WWU MÜNSTER

Introduction to System Theory and Model Order Reduction

Young Mathematicians in Model Order Reduction – YMMOR

Art Pelling (TU Berlin), Hendrik Kleikamp (WWU Münster)

July 18, 2022

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Technische Universität Berlin

MM Mathematics Münster Cluster of Excellence

Example I: Linear heat equation

$$\begin{aligned} \partial_t u(x,t) &= \nabla \cdot (a(x)\nabla u(x,t)) + f(x,t), && x \in \Omega, t \in [0,\infty), \\ u(x,0) &= u_0(x), && x \in \Omega \end{aligned}$$

- Conductivity $a\colon \Omega \to \mathbb{R}$
- Source term $f\colon \Omega \times [0,\infty) \to \mathbb{R}$
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$$E\dot{x}(t) = Ax(t) + F(t).$$

If the source term can be controlled via $F(t) = Bu(t)$, we obtain

$$\dot{x}(t) = Ax(t) + Bu(t).$$

(mass matrix omitted for simplicity, multiply by E^{-1})

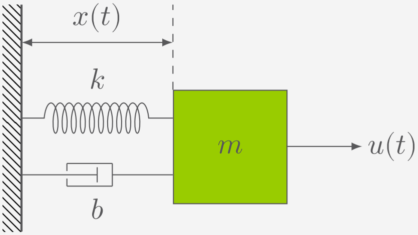
Example II: Mass-spring-damper system

Using Newton's second law, we model a mass-spring-damper system by

$$u(t) - b\dot{x}(t) - kx(t) = m\ddot{x}(t),$$

with

- the position $x\colon [0,T] \to \mathbb{R}$ of the moving mass,
- the mass $m > 0$,
- the friction coefficient $b > 0$,
- the spring constant $k > 0$,
- and an external force/control $u\colon [0,T] \to \mathbb{R}$ acting on the mass.



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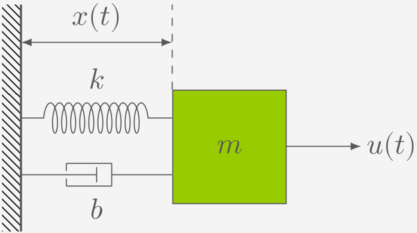
Rewriting this as a first order system by introducing the velocity v yields

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{=A} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{=B} u(t).$$

Example II: Mass-spring-damper system with output and feedback control



$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + Bu(t)$$

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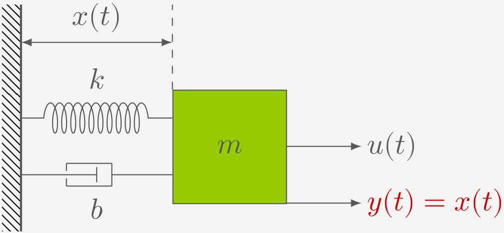

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + Bu(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=C} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

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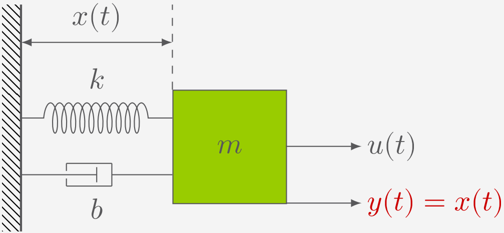

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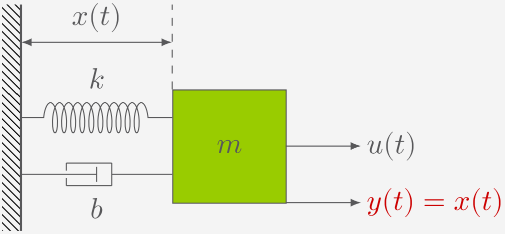

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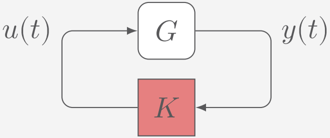

Output $y(t)$ can be used to dynamically determine control $u(t)$ to steer the system to a certain output of interest y_d!

For instance, set $u(t) = Ky(t)$ for some well-chosen matrix K.

