) \xrightarrow{R_w \circ q_w \circ \text{pr}_1} G/B]$$

between (3) and (4). Here  $\text{pr}_1$  denotes the projection

$$\text{pr}_1 : G/(B \cap B_w) \times [0, 1] \rightarrow G/(B \cap B_w),$$

i.e., the right-hand map and the  $K$ -theory class in (5) are constant in  $t$ . To verify that (5) is a well-defined correspondence we need to check that the  $K$ -theory class  $\text{pr}_1^* \tau(q_w)$  has compact support along the fibers of  $h$ .

Let  $kB \in G/B$ ; note that we may take  $k \in K$ . Suppose  $g(B \cap B_w) \in \text{supp}(\tau(q_w)) \cap h_t^{-1}(kB)$ . The support of the Thom class  $\tau(q_w)$  is the zero section  $\zeta'_w(G/B_w) = K \cdot (B \cap B_w) \subseteq G/B \cap B_w$ , so we may take  $g = k' \in K$ . Then

$$kB = h_t(k'(B \cap B_w)) = I_w \circ q_w \circ \gamma_t(k'(B \cap B_w)) = I_w(k'B_w) = k'B.$$

Therefore, the support of  $\text{pr}_1^* \tau(q_w)$  in the fiber  $h^{-1}(kB)$  is  $\{kB\} \times [0, 1]$ . Hence (5) is indeed a bordism between the correspondences (3) and (4). This completes the proof.  $\square$

**Corollary 3.12.** *The map  $w \mapsto \Lambda(w)$  is a group homomorphism from the Weyl group into the invertible elements of the ring  $\widehat{\text{KK}}^K(G/B, G/B)$ .*

*Proof.* One just needs to check that  $R_{w_1} \circ I_{w_1} \circ R_{w_2} \circ I_{w_2} = R_{w_1 w_2} \circ I_{w_1 w_2}$ . This is immediate if one represents elements of  $G/B$  as  $kB$  with  $k \in K$ .  $\square$

**3.13. Commutation relations in  $\widehat{\text{KK}}^K(G/B, G/B)$ .** We begin with some generalities on pullbacks of induced bundles.

Let  $H_2 \leq H_1 \leq G$  be a nested sequence of closed Lie subgroups, and let  $V$  be a vector space with a representation of  $H_1$ . If  $p : G/H_2 \rightarrow G/H_1$  denotes the canonical fibration map, then there is an equivariant bundle isomorphism

$$(6) \quad p^*(G \times_{H_1} V) \cong G \times_{H_2} V,$$

given by the following pullback diagram:

$$\begin{array}{ccc} G \times_{H_2} V & \longrightarrow & G \times_{H_1} V \\ \downarrow & & \downarrow \\ G/H_2 & \xrightarrow{p} & G/H_1, \end{array} \quad \begin{array}{ccc} [g, v] & \longmapsto & [g, v] \\ \downarrow & & \downarrow \\ gH_2 & \longmapsto & gH_1. \end{array}$$

Recall that each weight  $\mu$  defines a one-dimensional holomorphic representation of  $B$ . It will be convenient to use an explicit notation for this in the next few paragraphs, so we denote it by  $\sigma_\mu : B \rightarrow \text{End}(\mathbb{C}_\mu)$ . We shall denote by  $\sigma_\mu^w$  the representation of  $B_w$  defined by conjugating by  $w \in W$ :

$$\sigma_\mu^w(wbw^{-1}) := \sigma_\mu(b).$$

Then there is a  $G$ -equivariant bundle isomorphism

$$R_w^*(G \times_B \mathbb{C}_\mu) = G \times_{B_w} \mathbb{C}_\mu,$$

where the representation of  $B_w$  on the right-hand side is  $\sigma_\mu^w$ . The appropriate pullback diagram is:

$$\begin{array}{ccc} G \times_{B_w} \mathbb{C}_{w(\mu)} & \longrightarrow & G \times_B \mathbb{C}_\mu \\ \downarrow & & \downarrow \\ G/B_w & \xrightarrow{R_w} & G/B, \end{array} \quad \begin{array}{ccc} [g, v] & \longmapsto & [gw, v] \\ \downarrow & & \downarrow \\ gB_w & \longmapsto & gwB. \end{array}$$

