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Amenability for unitary groups of simple monotracial C*-algebras

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Abstract. We prove the following two results. First, the isometry semigroup of a unital properly infinite nuclear C*-algebra is right amenable. Second, the unitary group of a unital simple monotracial C*-algebra whose tracial GNS representation is hyperfinite is skew-amenable in the weak topology. This answers in part a conjecture of Alekseev, Schmidt, and Thom and a question of Pestov.

1. Introduction

We recall the cornerstone of the C*-algebra theory that a C*-algebra A is nuclear (or amenable) if and only if the enveloping von Neumann algebra A^{**} is hyperfinite [5, 7, 9], which is equivalent to amenability of $\mathcal{U}(A^{**})$ in the ultraweak topology [11]. In turn, amenability property of the unitary group $\mathcal{U}(A)$ of a unital C*-algebra A has been drawing considerable attention of researchers. Recall that a topological group G is said to be *amenable* (resp. *skew-amenable*) if there is a left-invariant (resp. right-invariant) mean on the space of right uniformly continuous bounded functions on G. It is known that A is nuclear if and only if $\mathcal{U}(A)$ is amenable in the weak topology [16], essentially because $\mathcal{U}(A)$ is dense in $\mathcal{U}(A^{**})$ in the ultraweak topology. On the other hand, it is not clear when $\mathcal{U}(A)$ with the norm topology is amenable. Note that norm amenability of $\mathcal{U}(A)$ implies, in addition to nuclearity [6], that A has the QTS property (*i.e.*, every nonzero quotient of A admits a tracial state). The converse is believed to hold, probably under some regularity assumptions. See [1] around this problem and the progress toward it. The purpose of this note is to prove the following two results.

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Theorem 1.1. *Let A be a unital properly infinite* C**-algebra. Then the isometry semigroup $\mathcal{I}(A)$ of A is right amenable in the norm topology if and only if A is nuclear.*

Here right amenability of $\mathcal{I}(A)$ means existence of a right invariant mean on the space of uniformly continuous bounded functions on $\mathcal{I}(A)$. We note that the assumption on proper infiniteness cannot be removed. Indeed, if A is finite, then amenability of $\mathcal{I}(A) = \mathcal{U}(A)$ implies existence of a tracial state, which is not always the case [22]. For the same reason, the conclusion of right amenability cannot be replaced with left amenability because it also implies existence of a tracial state (see Section 2).

We turn our attention to the finite case. For a unital C*-algebra A, we denote by $T(A)$ the compact convex space of tracial states on A. The C*-algebra A is said to be *monotracial* if $|T(A)| = 1$. We write $\|a\|_\tau := \tau(a^*a)^{1/2}$ for $a \in A$ and $\tau \in T(A)$ and define the *uniform* 2*-norm* on A by

$$\|a\|_{T(A)} := \sup\{\|a\|_\tau \mid \tau \in T(A)\}.$$

Theorem 1.2. *Let A be a unital* C**-algebra with the QTS property and denote by $\mathcal{U}(A)$ the unitary group of A. Consider the following conditions.*
(i) *$\mathcal{U}(A)$ is amenable in the uniform* 2*-norm topology.*
(ii) *$\mathcal{U}(A)$ is skew-amenable in the weak topology.*
(iii) *For every $\tau \in T(A)$, the von Neumann algebra $\pi_\tau(A)''$ generated by the GNS representation π_τ for τ is hyperfinite.*
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *holds. If A has only finitely many extremal tracial states, then* (iii) $\Rightarrow$ (i) *holds.*

This partly confirms/refutes a conjecture raised in [1], where it is proved that (ii) $\Rightarrow$ (iii). We note that weak skew-amenability of $\mathcal{U}(A)$ implies the QTS property of A (see [19, 1]). The following corollary answers in the negative a question in [18, 12, 19] asking if skew-amenability implies amenability.

