Universität Münster

Two-Stage Model Reduction Approaches for Parametrized Optimal Control Problems

SIAM CSE 2025 – Minisymposium on “Modern Approaches in Machine Learning for Model Order Reduction”

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MM Mathematics Münster Cluster of Excellence

Parametrized optimal control problems

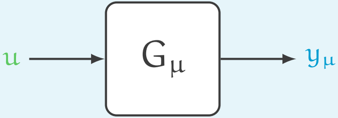
Control systems

$$u \longrightarrow \boxed{G_\mu} \longrightarrow y_\mu$$

- Parameter $\mu \in \mathcal{P}$
- Input/control $u\colon [0,T] \to U$
- Output $y_\mu\colon [0,T] \to Y$

Parametrized linear time-varying systems
General setting

Control systems



- Parameter $\mu \in \mathcal{P}$
- Input/control $u\colon [0,T] \to U$
- Output $y_\mu\colon [0,T] \to Y$

- State system relating controls to outputs:

$$G_\mu : \begin{cases} E\frac{d}{dt}x_\mu(t) &= A(\mu;t)x_\mu(t) + B(\mu;t)u(t) \\ y_\mu(t) &= Cx_\mu(t) \end{cases}$$

- (High-dimensional) state $x_\mu\colon [0,T] \to X$

Parametrized optimal control problems
General setting

$$\min_{u \in L^2([0,T];U)} \quad \frac{1}{2}\left[\underbrace{\| y_\mu(T) - y_\mu^T \|_Y^2}_{\substack{\text{deviation from target}\\ \text{at final time}}} + \underbrace{\int_0^T \langle u(t), R(t)u(t) \rangle_{U \times U'} \, dt}_{\text{control energy}} \right],$$

General setting

$$\min_{u\in L^2([0,T];U)} \frac{1}{2}\Bigg[\underbrace{\left\| y_\mu(T) - y_\mu^T \right\|_Y^2}_{\substack{\text{deviation from target}\\ \text{at final time}}} + \underbrace{\int_0^T \langle u(t), R(t)u(t)\rangle_{U\times U'}\,dt}_{\text{control energy}}\Bigg],$$

$$\text{such that}\quad \begin{cases} E\frac{d}{dt}x_\mu(t) = A(\mu;t)x_\mu(t) + B(\mu;t)u(t) & \text{for } t\in[0,T],\\ y_\mu(t) = Cx_\mu(t) & \text{for } t\in[0,T],\\ x_\mu(0) = x_\mu^0, & \text{(initial data)} \end{cases}$$

Target output: $y_\mu^T = Cx_\mu^T$

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Optimality system and final time adjoints

Theorem (K./Renelt'24)

The optimal state x_μ^*, *control* u_μ^* *and adjoint* φ_μ^* *can be characterized as solution to the system:*

$$
\begin{aligned}
E\frac{d}{dt}x_\mu(t) &= A(\mu;t)x_\mu(t) + B(\mu;t)u_\mu(t), && \textit{(state equation)}\\
u_\mu(t) &= -R(t)^{-1}B(\mu;t)^*\varphi_\mu(t), && \textit{(control equation)}\\
-E_{\mathrm{ad}}\frac{d}{dt}\varphi_\mu(t) &= A(\mu;t)^*\varphi_\mu(t), && \textit{(adjoint equation)}
\end{aligned}
$$

for almost all $t \in [0,T]$ *with initial condition* $x_\mu(0) = x_\mu^0$ *and terminal condition*

$$\mathcal{R}_X E^* \varphi_\mu(T) = M\left(x_\mu(T) - x_\mu^T\right),$$

where $\mathcal{R}_X \colon X \to X'$ *is the Riesz map,* $E_{\mathrm{ad}} \coloneqq \mathcal{R}_X E^* \mathcal{R}_X^{-1}$ *and* $M \coloneqq C^* \mathcal{R}_Y C$.