Basic notation

- Most general setting:

$$G : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ y(t) = g(t, x(t), u(t)) \end{cases}$$

Components

- State $x\colon [0, \infty) \to \mathbb{R}^n$
- Input/control $u\colon [0, \infty) \to \mathbb{R}^m$
- Output $y\colon [0, \infty) \to \mathbb{R}^p$

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- Linear time-invariant setting:

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

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Linearity and Time-Invariance

Linearity:

$$G(\alpha u_1 + \beta u_2) = \alpha G(u_1) + \beta G(u_2) = \alpha y_1 + \beta y_2$$

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Time-Invariance:

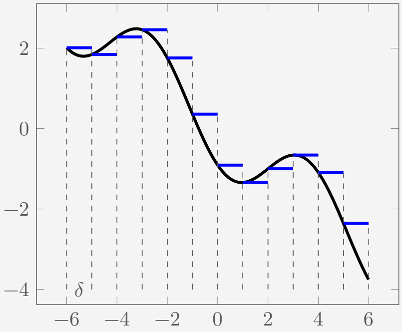
System commutes with the shift operator $\mathcal{S}_\tau(u(t)) = u(t+\tau)$:

$$G(\mathcal{S}_\tau(u)) = \mathcal{S}_\tau G(u)$$

Linearity and Time-Invariance

Idea:

If we know system response of an impulse, we can decompose input signal u into scaled and delayed “impulses” and calculate the total response y by superimposing scaled and delayed impulse responses.



Solution of LTI systems

The general LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$

$$y(t) = Cx(t) + Du(t) \quad (2)$$

Solution by multiplying (1) by e^{-At} and integration with respect to time:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

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Plugging $x(t)$ into (2) yields:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

Convolution with the Impulse Response

Assume $x_0 = 0$ (zero-state case) and $D = 0$ (no feedthrough), then the output is given by

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$$y(t) = C \int_0^t e^{A(t-\tau)} B u(\tau) \, d\tau = \int_0^t h(t-\tau) u(\tau) d\tau = (h * u)(t),$$

which is a convolution of u with the impulse response

$$h(t) = C e^{At} B.$$

Laplace transform and properties

Definition: Laplace transform

Given a signal $y\colon [0,\infty) \to \mathbb{R}^m$ in time-domain, its *Laplace transform* is defined as

$$\mathcal{L}\{y\}(s) = \hat{y}(s) = \int_0^\infty y(t)e^{-st}\,dt.$$

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Properties of the Laplace transform:

- Linearity: $\mathcal{L}\{\alpha y_1 + \beta y_2\}(s) = \alpha\mathcal{L}\{y_1\}(s) + \beta\mathcal{L}\{y_2\}(s)$
- Derivatives: $\mathcal{L}\{\dot{y}\}(s) = s\mathcal{L}\{y\}(s) - y(0)$

The Transfer function

- Applying the Laplace transform to a system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

gives

$$\begin{aligned} s\hat{x}(s) - x_0 &= A\hat{x}(s) + B\hat{u}(s), \\ \hat{y}(s) &= C\hat{x}(s) + D\hat{u}(s). \end{aligned}$$

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$$\begin{aligned} (sI - A)\hat{x}(s) &= x_0 + B\hat{u}(s), \\ \hat{y}(s) &= C\hat{x} + D\hat{u}(s). \end{aligned}$$

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- Assume that $x_0 = 0$ (zero-state case). Then it holds

$$\hat{y}(s) = H(s)\hat{u}(s)$$

with the *Transfer function* H, given as $H(s) = C(sI - A)^{-1}B + D$.

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- **Internal description:** Matrix quadruple (A, B, C, D) is called a realization.
- **Realization problem:** Find the (smallest) system realizing a given impulse response or transfer function.

State Transformations

- Realizations are not unique!
- For regular $T \in \mathbb{R}^{n \times n}$, define the new state variable $z = Tx$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

- (A, B, C) and (TAT^{-1}, TB, CT^{-1}) are equivalent systems.