Corollary 1.3. *Let A be a unital simple monotracial* C**-algebra and let $\pi_\tau(A)''$ be the* II$_1$*-factor generated by the GNS representation π_τ for the unique tracial state τ on A. Then $\mathcal{U}(A)$ is skew-amenable in the weak topology if and only if $\pi_\tau(A)''$ is hyperfinite.*

In particular, the unitary group $\mathcal{U}(\mathcal{R})$ of the hyperfinite II$_1$ *factor $\mathcal{R}$ (of any cardinality) is skew-amenable but not amenable in the weak topology.*

2. Proof of Theorem 1.1

Definition 2.1. Given a unital C*-algebra A, a finite sequence $a = (a_1, \ldots, a_n)$ in A is called a *column isometry* if $a^*a := \sum_i a_i^* a_i = 1$, *i.e.*, if a is an isometry in $\mathbb{M}_{n,1}(A)$. We identify a finite sequence $(a_1, \ldots, a_n)$ with $(a_1, \ldots, a_n, 0, \ldots, 0)$. The set of isometries (resp. column isometries) of A is denoted by $\mathcal{I}(A)$ (resp. $\mathcal{CI}(A)$). For $a \in \mathcal{CI}(A)$ and $s \in \mathcal{I}(A)$, write $as := (a_1 s, \ldots, a_n s) \in \mathcal{CI}(A)$.

For a finite sequence $a = (a_1, \dots, a_n)$ in A, we put

$$\|a\|_{\mathrm{C}} := \Big\|\sum_i a_i^* a_i\Big\|^{1/2} \quad \text{and} \quad \|a\|_{\mathrm{RC}} := \Big\|\sum_i a_i a_i^*\Big\|^{1/2} + \Big\|\sum_i a_i^* a_i\Big\|^{1/2}.$$

We note that the distance $\|a - b\|_{\mathrm{C}}$ makes sense for any finite sequences a and b, by padding them out with zeros as necessary.

The following theorem is proved in [14] and [24, Sec. 5], where it is proved for $E \subset \mathcal{U}(A)$, but the proof works verbatim for $E \subset \mathcal{I}(A)$.

Theorem 2.2. *A unital* C^**-algebra* A *is nuclear if and only if, for every finite subset* $E \subset \mathcal{I}(A)$ *and every* $\varepsilon > 0$*, there are a nonempty finite subset* $F \subset \mathcal{CI}(A)$ *and permutations* $\{\rho_s \mid s \in E\}$ *on* F *such that* $\|as - \rho_s(a)\|_{\mathrm{C}} < \varepsilon$ *for every* $s \in E$ *and* $a \in F$*.*

If A is moreover properly infinite, then one can take isometries $s_1, s_2, \dots$ with mutually orthogonal ranges and replace $\mathcal{CI}(A)$ with $\mathcal{I}(A)$ via the right $\mathcal{I}(A)$-equivariant isometric map

$$\mathcal{CI}(A) \ni (a_1, \dots, a_n) \mapsto \sum_i s_i a_i \in \mathcal{I}(A).$$

By taking a limit point of the uniform probability measures on suitable Fs, one obtains a right invariant mean on $\mathcal{I}(A)$. This proves the "if" part of Theorem 1.1. The proof of the "only if" part is standard [6]. Take a universal representation $A \subset \mathbb{B}(\mathcal{H})$ and consider for each $x \in \mathbb{B}(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$ the uniformly continuous bounded function

$$f_{x,\xi,\eta} : \mathcal{I}(A) \ni s \mapsto \langle s^* x s \xi, \eta \rangle \in \mathbb{C}.$$

If $\mathcal{I}(A)$ admits a right invariant mean m, then by the Riesz representation theorem, there is $\Phi(x) \in \mathbb{B}(\mathcal{H})$ that satisfies $m(f_{x,\xi,\eta}) = \langle \Phi(x)\xi, \eta \rangle$ for every $\xi, \eta \in \mathcal{H}$. It is not hard to see that $x \mapsto \Phi(x)$ is a conditional expectation from $\mathbb{B}(\mathcal{H})$ onto A'. This implies that $A^{**} \cong (A')^{\mathrm{op}}$ is injective (hyperfinite) and hence that A is nuclear [5, 7]. In passing, we observe that if $\mathcal{I}(A)$ admits a left invariant mean m', then for any unit vector ξ, the map $x \mapsto m'(f_{x,\xi,\xi})$ defines a tracial state on A (in fact an A-central state on $\mathbb{B}(\mathcal{H})$).