**Lemma 3.14.** *For any  $\mu \in \Gamma_W$  and  $w \in W$  we have  $q_w^* R_w^* E_\mu \cong p_w^* E_{w(\mu)}$  as  $G$ -equivariant complex line bundles over  $G/(B \cap B_w)$ .*

*Proof.* As described above,  $q_w^* R_w^* E_\mu \cong q_w^*(G \times_{B_w} \mathbb{C}_\mu)$ . Restricting the conjugated representation  $\sigma_\mu^w$  to  $B \cap B_w$  yields a representation which is trivial on  $N \cap N_w$  and given by  $e^{w(\mu)}$  on  $T \cdot A$ . Thus (6) gives  $q_w^* R_w^* E_\mu \cong G \times_{B \cap B_w} \mathbb{C}_{w(\mu)}$ . This is isomorphic to  $p_w^* E_{w(\mu)}$  by (6) again.  $\square$

**Proposition 3.15.** *For any  $\mu \in \Gamma_W$  and  $w \in W$ ,*

$$\Lambda(w) \otimes_{G/B} [[\mu]] = [[w(\mu)]] \otimes_{G/B} \Lambda(w).$$

*Proof.* We calculate

$$\begin{aligned} & \Lambda(w) \otimes_{G/B} [[\mu]] \\ &= G/B \xleftarrow{p_w} (G/B \cap B_w, \tau(q_w)) \xrightarrow{R_w \circ q_w} G/B \xleftarrow{\text{id}} (G/B, [E_\mu]) \xrightarrow{\text{id}} G/B \\ &= G/B \xleftarrow{p_w} (G/B \cap B_w, \tau(q_w) \cdot q_w^* R_w^* [E_\mu]) \xrightarrow{R_w \circ q_w} G/B \\ &= G/B \xleftarrow{p_w} (G/B \cap B_w, \tau(q_w) \cdot p_w^* [E_{w(\mu)}]) \xrightarrow{R_w \circ q_w} G/B \\ &= G/B \xleftarrow{\text{id}} (G/B, [E_{w(\mu)}]) \xrightarrow{\text{id}} G/B \xleftarrow{p_w} (G/B \cap B_w, \tau(q_w)) \xrightarrow{R_w \circ q_w} G/B \\ &= [[w(\mu)]] \otimes_{G/B} \Lambda(w). \end{aligned}$$

This completes the proof.  $\square$

**3.16. Comparing Thom classes.** We begin this section by comparing the two Thom classes  $\tau(p_w)$  and  $\tau(q_w)$  on the space  $G/B \cap B_w$  (see Section 3.3). It will suffice to consider the case where  $w$  is the reflection in a simple root  $\alpha$ . In that case we have  $\mathfrak{n} \cap \bar{\mathfrak{n}}_w = \mathfrak{g}_\alpha$  and  $\mathfrak{n}_w \cap \bar{\mathfrak{n}} = \mathfrak{g}_{-\alpha}$ .

Recall that  $\tau(p_w)$  is the pushforward of the Thom class of

$$|K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)| = |K \times_T \mathfrak{g}_\alpha|$$

via the bundle isomorphism of (1). Taking advantage of the complex structure on the fibers, the corresponding spinor bundle is  $K \times_T \bigwedge_{\mathbb{C}}^\bullet \mathfrak{g}_\alpha$ . There is an  $\text{Ad}(T)$ -invariant inner product on  $\mathfrak{g}_\alpha$  via the Killing form. Letting  $\lambda_X$  denote the exterior product by  $X \in \mathfrak{g}_\alpha$ , we have a Clifford algebra representation

$$c : \mathfrak{g}_\alpha \rightarrow \text{End}(\bigwedge_{\mathbb{C}}^\bullet \mathfrak{g}_\alpha), \quad c(X) := \lambda_X - \lambda_X^*.$$

The Thom class of  $|K \times_T \mathfrak{g}_\alpha|$  is the pullback of the spinor bundle along the bundle projection  $\pi_w : K \times_T \mathfrak{g}_\alpha \rightarrow K/T$ , equipped with the bundle endomorphism which at each point is the Clifford representation of that point.