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$$T^{-1}\dot{z}(t) = AT^{-1}z(t) + Bu(t)$$
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System Invariants

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3. Markov parameters:

$$h_i = \frac{d^i}{dt^i}h(t)\bigg|_{t=0} = \frac{d^i}{ds^i}H(s)\bigg|_{s=\infty} = CA^{i-1}B, \quad i \in \mathbb{N}$$

Controllability of a system

Definition: Controllability

The system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is called *controllable* if for any initial state x_0 and desired state x_1 there exists a control u that transfers x_0 into x_1 in finite time.

Controllability and Gramian

How to determine whether a system is controllable?

The following statements are equivalent:

- The system $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable.
- The controllability matrix

  $$\mathcal{C} = \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}$$

  has full rank. ($\rightarrow$ Cayley-Hamilton theorem)

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has full rank. ($\rightarrow$ Cayley-Hamilton theorem)
- If A is stable, i.e. all eigenvalues of A have negative real part, then the *Controllability Gramian* W_c, given as
$$W_c = \int_0^\infty e^{At} BB^\top e^{A^\top t}\, dt,$$
is positive definite.

Observability of a system

Definition: Observability

The system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$

is called *observable* if for any (unknown) initial state x_0 there exists a point in time $T > 0$ such that one can uniquely determine x_0 from $y(t)$, $t \in [0, T]$, and $u(t)$, $t \in [0, T]$.

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$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

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- If A is stable, then the *Observability Gramian* W_o, given as

$$W_o = \int_0^\infty e^{A^\top t} C^\top C e^{At} \, dt,$$

  is positive definite.

Computing the Gramians via Lyapunov equations

If A is stable, then the controllability gramian W_c and the observability gramian W_o are the unique solutions to the Lyapunov equations

$$AW_c + W_c A^\top = -BB^\top,$$
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First equation:

$$\begin{aligned} AW_c + W_c A^\top &= A \int_0^\infty e^{At} BB^\top e^{A^\top t} \, dt + \int_0^\infty e^{At} BB^\top e^{A^\top t} \, dt A^\top \\ &= \int_0^\infty \frac{d}{dt} \left(e^{At} BB^\top e^{A^\top t} \right) \, dt \\ &= \left[e^{At} BB^\top e^{A^\top t} \right]_0^\infty = \underbrace{0}_{A \text{ is stable}} - BB^\top \end{aligned}$$

Proceed similarly for the second equation.

Projection-based Model Order Reduction

Goal: Replace the system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$

$$\dot{x}(t) \quad = \quad A \quad x(t) \quad + \quad B \quad u(t)$$
$$y(t) \quad = \quad C \quad x(t) \quad + \quad D \quad u(t)$$

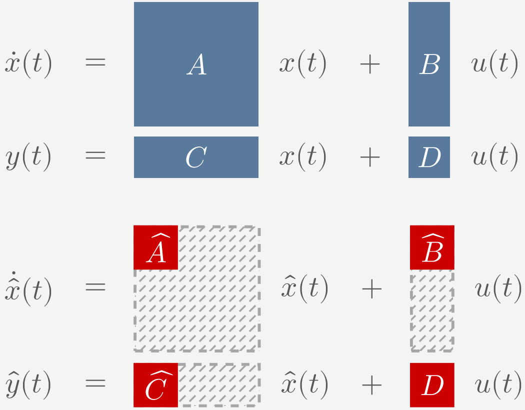
Projection-based Model Order Reduction

Goal: Replace the system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$

by a reduced system

$$\dot{\hat{x}}(t) = \widehat{A}\hat{x}(t) + \widehat{B}u(t),$$
$$y(t) = \widehat{C}\hat{x}(t) + Du(t).$$



Motivation for Balanced Truncation

Idea: Remove states x that are at the same time

1. hard to reach (i.e. it requires a lot of energy to control the system to that state)
2. hard to observe (i.e. have small observation energy)

In other words: Keep those states that retain the largest amount of input-output energy.

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In other words: Keep those states that retain the largest amount of input-output energy.

Use a basis transformation such that controllability and observability are equivalent concepts, i.e. the system is balanced.

