# Classical Mechanics

Classical Newtonian mechanics treat the motion of finitely many rigid objects, in the simplest case mass points (i.e. points with zero spatial extension but nonzero mass m). These objects can interact with each other (e.g. they can attract or repel each other) or the environment. "Classical" means that quantum effects do not occur. Classical mechanics can for instance be used to describe planetary motion or systems of interacting particles (e.g. to some extent molecules in a gas).

- a classic : Landau, Lifschitz : Mechanics
- introductory text : Bühler : A brief Introdution to Classical, Statistical and Quantum Mechanics
- more advanced : Arnold : Mathematical Methods of Classical Mechanics

Classical mechanics are based on <u>Newton's law of motion</u>, an axiom from physics, which holds for any particle and in words can be expressed as "mass times acceleration equals force", or, using the <u>momentum $p=mv$</u> of a particle with mass $m>0$ and velocity $v\in\mathbb{R}^n$,

"the change of momentum equals the force" , $\boxed{\frac{dp}{dt}=F}$, (∗)

where $t$ is time, $F\in\mathbb{R}^n$ is the <u>force</u> acting on the particle, which may depend on particle position, velocity, on other particles, etc.
For a set of particles with positions $q_i(t)\in\mathbb{R}^n$, Newton's law produces a <u>system of odes</u> for the $q_i$ that can be solved given <u>initial values</u> such as an initial position $q_i(0)$ and velocity $\dot q_i(0)\in\mathbb{R}^n$.
There are three different perspectives on this problem which all shed light on different aspects:

1) ODE system as introduced above (Newton 1687: Principia Mathematica)
2) Lagrangian viewpoint / action principle: The motion is interpreted as a high-dimensional variational problem (Lagrange 1788: reformulation of Newton's law in "generalized coordinates", Lagrange multipliers for constrained motion; action principle has longer history)
3) Hamiltonian viewpoint: volume-preserving flow in phase space (formulation by Hamilton 1833)

## <u>Guiding examples</u>

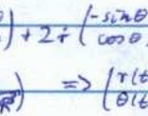
PM) point mass in potential field (e.g. gravitational field)

Here, force is given as gradient of a fixed external potential $V$, $F=-\nabla V$, i.e.

$$\frac{dp}{dt}=\frac{d}{dt}(m\dot q)=m\ddot q=-\nabla_q V(q)$$

a) gravitational field on earth : $V(q)=mgq_3 \quad\Rightarrow\quad \ddot q=\begin{pmatrix}0\\0\\-g\end{pmatrix} \quad\Rightarrow q=q(0)+t\dot q(0)-\frac{g}{2}\begin{pmatrix}0\\0\\t^2\end{pmatrix}$

e.g. height $q_3$ of a ball thrown in the air



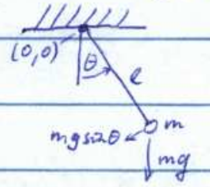
b) earth (mass m) in sun's gravitational field (assuming sun of mass M to be stationary):

$V=-\frac{\gamma mM}{|q|} \quad\Rightarrow\quad \ddot q=-\frac{\gamma M}{|q|^2}\frac{q}{|q|}$ (acceleration towards center)

in polar coord. (restricting to 2D): $\ddot r\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}+2\dot r\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}\dot\theta+r\ddot\theta\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}-r\dot\theta^2\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}=-\frac{\gamma M}{r^2}\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$

example solution: $\begin{pmatrix}r(0)\\\theta(0)\end{pmatrix}=\begin{pmatrix}R\\0\end{pmatrix}$, $\begin{pmatrix}\dot r(0)\\\dot\theta(0)\end{pmatrix}=\begin{pmatrix}0\\\sqrt{\gamma M/R^3}\end{pmatrix} \quad\Rightarrow \begin{pmatrix}r(t)\\\theta(t)\end{pmatrix}=\begin{pmatrix}R\\\sqrt{\gamma M}\,t/R^{3/2}\end{pmatrix}$

P) pendulum



$$\underbrace{m\ddot q\cdot\begin{pmatrix}-\cos\theta\\-\sin\theta\end{pmatrix}}_{\underset{q=\ell\binom{\sin\theta}{-\cos\theta}}{\underset{\uparrow}{=}}\,-m\ell\ddot\theta}=mg\sin\theta \quad\Rightarrow\quad \ddot\theta+\frac{g}{\ell}\sin\theta=0$$

for small $\theta$, $\sin\theta\approx\theta$, thus $\theta(t)\approx a\sin\left(\sqrt{g/\ell}\,t\right)+b\cos\left(\sqrt{g/\ell}\,t\right)$

MB) multibody problem : attracting particles (e.g. planets)

$V(q_i)=-\gamma m_i\sum_{j\neq i}\frac{m_j}{|q_i-q_j|} \quad\Rightarrow\quad \ddot q_i=-\gamma\sum_{j\neq i}\frac{m_j}{|q_j-q_i|^2}\frac{q_j-q_i}{|q_j-q_i|}$

no closed form solution for ≥ 3 particles (sun, earth, moon)