On the other hand, if A is moreover with the QTS property (as opposed to proper infiniteness), then by Dixmier's averaging [15, Thm. 1], there is a finite sequence $w_1, \dots, w_k \in \mathcal{U}(A)$ that satisfies

$$\Big\|k^{-1} \sum_{i,j} w_j a_i a_i^* w_j^* - 1\Big\| < \varepsilon,$$

$$\Big\|k^{-1} \sum_{i,j} w_j (a_i s - \rho_s(a)_i)(a_i s - \rho_s(a)_i)^* w_j^*\Big\| < \varepsilon$$

for every $s \in E$ and $a \in F$. Thus, by replacing each $a \in F$ with $(k^{-1/2} w_j a_i)_{i,j}$ and retaining $\{\rho_s\}$, we obtain the following.

Corollary 2.3. *Let A be a unital nuclear* C*-*algebra with the QTS property, Then, for every finite subset $E \subset \mathcal{U}(A)$ and every $\varepsilon > 0$, there are a nonempty finite subset $F \subset \mathcal{CI}(A)$ and permutations $\{\rho_u \mid u \in E\}$ on F such that*

$$\Big\|\sum_i a_i a_i^* - 1\Big\| < \varepsilon \quad \textit{and} \quad \|au - \rho_u(a)\|_{\mathrm{RC}} < \varepsilon$$

for every $u \in E$ and $a \in F$.

3. Proof of Theorem 1.2

Proof. We consider the following strengthening of condition (ii):
(ii)′ $\mathcal{U}(A)$ is skew-amenable in the skew-strong topology.
Here the *skew-strong topology* on A is given by the directed family of semi-norms $\|\cdot\|_\varphi$, $\varphi \in A_+^*$, where $\|a\|_\varphi := \varphi(aa^*)^{1/2}$ for $a \in A$. Be aware that it is *not* the more common $\|a\|_\varphi = \varphi(a^*a)^{1/2}$. As A^* is spanned by A_+^*, the Cauchy–Schwarz inequality implies that the skew-strong topology is finer than the weak topology. Thus (ii)′ ⇒ (ii) holds.

Let us assume (i) and prove (ii)′. By [25, Thm. 4.5], condition (i) means that, for every finite subset $E \subset \mathcal{U}(A)$ and $\varepsilon > 0$, there are a nonempty finite subset $F \subset \mathcal{U}(A)$ and permutations $\{\rho_u \mid u \in E\}$ on F that satisfy

$$|F|^{-1} \sum_{v\in F}\sum_{u\in E} \|vu - \rho_u(v)\|_{T(A)}^2 < \varepsilon.$$

Hence, by Dixmier's averaging [15, Thm. 1], there is a finite sequence $w_1, \dots, w_k$ in $\mathcal{U}(A)$ such that

$$\Big\|k^{-1}\sum_{j=1}^{k} w_j\Big(|F|^{-1}\sum_{u\in E,\, v\in F}(vu-\rho_u(v))(vu-\rho_u(v))^*\Big)w_j^*\Big\| < \varepsilon.$$

It follows that, for every state φ on A, there is j such that

$$|F|^{-1}\sum_{u\in E,\, v\in F}\|w_j vu - w_j\rho_u(v)\|_\varphi^2 < \varepsilon.$$

Replacing F with $\{w_j v \mid v \in F\}$ and retaining $\{\rho_u\}$, one sees skew-amenability of $\mathcal{U}(A)$ in $\|\cdot\|_\varphi$. Since the semi-norms $\|\cdot\|_\varphi$, $\varphi \in A_+^*$, are directed, this proves (ii)′.