Since  $K \times_T \bigwedge_{\mathbb{C}}^\bullet \mathfrak{g}_\alpha \cong \mathbb{C}_0 \oplus \mathbb{C}_\alpha$ , we can identify the spinor bundle over  $K/T$  with  $G \times_B (\mathbb{C}_0 \oplus \mathbb{C}_\alpha)$ . The space  $\mathbb{C}_\alpha$  here identifies naturally with  $\mathfrak{g}_\alpha$  as a  $T$ -space, but not as a  $B$ -space: we have made an arbitrary extension to a  $B$ -representation.

Using equation (6), the push-forward of the Thom class by  $\varphi_w$  is then

$$\tau(p_w) = (G \times_{B \cap B_w} (\mathbb{C}_0 \oplus \mathbb{C}_\alpha), \mathbf{C}_w),$$

where  $\mathbf{C}_w$  is the bundle endomorphism defined at each point of  $G/(B \cap B_w)$  by

$$\mathbf{C}_w(k \exp(X)(B \cap B_w)) = c(X) \quad \text{for } k \in K, X \in \mathfrak{g}_\alpha.$$

A similar calculation shows that the Thom class  $\tau(q_w)$  associated to the other projection is

$$\tau(q_w) = (G \times_{B \cap B_w} (\mathbb{C}_0 \oplus \mathbb{C}_{-\alpha}), \mathbf{C}'_w),$$

where

$$C'_w(k \exp(X')(B \cap B_w)) = c(X') \quad \text{for } k \in K, X' \in \mathfrak{g}_{-\alpha}.$$

To compare these two classes, we define a homotopy. For  $t \in [0, 1]$ , define a map

$$\begin{aligned} \gamma_t &: G/(B \cap B_w) \rightarrow G/(B \cap B_w), \\ k \exp(X)(B \cap B_w) &\mapsto k \exp(tX)(B \cap B_w) \quad \text{for } k \in K, X \in \mathfrak{g}_{\alpha}. \end{aligned}$$

This is just the pushforward by  $\varphi_w$  of the retraction of the bundle  $K \times_T \mathfrak{g}_{\alpha}$  to the zero section.

Consider the smooth family  $\Phi_t$  of bundle endomorphisms of the vector bundle  $G \times_{B \cap B_w} (\mathbb{C}_0 \oplus \mathbb{C}_{-\alpha})$  defined by

$$\Phi_t(z) := C'(\gamma_t(x)), \quad z \in G/B \cap B_w.$$

Since  $\gamma_0$  has image the zero section,  $\Phi_0$  is the zero endomorphism. By smoothness and the compactness of  $G/B$ , the family

$$\Psi_t := \frac{1}{t} \Phi_t \quad (t \neq 0)$$

has a well-defined limit at  $t = 0$ , which we denote by  $\Psi_0$ .

**Lemma 3.17.** *Let  $\theta$  denote the Cartan involution on  $\mathfrak{g}$ . At a point  $k \exp(X)(B \cap B_w)$  of  $G/(B \cap B_w)$ , where  $k \in K$  and  $X \in \mathfrak{g}_{\alpha}$ , the limit  $\Psi_0(k \exp(X)B \cap B_w)$  is the endomorphism of the fiber  $\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{-\alpha}$  defined by*

$$\Psi_0(k \exp(X)B \cap B_w) = c(-\theta X).$$

*Proof.* We have  $\gamma_t(k \exp(X)(B \cap B_w)) = k \exp(tX)(B \cap B_w)$ . By the Campbell–Baker–Hausdorff formula,

$$\exp(tX) = \exp(t(X + \theta X)) \exp(-t\theta X) \exp(o(t)).$$

Since  $\exp(t(X + \theta X)) \in K$ , we have that  $\Psi_t$  acts on the fiber at  $k \exp(X)B \cap B_w$  by