Linear system for optimal final time adjoints

Lemma (K./Renelt'24)

The optimal final time adjoint $\varphi_\mu^*(T) \in X$ solves the linear system

$$\boxed{S(\mu)\varphi_\mu^*(T) = g(\mu)}$$

where

- $S(\mu)$ *involves solving high-dimensional parametric systems,*
- $g(\mu)$ *is a parameter-dependent right hand side.*

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$$S(\mu)\varphi_\mu^*(T) = g(\mu) \qquad \Longleftrightarrow \qquad \varphi_\mu^*(T) = \operatorname*{arg\,min}_{p \in X} \| g(\mu) - S(\mu)p \|_X^2$$

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Reduced order models

Overview of the involved reduced order models

FOM

Optimal final time adjoint

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Primal and adjoint systems

$$E \frac{d}{dt} x_\mu(t) = A(\mu; t) x_\mu(t) + B(\mu; t) u_\mu(t)$$

$$-E_{ad} \frac{d}{dt} \varphi_\mu(t) = A(\mu; t)^* \varphi_\mu(t)$$

Overview of the involved reduced order models

[Lazar/Zuazua'16, K./Lazar/Molinari'24]

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RB-ROM

Least-squares on reduced subspace

$$\tilde{\varphi}_\mu^N = \arg\min_{p \in X^N} \| g(\mu) - S(\mu)p \|_X^2$$

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[Haasdonk/Ohlberger'11]

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$$\hat{E}_{pr} \frac{d}{dt} \hat{x}_\mu(t) = \hat{A}_{pr}(\mu; t) \hat{x}_\mu(t) + \hat{B}_{pr}(\mu; t) u_\mu(t)$$

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[Fabrini/Iapichino/Volkwein'18, K./Renelt'24]

F-ROM

Combination of RB-ROM and system reduction

$$\hat{\varphi}_\mu^{N,\mathrm{red}} = \arg\min_{p\in X^N} \left\| \hat{g}(\mu) - \hat{S}(\mu)p \right\|_X^2$$

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[Hesthaven/Ubbiali'18, K.'25]

ML-F-ROM

Approximate $\hat{\varphi}_\mu^{N,\mathrm{red}}$ as a function of the parameter using machine learning

Primal and adjoint systems

$$E\frac{d}{dt}x_\mu(t) = A(\mu;t)x_\mu(t) + B(\mu;t)u_\mu(t)$$

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A posteriori error estimation for the reduced models

- Reliable a posteriori error estimators for RB-ROM, F-ROM and ML-F-ROM available

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Theorem (Reliable a posteriori error estimator for the F-ROM)

For an approximate final time adjoint φ^N we have

$$\| \varphi_\mu^*(T) - \varphi^N \|_X \leqslant \eta_\mu^{\mathrm{red}}(\varphi^N).$$

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For an approximate final time adjoint φ^N we have

$$\left\| \varphi_\mu^*(T) - \varphi^N \right\|_X \leqslant \eta_\mu^{\text{red}}(\varphi^N).$$

Main idea of the proof: Derive the bound

$$\begin{aligned}
\left\| \varphi_\mu^*(T) - \varphi^N \right\|_X \leqslant \boxed{\eta_\mu(\varphi^N)} &= C_{\text{op}} \left\| g(\mu) - S(\mu)\varphi^N \right\|_X \\
&= C_{\text{op}} \left\| \Big(g(\mu) - \hat{g}(\mu)\Big) + \Big(\hat{g}(\mu) - \hat{S}(\mu)\varphi^N\Big) + \Big(\hat{S}(\mu)\varphi^N - S(\mu)\varphi^N\Big) \right\|_{X'} \\
&\leqslant C \left(\Delta_\mu^{\text{pr}}[0](T) + \hat{\eta}_\mu(\varphi^N) + \Delta_\mu^\Lambda[\varphi^N]\right) \\
&=: \boxed{\eta_\mu^{\text{red}}(\varphi^N)}.
\end{aligned}$$