The implication (ii) ⇒ (iii) is [1, Prop. 4.4]. We prove (iii) ⇒ (i) assuming that A has only finitely many extremal tracial states $\tau_1, \dots, \tau_k$. We put $\tau := k^{-1}\sum_{j=1}^k \tau_j \in T(A)$ and observe that $\|a\|_{T(A)}^2 \le k\|a\|_\tau^2$ for every $a \in A$. By Kaplansky's density theorem, $\mathcal{U}(A)$ is $\|\cdot\|_\tau$-dense in $\mathcal{U}(\pi_\tau(A)'')$. The rest is standard: amenability of $\mathcal{U}(\pi_\tau(A)'')$ is inherited by the dense subgroup $\mathcal{U}(A)$. We include the proof for completeness. Let a finite subset $E \subset \mathcal{U}(A)$ and $\varepsilon > 0$ be given. Since $\pi_\tau(A)''$ is hyperfinite, there is a finite-dimensional von Neumann subalgebra $B \subset \pi_\tau(A)''$ such that $\mathrm{dist}_\tau(u, \mathcal{U}(B)) < \varepsilon$ for every $u \in E$, *i.e.*, for each $u \in E$, there is $\tilde{u} \in \mathcal{U}(B)$ with $\|\tilde{u} - u\|_\tau < \varepsilon$. Since $\mathcal{U}(B)$ is a compact group, there are a finite subset $\{\tilde{v}_1, \dots, \tilde{v}_n\} \subset \mathcal{U}(B)$ and permutations $\{\rho_u \mid u \in E\}$ on $\{1, \dots, n\}$ such that $\|\tilde{v}_i\tilde{u} - \tilde{v}_{\rho_u(i)}\|_\tau < \varepsilon$ for every $\tilde{u} \in \tilde{E}$

and $i \in \{1, \dots, n\}$. For each i, take $v_i \in \mathcal{U}(A)$ with $\|v_i - \tilde{v}_i\|_\tau < \varepsilon$ and put $F := \{v_1, \dots, v_n\}$. Since

$$\|v_i u - \tilde{v}_i \tilde{u}\|_\tau \leq \|(v_i - \tilde{v}_i)u\|_\tau + \|\tilde{v}_i(u - \tilde{u})\|_\tau \leq 2\varepsilon$$

(this is where the tracial property is indispensable), one has $\|v_i u - v_{\rho_u(i)}\|_\tau \leq 4\varepsilon$ for every $u \in E$ and i. By adjusting $\varepsilon > 0$, we are done. □

Remark 3.1. Since every norm separable subset of $\mathcal{R}$ is contained in a separable simple monotracial C*-subalgebra of $\mathcal{R}$ (see, *e.g.*, [15, Lem. 9]), there exists a unital separable simple monotracial non-exact [13] C*-algebra A whose unitary group $\mathcal{U}(A)$ is skew-amenable but not amenable in the weak topology.

4. Further results that may be useful in the future

The rest of this note handles the case of C*-algebras with infinitely many extremal tracial states. Exactness plays a crucial role, as it assures certain commutativity of ultraproduct and tensor product [13]. We first collect some useful facts about the free semi-circular systems.

We recall the free semi-circular system $\{s_i \mid i = 1, 2, \dots\}$. Let $\mathcal{O}_\infty$ be the Cuntz algebra generated by isometries l_i with mutually orthogonal ranges and let $s_i := l_i + l_i^*$. Then $\mathcal{C} := \mathrm{C}^*(\{s_i \mid i = 1, 2, \dots\})$ is $*$-isomorphic to the reduced free product of the copies of $C([-2, 2])$ with respect to the Lebesgue measure (see [26, Sec. 2.6]), and the corresponding tracial state $\tau_\mathcal{C}$ coincides with the restriction to $\mathcal{C}$ of the vacuum state on $\mathcal{O}_\infty$. We note that the reduced free group C*-algebra $\mathrm{C}^*_\mathrm{r}\mathbb{F}_d$ and the free semi-circular C*-algebra $\mathcal{C}$ embed into each other because $\mathrm{C}^*_\mathrm{r}\mathbb{Z}$ and $C([-2, 2])$ embed into each other. Also, the countable free groups of rank at least two embed into each other as groups.