$$\Psi_t(k \exp(X)B \cap B_w) = \frac{1}{t} c(-t\theta X + o(t)),$$

which has limit  $c(-\theta X)$  as  $t \rightarrow 0$ . □

In the next lemma, we fix identifications of  $\mathfrak{g}_{\pm\alpha}$  with  $\mathbb{C}$  by identifying some arbitrary unit vector  $Y \in \mathfrak{g}_{-\alpha}$  with 1, and likewise with  $\theta Y \in \mathfrak{g}_{\alpha}$ . Ultimately the choice of this  $Y$  makes no difference.

**Lemma 3.18.** *Fix  $Y \in \mathfrak{g}_{-\alpha}$  with  $\|Y\| = 1$ . Define a grading-reversing map  $\beta : \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{-\alpha} \rightarrow \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\alpha}$  by*

$$\begin{aligned} \beta : \omega &\mapsto \omega \cdot \theta Y && \text{for } \omega \in \bigwedge_{\mathbb{C}}^0 \mathfrak{g}_{-\alpha} = \mathbb{C}, \\ \beta : X' &\mapsto \langle Y, X' \rangle && \text{for } X' \in \bigwedge_{\mathbb{C}}^1 \mathfrak{g}_{-\alpha} = \mathfrak{g}_{-\alpha}. \end{aligned}$$

Then for any  $X \in \mathfrak{g}_{\alpha}$ ,

$$\beta^{-1} c(X) \beta = c(-\theta X).$$

**Remark 3.19.** Equivalently,  $\beta = \theta \circ \otimes$ , where  $\otimes$  is the (anti-linear) Hodge  $*$ -operator on  $\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{-\alpha}$ .

*Proof.* We calculate

$$\begin{aligned} \beta^{-1} \lambda_X \beta : \omega &\mapsto 0 && \text{for } \omega \in \mathbb{C}, \\ \beta^{-1} \lambda_X \beta : \omega Y &\xrightarrow{\beta} \omega \xrightarrow{\lambda_X} \omega X \xrightarrow{\beta^{-1}} \langle \theta Y, \omega X \rangle && \text{for } \omega Y \in \mathfrak{g}_{-\alpha} \end{aligned}$$

and

$$\begin{aligned} \lambda_{\theta X}^* : \omega &\mapsto 0 && \text{for } \omega \in \mathbb{C}, \\ \lambda_{\theta X}^* : \omega Y &\mapsto \langle \theta X, \omega Y \rangle && \text{for } \omega Y \in \mathfrak{g}_{-\alpha}. \end{aligned}$$

These maps are equal since  $\theta$  is anti-unitary. Also  $\beta^{-1} \lambda_X^* \beta = \lambda_{\theta X}$ , by the unitarity of  $\beta$ . The result now follows from the definition  $\mathfrak{c}(X) := \lambda_X - \lambda_X^*$ .  $\square$

The map  $\beta$  is not  $T$ -equivariant—it alters the weights, since it maps  $\mathfrak{g}_{-\alpha}$  to  $\mathbb{C}_0$  and  $\mathbb{C}_0$  to  $\mathfrak{g}_{\alpha}$ . But if we alter it by defining

$$\beta' : \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{-\alpha} \rightarrow (\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\alpha}) \otimes \mathfrak{g}_{-\alpha}, \quad Z \mapsto \beta Z \otimes Y,$$

then it is weight-preserving, and hence  $T$ -equivariant. It induces a grading-reversing bundle isomorphism

$$\begin{aligned} \text{id} \times_{B \cap B_w} \beta' : G \times_{B \cap B_w} \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{-\alpha} &\rightarrow G \times_{B \cap B_w} ((\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\alpha}) \otimes \mathfrak{g}_{-\alpha}) \\ &\cong (G \times_{B \cap B_w} \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\alpha}) \otimes_{G/B \cap B_w} \mathfrak{p}^* E_{-\alpha}, \end{aligned}$$

which intertwines the bundle endomorphisms  $\Psi_0$  and  $\mathbb{C}_w \otimes \text{id}$ . Combining this with the fact that  $\Psi_1 = \mathbb{C}_w$ , we have proven the following fact.