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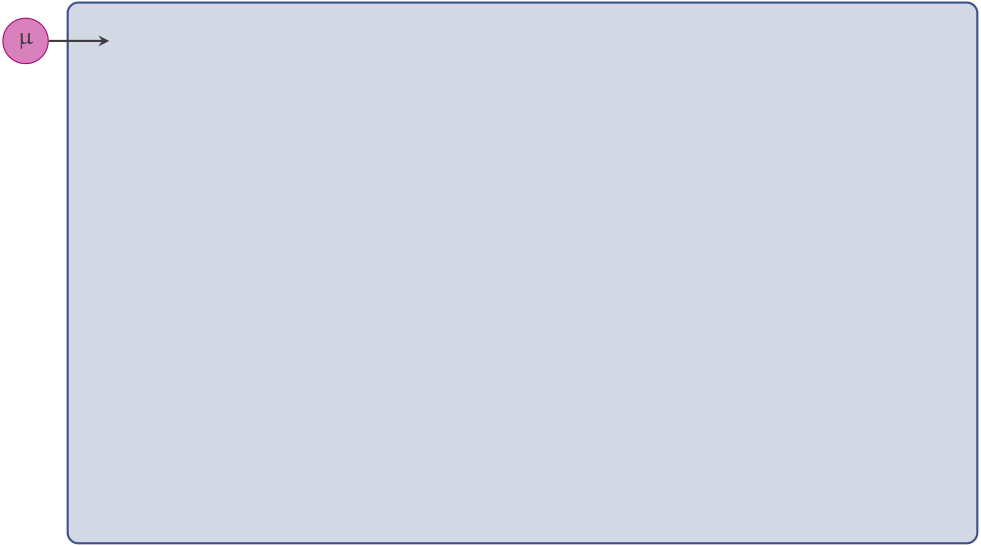
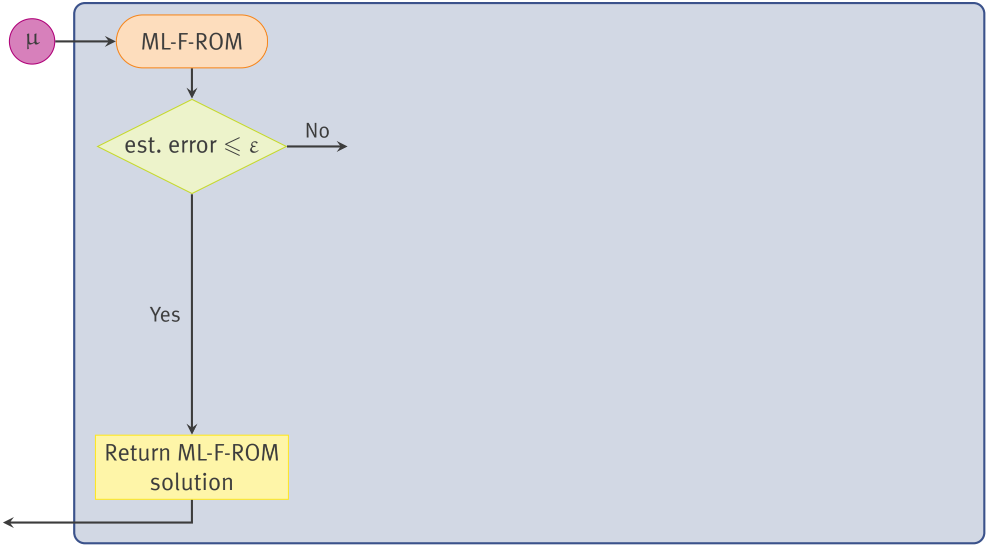
Adaptive model hierarchies

Four-stage hierarchy for optimal control problems



Four-stage hierarchy for optimal control problems



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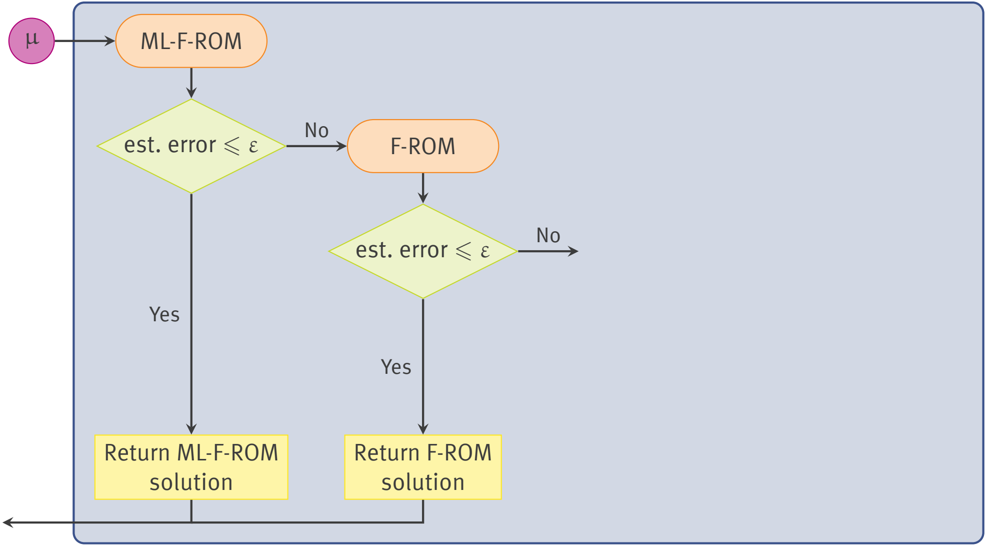