Theorem 4.1. *Let $\mathcal{Z}$ denote the Jiang–Su algebra and $\mathcal{Z}^\omega$ the ultrapower with respect to a free ultrafilter on $\mathbb{N}$. Then the free semi-circular* C*-*algebra $\mathcal{C}$ embeds into $\mathcal{Z}^\omega$.*

Proof. Recall that the Jiang–Su algebra $\mathcal{Z}$ is a (unique) simple monotracial C*-algebra which arises as an inductive limit of some prime dimension drop C*-algebras

$$\{f \in C([0, 1], \mathbb{M}_{p(n)} \otimes \mathbb{M}_{q(n)}) \mid f(0) \in \mathbb{M}_{p(n)} \otimes \mathbb{C}1,\ f(1) \in \mathbb{C}1 \otimes \mathbb{M}_{q(n)}\},$$

where $(p(n), q(n))$ are pairs of relatively prime numbers. We may assume $p(n) \gg q(n)$. We take i.i.d. G.U.E. random matrices $x_1(n), x_2(n) \in \mathbb{M}_{p(n)}$ and $y_1(n), y_2(n) \in \mathbb{M}_{q(n)}$. Then, by [10, 4, 20] (or more advanced [2, 3] if we do not want to assume $p(n) \gg q(n)$), the tuple

$$(x_1(n) \otimes 1_{q(n)}, x_2(n) \otimes 1_{q(n)}, 1_{p(n)} \otimes y_1(n), 1_{p(n)} \otimes y_2(n))$$

strongly converges to $(s_1 \otimes 1, s_2 \otimes 1, 1 \otimes s_1, 1 \otimes s_2)$ in $\mathcal{C}_2 \otimes \mathcal{C}_2$, where $\mathcal{C}_2$ is the C*-algebra generated by the free semicircular system $\{s_1, s_2\}$; in other words, there is an embedding

$$\mathcal{C}_2 \otimes \mathcal{C}_2 \hookrightarrow \Big(\prod_n \mathbb{M}_{p(n)} \otimes \mathbb{M}_{q(n)}\Big)/\omega$$

of $\mathcal{C}_2 \otimes \mathcal{C}_2$ into the norm ultraproduct $(\prod_n \mathbb{M}_{p(n)} \otimes \mathbb{M}_{q(n)})/\omega$. Since $C([0,1])$ is exact, we may view by [13] that

$$C\Big([0,1], \Big(\prod_n \mathbb{M}_{p(n)} \otimes \mathbb{M}_{q(n)}\Big)/\omega\Big) \subset \Big(\prod_n C([0,1], \mathbb{M}_{p(n)} \otimes \mathbb{M}_{q(n)})\Big)/\omega$$

and thus

$$B := \{f \in C([0,1], \mathcal{C}_2 \otimes \mathcal{C}_2) \mid f(0) \in \mathcal{C}_2 \otimes \mathbb{C}1,\ f(1) \in \mathbb{C}1 \otimes \mathcal{C}_2\} \hookrightarrow \mathcal{Z}^\omega.$$

For each $i = 1, 2$, we take $h_i = h_i^* \in \mathrm{C}^*(\{1, s_i\})$ such that the unitary element $u_i := \exp(\sqrt{-1}h_i)$ satisfies $\tau_{\mathcal{C}_2}(u_i^n) = 0$ for all $n \neq 0$. Put $u_i(t) := \exp(t\sqrt{-1}h_i)$ which connects $u_i(0) = 1$ to $u_i(1) = u_i$. We define $g_i \in C([0,1], \mathcal{C}_2 \otimes \mathcal{C}_2)$ by

$$g_i(t) = \begin{cases} u_i(1) \otimes u_i(2t) & \text{for } t \in [0, 1/2], \\ u_i(2-2t) \otimes u_i(1) & \text{for } t \in [1/2, 1]. \end{cases}$$