## kinetic & potential energy, Lagrangian, Hamiltonian

- Often we consider the case that the force only depends on position $q$ and is the gradient of a potential energy $V$, $F = -\nabla_q V(q)$. Also, the momentum $p_i$ of each particle is typically given by $m_i \dot{q}_i = \frac{\partial T}{\partial \dot{q}_i}$ for the so-called kinetic energy $T = \sum_i \frac{m_i}{2}|\dot{q}_i|^2$. The total physical energy then is $H = T+V$. Note that the potential energy is always relative to a reference state, i.e. $V + \text{const.}$ is also a valid potential energy.
- We then introduce the Hamiltonian $H = T+V$ and the Lagrangian $L = T - V$. Obviously, $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and $F_i = \frac{\partial L}{\partial q_i}$.
- In a more abstract setting we assume that at anytime $t$ our system is specified by a position vector or generalized coordinate $q(t) \in \mathbb{R}^N$ (which can e.g. be composed of the particle positions, $q = (q_1, q_2, \dots)$) and the velocity vector $\dot{q}(t) \in \mathbb{R}^N$. Furthermore, we assume there is a Lagrangian fcn $L(t,q,\dot{q})$ s.t. the momentum vector of the system is given by $p = \frac{\partial L}{\partial \dot{q}}$ and the force vector by $F = \frac{\partial L}{\partial q}$. (∗) then reads

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}} \qquad (**)$$

The Hamiltonian is then defined as the Legendre-Fenchel dual of $L$ in the "$\dot{q}$"-argument,

$$H(t,q,p) = \sup_{\dot{q}\in\mathbb{R}^N} \dot{q}\cdot p - L(t,q,\dot{q}),$$

and it is interpreted as the total energy of the system.

Ex: PM a) $T = \frac{m}{2}|\dot{q}|^2 = \frac{1}{2m}|p|^2$ ; $V = mg q_3$ ; $L(t,q,\dot{q}) = \frac{m}{2}|\dot{q}|^2 - mgq_3$ ; $H(t,q,p) = \frac{1}{2m}|p|^2 + mgq_3$

PM b) —— " —— ; $V = -\frac{\gamma m M}{|q|}$ ; $L(t,q,\dot{q}) = \frac{m}{2}|\dot{q}|^2 + \frac{\gamma m M}{|q|}$ ; $H(t,q,p) = \frac{1}{2m}|p|^2 - \frac{\gamma m M}{|q|}$

P) $T = \frac{m}{2}\ell^2\dot{\theta}^2$ ; $V = -mg\ell\cos\theta$ ; $L(t,\theta,\dot{\theta}) = \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta$ ; $H(t,\theta,p) = \frac{1}{2m\ell^2}p^2 - mg\ell\cos\theta$

Note: Had we chosen other generalized coordinates than $q=\theta$, e.g. $q = \ell\theta$, then we would have got a different Lagrangian, momentum, and Hamiltonian. However, we will see that (∗∗) holds for any choice of coordinates

MB) $T = \sum_i \frac{m_i}{2}|\dot{q}_i|^2 = \sum_i \frac{1}{2m_i}|p_i|^2$ ; $V = -\sum_{\substack{i,j\\ i\neq j}} \frac{\gamma m_i m_j}{|q_i - q_j|}$ ; $L\left(t, \begin{pmatrix} q_1\\ q_2\\ \vdots\end{pmatrix}, \begin{pmatrix}\dot{q}_1\\ \dot{q}_2\\ \vdots\end{pmatrix}\right) = \sum_i \frac{m_i}{2}|\dot{q}_i|^2 + \sum_{\substack{i,j\\ i\neq j}} \frac{\gamma m_i m_j}{|q_i - q_j|}$ ;

$H\left(t, \begin{pmatrix} q_1\\ q_2\\ \vdots\end{pmatrix}, \begin{pmatrix} p_1\\ p_2\\ \vdots\end{pmatrix}\right) = \sum_i \frac{1}{2m_i}|p_i|^2 - \sum_{i\neq j}\frac{\gamma m_i m_j}{|q_i - q_j|}$

- We will here always assume $L$ to be
  - convex in "$\dot{q}$"-argument
  - differentiable
  - superlinear growth in $\dot{q}$

$\Rightarrow H(t,q,p) = \max_{\dot{q}} \dot{q}\cdot p - L(t,q,\dot{q})$ is
- differentiable
- convex

Lemma: Let $f(x,y)$ and $f^*(p,y) = \sup_x x\cdot p - f(x,y)$ be convex in $x, p$, resp. and differentiable, then

$$\left(\frac{\partial f^*}{\partial p}\right)_{\underset{\text{held fixed}}{y}} = x \qquad \text{and} \qquad \left(\frac{\partial f^*}{\partial y}\right)_p = -\left(\frac{\partial f}{\partial y}\right)_x$$

, where the relation between the coordinates $(x,y)$ and $(p,y)$ is determined by $p = \frac{\partial f}{\partial x}(x,y)$.

proof: $f^*(p,y) = p\cdot x - f(x,y)$ where $x = x(p,y)$ satisfies $p = \frac{\partial f}{\partial x}(x,y)$ (uniquely solvable due to convexity). Now $\frac{\partial f^*}{\partial p} = x + \frac{\partial x}{\partial p}\left(p - \frac{\partial f}{\partial x}(x,y)\right) = x$, and $\frac{\partial f^*}{\partial y} = -\frac{\partial f}{\partial y} + \frac{\partial x}{\partial y}\left(p - \frac{\partial f}{\partial x}\right) = -\frac{\partial f}{\partial y}$. ☐

Thm: (a) $L(t,q,\dot{q}) = \sup_p p\cdot\dot{q} - H(t,q,p)$

(b) (∗∗) $\Leftrightarrow$

$$\boxed{\dot{q} = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}} \qquad (***)$$

proof: (a) by properties of bidual for convex lsc fcns

(b)
- $H = p\cdot\dot{q} - L(t,q,\dot{q})$ with $p = \frac{\partial L}{\partial \dot{q}}(t,q,\dot{q})$
- $p = \frac{\partial L}{\partial \dot{q}} \Leftrightarrow \dot{p} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}$ by Lemma
- $\frac{\partial H}{\partial p} = \dot{q}$ by Lemma

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