**Proposition 3.20.** *If  $w \in W$  is the reflection in the simple root  $\alpha$ , then  $\tau(\mathfrak{q}_w) = -\tau(\mathfrak{p}_w) \otimes \mathfrak{p}_w^*[E_{-\alpha}]$  in  $\mathbb{K}_K^*(G/(B \cap B_w))$ .*

**Remark 3.21.** There is a more general formula: for any  $w \in W$ ,

$$\tau(\mathfrak{q}_w) = (-1)^{l(w)} \tau(\mathfrak{p}_w) \otimes \mathfrak{p}_w^*[E_{w(\rho)-\rho}],$$

where  $\rho$  is the half-sum of the positive roots. This can be proven along the same lines as above with significantly more work, or deduced from results to follow. We shall not need it.

### 3.22. The Borel–Bott–Weil theorem: Action on $\mathbb{K}$ -homology and indices.

*Proof of Theorem 1.2.* We wish to show

$$\Lambda(w) \otimes_{G/B} [G/B]_{\mu} = (-1)^{l(w)} [G/B]_{w(\mu+\rho)-\rho}.$$

By the multiplicativity of the map  $w \mapsto \Lambda(w)$ , it suffices to take a reflection  $w$  in a simple root  $\alpha$ .

Let  $\mu \in \Gamma_W$ . Using the fact that  $[G/B]_{\mu} = [[\mu]] \otimes_{G/B} [G/B]$ , Proposition 3.15 gives

$$\Lambda(w) \otimes_{G/B} [G/B]_{\mu} = [[w(\mu)]] \otimes_{G/B} \Lambda(w) \otimes_{G/B} [G/B].$$

From Proposition 3.20,

$$\Lambda(w) \otimes_{G/B} [G/B] = [G/B \xleftarrow{p_w} (G/(B \cap B_w), -\tau(p_w) \otimes p_w^*[E_{-\alpha}]) \rightarrow \star].$$

Since  $G/(B \cap B_w) \xrightarrow{p_w} G/B$  is  $K$ -equivariantly diffeomorphic to a vector bundle with Thom class  $\tau(p_w)$ , the latter correspondence is precisely the Thom modification of

$$[G/B \xleftarrow{\text{id}} (G/B, -[E_{-\alpha}]) \rightarrow \star] = -[G/B]_{-\alpha}.$$

So we get

$$\Lambda(w) \otimes_{G/B} [G/B]_{\mu} = -[G/B]_{w(\mu)-\alpha}.$$

Since  $w$  is the reflection in  $\alpha$ , we have  $\alpha = w(\rho) - \rho$ , which proves the result.  $\square$

We now pass to the index-theoretic application. Let  $\text{pt} : G/B \rightarrow \star$  denote the map of  $G/B$  to a point and  $\text{pt}^* \in \widehat{\text{KK}}^K(\mathbb{C}, G/B)$  its topological  $\text{KK}$ -theory class.

For a weight  $\mu$ , the topological  $K$ -index of the twisted fundamental class  $[G/B]_{\mu} \in \widehat{\text{KK}}^G(G/B, \star)$  is defined by

$$\text{Index}_K[G/B]_{\mu} := \text{pt}^* \otimes_{G/B} [G/B]_{\mu} \in \widehat{\text{KK}}^K(\mathbb{C}, \mathbb{C}).$$

We do not bother to use different notation for the *analytic* index

$$\text{Index}_K[G/B]_{\mu} \in \text{KK}^K(\mathbb{C}, \mathbb{C}) \cong \text{R}(K);$$

which one we are talking about will be made clear by the context. The analytic index, as a graded representation of  $K$ , is the same as the cohomology group  $H^*(G/B, E_{\mu})$  figuring in the classical Borel–Bott–Weil theorem, and it equals the image of the topological index under the map

$$\widehat{\text{KK}}^K(G/B, \star) \rightarrow \text{KK}^K(\mathcal{C}(G/B), \mathbb{C})$$

(for a proof see [7].)