Then $g_i \in B$ and, for each t, the pair $\{g_1(t), g_2(t)\}$ is unitarily equivalent to the standard generating pair of $\mathrm{C}^*_{\mathrm{r}}\mathbb{F}_2 \otimes \mathbb{C}1$, by Fell's absorption principle. Thus, $\{g_1, g_2\}$ itself generates a copy of $\mathrm{C}^*_{\mathrm{r}}\mathbb{F}_2$ inside $\mathcal{Z}^\omega$. □

For every finite sequence $a = (a_1, \ldots, a_n)$ in a C^*-algebra A, we write

$$\mathrm{s}(a) := \sum_i s_i \otimes a_i \in \mathcal{C} \otimes A,$$

where $\{s_i\}$ is a free semi-circular system. We note that

$$\mathrm{s}(a) = \sum_i l_i \otimes a_i + \Big(\sum_i l_i \otimes a_i^*\Big)^* =: S + T^*$$

in $\mathcal{O}_\infty \otimes A$ and that S and T satisfy

$$S^*S = 1 \otimes \sum_i a_i^* a_i \quad \text{and} \quad T^*T = 1 \otimes \sum_i a_i a_i^*.$$

Thus, in particular (see Definition 2.1 for $\|\cdot\|_{\mathrm{RC}}$), $\|\mathrm{s}(a)\| \leq \|a\|_{\mathrm{RC}}$.

Lemma 4.2. *For every $\varepsilon > 0$, there is $\delta > 0$ that satisfies the following. For every finite sequence a in a C^*-algebra A with $T(A) \neq \varnothing$, if $\|\sum_i a_i^* a_i - 1\| < \delta$ and $\|\sum_i a_i a_i^* - 1\| < \delta$, then*

$$\|1_{[0,\delta)}(|\mathrm{s}(a)|)\|_{T(\mathcal{C}\otimes A)} < \varepsilon.$$

Here $1_{[0,\delta)}(|\mathrm{s}(a)|) \in (\mathcal{C} \otimes A)^{**}$ is the spectral projection for $|\mathrm{s}(a)|$ corresponding to $[0, \delta)$.

Proof. Note that $T(\mathcal{C} \otimes A) = \{\tau_{\mathcal{C}} \otimes \tau_A \mid \tau_A \in T(A)\}$. Suppose that the conclusion were false. Then there is $\varepsilon > 0$ such that, for every n, there are a finite sequence a_n in A_n and $\tau_n \in T(A_n)$ that satisfy

$$\Big\|\sum_i a_{n,i}^* a_{n,i} - 1\Big\| < 1/n, \quad \Big\|\sum_i a_{n,i} a_{n,i}^* - 1\Big\| < 1/n,$$

and $(\tau_{\mathcal{C}} \otimes \tau_{A_n})(1_{[0,1/n)}(|\mathrm{s}(a_n)|)) \geq \varepsilon$. We define a continuous function f by $f = 1$ on $[0, 1/n]$, $f = 0$ on $[1/(n-1), \infty)$, and linear on $[1/n, 1/(n-1)]$. Let

ψ denote the vacuum state on $\mathcal{O}_\infty$ that extends $\tau_{\mathcal{C}}$ and put $\varphi_n := \psi \otimes \tau_{A_n}$ on $\mathcal{O}_\infty \otimes A_n$. Then $S_n := \sum_i l_i \otimes a_{n,i}$, $T_n := \sum_i l_i \otimes a_{n,i}^*$ satisfy $\|S_n^* S_n - 1\| < 1/n$, $\|T_n^* T_n - 1\| < 1/n$, $\varphi_n(T_n T_n^*) = 0$, and $\varphi_n(f_m(|S_n + T_n^*|)) \geq \varepsilon$ for all n and m with $n \geq m$. Thus, by passing to an ultralimit, one obtains isometries S, T and a state φ such that $\varphi(TT^*) = 0$ and $\varphi(f_m(|S + T^*|)) \geq \varepsilon$ for all m. By the GNS construction, we may assume that φ is the vector state associated with a unit vector ξ. Then $\lim_m \varphi(f_m(|S + T^*|)) = \|P\xi\|^2$, where P is the orthogonal projection onto the kernel of $S + T^*$. Since S and T are isometries, $(S + T^*)\eta = 0$ is equivalent to $-TS\eta = \eta$. Hence P is a WOT-limit point of $(k^{-1}\sum_{j=1}^{k}(-TS)^j)_k$. However, since $\|T^*\xi\|^2 = \varphi(TT^*) = 0$, this implies that $P\xi = 0$, contradicting $\|P\xi\|^2 \geq \varepsilon$. $\square$