$\Longrightarrow$ Balanced Truncation

Basis transformations and how they change a system

Apply basis transformation via

$$x = Tz, \qquad T\colon \mathbb{R}^n \to \mathbb{R}^n.$$

The system changes according to

$$\left.\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t)\end{aligned}\right\} \implies \left\{\begin{aligned}\dot{z}(t) &= T^{-1}ATz(t) + T^{-1}Bu(t),\\ y(t) &= CTz(t) + Du(t).\end{aligned}\right.$$

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The Gramians change according to

$$\left. W_c = \int_0^\infty e^{At}BB^\top e^{A^\top t}\,dt \right\} \Longrightarrow \left\{ \int_0^\infty e^{T^{-1}ATt}T^{-1}BB^\top T^{-\top}e^{T^\top A^\top T^{-\top}t}\,dt \right.$$

$$\left. W_o = \int_0^\infty e^{A^\top t}C^\top Ce^{At}\,dt \right\} \Longrightarrow \left\{ \int_0^\infty e^{T^\top A^\top T^{-\top}t}T^\top C^\top CTe^{T^{-1}ATt}\,dt \right.$$

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Basis transformations and how they change a system

Apply basis transformation via

$$x = Tz, \qquad T \colon \mathbb{R}^n \to \mathbb{R}^n.$$

The system changes according to

$$\left.\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t)\end{aligned}\right\} \Longrightarrow \left\{\begin{aligned}\dot{z}(t) &= T^{-1}ATz(t) + T^{-1}Bu(t),\\ y(t) &= CTz(t) + Du(t).\end{aligned}\right.$$

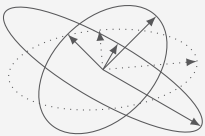
The Gramians change according to

$$\left.W_c = \int_0^\infty e^{At}BB^\top e^{A^\top t}\,dt\right\} \Longrightarrow \left\{T^{-1}W_cT^{-\top} =: \widehat{W}_c\right.$$

$$\left.W_o = \int_0^\infty e^{A^\top t}C^\top Ce^{At}\,dt\right\} \Longrightarrow \left\{T^\top W_oT =: \widehat{W}_o\right.$$

Balanced Truncation

- Goal: Find transformation T, such that
$$\widehat{W}_c = \widehat{W}_o = \Sigma$$
for a diagonal matrix $\Sigma \in \mathbb{R}^{n \times n}$.



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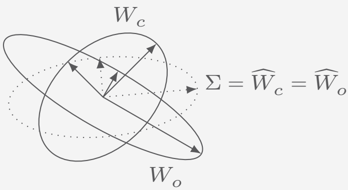

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$$\widehat{W}_c \widehat{W}_o = T^{-1} W_c T^{-\top} T^\top W_o T = T^{-1} W_c W_o T = \Sigma^2,$$
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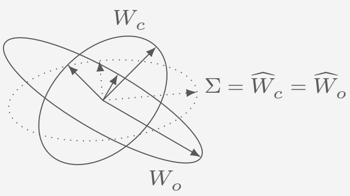

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- T is given by the eigenvectors of $W_c W_o$.

Balanced Truncation

To obtain a reduced order model:

1. Solve Lyapunov equations for W_c and W_o.
2. Compute $W_c W_o$ and its eigenpairs (T, Σ^2).
3. Sort eigenvalues in Σ^2 by decreasing magnitude and truncate at some point.
4. Project all system matrices onto the resulting reduced space.

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Note:
In practice, one usually uses (low-rank) Cholesky factorizations of W_c and W_o to compute the balancing transformation T.

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A-priori error bounds:

$$||H - \hat{H}||_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} ||H(i\omega) - \hat{H}(i\omega)||_2 \leq 2 \sum_{k=r+1}^{n} \sigma_k$$

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Hankel Matrix

The entries of Σ are the singular values of

$$W_c W_o = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ h_2 & h_3 & & \iddots & \\ h_3 & & \iddots & & \vdots \\ & \iddots & & & \\ h_n & & \cdots & & h_{2n-1} \end{bmatrix} = \mathcal{H}$$

and are referred to as the *Hankel singular values* of the system.

Example: Balanced Truncation