*Proof of Theorem 1.1.* We note that a Thom modification yields

$$\begin{aligned} (7) \quad \text{pt}^* \otimes_{G/B} \Lambda(w) &= [\star \leftarrow (G/(B \cap B_w), \tau(q_w)) \xrightarrow{R_w \circ q_w} G/B] \\ &= [\star \leftarrow G/B_w \xrightarrow{R_w} G/B] \\ &= \text{pt}^*. \end{aligned}$$

Composing with the  $\mu$ -twisted fundamental class on the right and applying Theorem 1.2 gives

$$(-1)^{l(w)} \text{Index}_K[G/B]_{w(\mu+\rho)-\rho} = \text{Index}_K[G/B]_{\mu}. \quad \square$$

**Remark 3.23.** Let us also record the action of the Borel–Bott–Weil classes on equivariant  $K$ -theory. The induction isomorphism  $\text{R}(T) \xrightarrow{\cong} \text{K}_K(K/T)$  associates to  $[\mu]$  the correspondence

$$[E_{\mu}] := [\star \leftarrow (K/T, [E_{\mu}]) \xrightarrow{\text{id}} K/T] = \text{pt}^*[[\mu]].$$

Thus, if we compose the commutation relation of Proposition 3.15 on the left by  $\text{pt}^*$  and use equation (7), we get the right action:

$$[E_{w(\mu)}] \otimes_{G/B} \Lambda(w) = [E_\mu].$$

#### REFERENCES

- [1] M. F. Atiyah and I. M. Singer, The index of elliptic operators. I, *Ann. of Math. (2)* **87** (1968), 484–530. MR0236950
- [2] B. Blackadar, *K-theory for operator algebras*, second edition, Math. Sci. Res. Inst. Publ., 5, Cambridge Univ. Press, Cambridge, 1998. MR1656031
- [3] J. Block and N. Higson, Weyl character formula in KK-theory, in *Noncommutative geometry and physics. 3*, 299–334, Keio COE Lect. Ser. Math. Sci., 1, World Sci. Publ., Hackensack, NJ, 2013. MR3098596
- [4] R. Bott, Homogeneous vector bundles, *Ann. of Math. (2)* **66** (1957), 203–248. MR0089473
- [5] A. Connes and G. Skandalis, The longitudinal index theorem for foliations, *Publ. Res. Inst. Math. Sci.* **20** (1984), no. 6, 1139–1183. MR0775126
- [6] H. Emerson and R. Meyer, Bivariant  $K$ -theory via correspondences, *Adv. Math.* **225** (2010), no. 5, 2883–2919. MR2680187
- [7] H. Emerson and R. Meyer, Equivariant embedding theorems and topological index maps, *Adv. Math.* **225** (2010), no. 5, 2840–2882. MR2680186
- [8] A. W. Knap, *Lie groups beyond an introduction*, second edition, *Progr. Math.*, 140, Birkhäuser Boston, Boston, MA, 2002. MR1920389
- [9] H.-H. Leung, Divided difference operators in equivariant  $KK$ -theory, *J. Topol. Anal.* **6** (2014), no. 2, 237–261. MR3191651
- [10] J.-L. Tu, La conjecture de Baum–Connes pour les feuilletages moyennables, *K-Theory* **17** (1999), no. 3, 215–264. MR1703305

Received August 8, 2015; accepted June 11, 2016

Heath Emerson

Department of Mathematics and Statistics, University of Victoria,  
PO Box 3045 STN CSC, Victoria, B.C., Canada V8W 3P4  
E-mail: [hemerson@math.uvic.ca](mailto:hemerson@math.uvic.ca)

Robert Yuncken

Laboratoire de Mathématiques,  
Université Blaise Pascal, Clermont-Ferrand II,  
Campus des Cézeaux, 63177 Aubière cedex, France  
E-mail: [yuncken@math.univ-bpclermont.fr](mailto:yuncken@math.univ-bpclermont.fr)