In particular, if $\sum_i a_i^* a_i = 1 = \sum_i a_i a_i^*$, then $\|1_{\{0\}}(|\mathrm{s}(a)|)\|_{T(\mathcal{C}\otimes A)} = 0$. However, $\mathrm{s}(a) = S + T^*$ is not invertible since S and T are proper isometries (in which case -1 is an approximate eigenvalue of TS). The author does not know whether $\|1_{\{0\}}(|\mathrm{s}(a)|)\|_{T(\mathcal{C}\otimes A)} = 0$ holds as soon as $\|\sum_i a_i^* a_i - 1\| < 1/2$ and $\|\sum_i a_i a_i^* - 1\| < 1/2$.

Theorem 4.3. *Let A be a unital simple finite nuclear $\mathcal{Z}$-stable* C*-*algebra. Then, for every finite subset $E \subset \mathcal{U}(A)$ and $\varepsilon > 0$, there are a nonempty finite subset $F \subset \mathcal{U}(A)$ and permutations $\{\rho_u \mid u \in E\}$ on F such that*

$$\|vu - \rho_u(v)\|_{T(A)} < \varepsilon$$

for every $u \in E$ and $v \in F$. In particular, $\mathcal{U}(A)$ is amenable in the uniform 2-norm topology.

Proof. Let a finite subset $E \subset \mathcal{U}(A)$ and $\varepsilon > 0$ be given. We may assume that $A = \mathcal{Z} \otimes A_0$, $A_0 \cong A$, and $E \subset \mathbb{C}1 \otimes A_0$. We embed $\mathcal{Z} \otimes A_0$ into the norm ultrapower $(\mathcal{Z} \otimes A_0)^\omega$. We take an embedding of the free semi-circular C*-algebra $\mathcal{C}$ into $\mathcal{Z}^\omega$ and view (by exactness of A_0)

$$\mathcal{C} \otimes A_0 \subset \mathcal{Z}^\omega \otimes A_0 \subset (\mathcal{Z} \otimes A_0)^\omega.$$

Take $\delta > 0$ as in Lemma 4.2 and take $F \subset \mathcal{CI}(A_0)$ and $\{\rho_u \mid u \in E\}$ as in Corollary 2.3 for E and κ (instead of ε there), where $\kappa > 0$ is a sufficiently small number which is specified later. Hence $\|1_{[0,\delta)}(|s(a)^*|)\|_{T(\mathcal{C}\otimes A_0)} < \varepsilon$ for every $a \in F$. Further, $\mathrm{s}(a) \in \mathcal{C} \otimes A_0$ satisfies, for every $u \in E$ and $a \in F$, that

$$\|\mathrm{s}(a)\| \leq \|a\|_{\mathrm{RC}} \leq 3 \quad \text{and} \quad \|\mathrm{s}(a)u - s(\rho_u(a))\| \leq \|au - \rho_u(a)\|_{\mathrm{RC}} < \kappa.$$

Consider the polar decomposition $\mathrm{s}(a) = |\mathrm{s}(a)^*|w(a)$ of $\mathrm{s}(a)^*$. Note that $w(a) \in (\mathcal{C} \otimes A_0)^{**}$ may not belong to $\mathcal{C} \otimes A_0$, but they satisfy $w(au) = w(a)u$. Since A_0 is simple, finite, and $\mathcal{Z}$-stable, $\mathcal{C} \otimes A_0$ has stable rank one by [23, Thm. 6.7]. Hence, $\mathrm{s}(a) \in \overline{\mathrm{GL}(\mathcal{C} \otimes A_0)}$, and by [17, 21], there is a unitary element $v(a)$ in $\mathcal{C} \otimes A_0$ such that

$$1_{[\delta,\infty)}(|\mathrm{s}(a)^*|)v(a) = 1_{[\delta,\infty)}(|\mathrm{s}(a)^*|)w(a).$$

Let $g(t) = \min\{\delta^{-1}t, 1\}$ be the continuous function on $[0, \infty)$. We may assume that $\kappa > 0$ is small enough so that $\|g(|x^*|)w(x) - g(|y^*|)w(y)\| < \varepsilon$ whenever

$x = |x^*|w(x)$ and $y = |y^*|w(y)$ (polar decompositions) satisfy $\|x\|, \|y\| \le 3$ and $\|x - y\| < \kappa$. We write $x \approx_\gamma y$ if $\|x - y\|_{T(\mathcal{C}\otimes A_0)} < \gamma$. Since $\mathrm{s}(au) = \mathrm{s}(a)u = |\mathrm{s}(a)^*|w(a)u$, $\mathrm{s}(\rho_u(a)) = |\mathrm{s}(\rho_u(a))^*|w(\mathrm{s}(\rho_u(a)))$, and $\|\mathrm{s}(au) - \mathrm{s}(\rho_u(a))\| < \kappa$, one has

$$v(a)u \approx_{2\varepsilon} g(|\mathrm{s}(a)^*|)w(a)u \approx_\varepsilon g(|\mathrm{s}(\rho_u(a))^*|)w(\rho_u(a)) \approx_{2\varepsilon} v(\rho_u(a)),$$

or equivalently

$$\|v(a)u - v(\rho_u(a))\|_{T(\mathcal{C}\otimes A_0)} < 5\varepsilon$$

for every $u \in E$ and $a \in F$.

Recall that $\mathcal{C} \otimes A_0 \subset (\mathcal{Z} \otimes A_0)^\omega$ and write $v(a) = (v_n(a))_{n\to\omega} \in (\mathcal{Z} \otimes A_0)^\omega$, where $v_n(a) \in \mathcal{U}(\mathcal{Z} \otimes A_0) = \mathcal{U}(A)$ for every $a \in F$ and $n \in \mathbb{N}$. For every $u \in E$ and $a \in F$, one has

$$\{n \in \mathbb{N} \mid \|v_n(a)u - v_n(\rho_u(a))\|_{T(A)} < 5\varepsilon\} \in \omega,$$

for otherwise $\{n \in \mathbb{N} \mid \|v_n(a)u - v_n(\rho_u(a))\|_{\tau_n} \ge 5\varepsilon\} \in \omega$ for some $(\tau_n)_n \in T(A)^{\mathbb{N}}$, which implies $\|v(a)u - v(\rho_u(a))\|_{\tau_\omega} \ge 5\varepsilon$ for the ultraproduct tracial state τ_ω of $(\tau_n)_n$. It follows that there is $n \in \mathbb{N}$ that satisfies $\|v_n(a)u - v_n(\rho_u(a))\|_{T(A)} < 5\varepsilon$ simultaneously for all $u \in E$ and $a \in F$. □

The simplicity assumption in lieu of the QTS property is used in the above proof only to have stable rank one. This begs the question (vaguely related to [8]): is it true that $\mathrm{s}(a) \in \overline{\mathrm{GL}(\mathcal{C} \otimes A)}$ for every finite sequence a in a C*-algebra with the QTS property? Incidentally, it seems that one can push the above proof and replace the simplicity assumption with the QTS property, by manipulating on a large matrix algebra $\mathbb{M}_n(\mathcal{C} \otimes A_0)$ and by suitably embedding $C_0((0,1], \mathbb{M}_n(\mathcal{C} \otimes A_0))$ into $\mathcal{Z} \otimes \mathcal{C} \otimes A_0$, without much affecting the uniform 2-norm.

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