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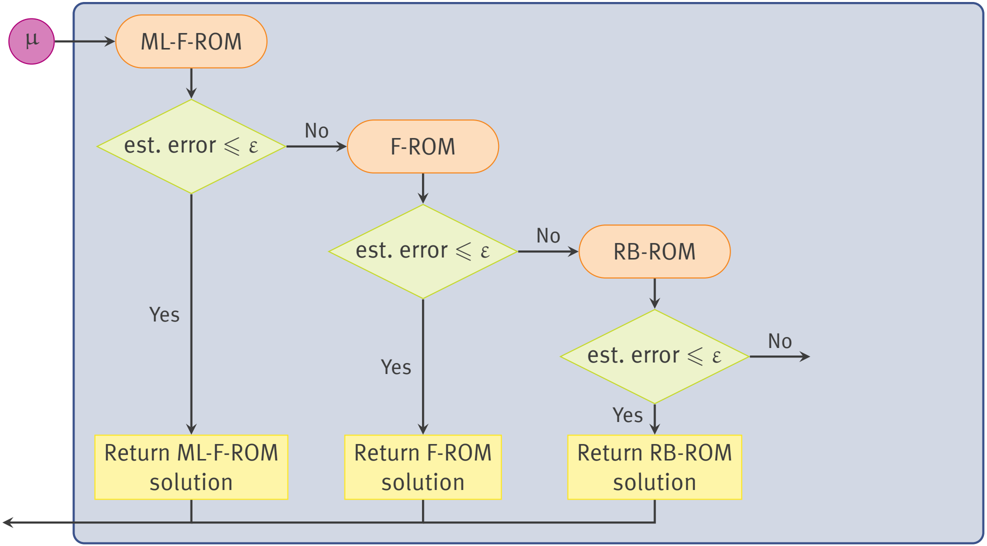

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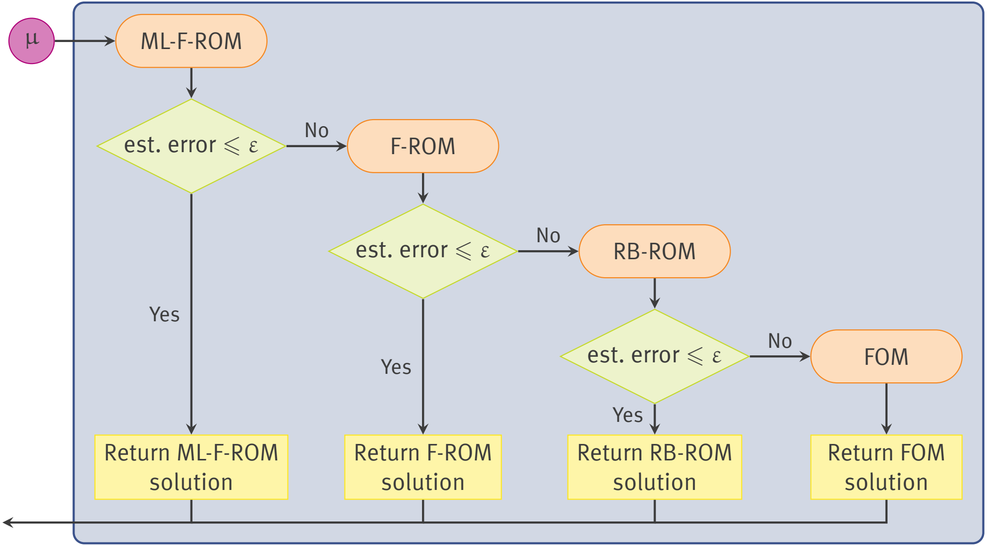

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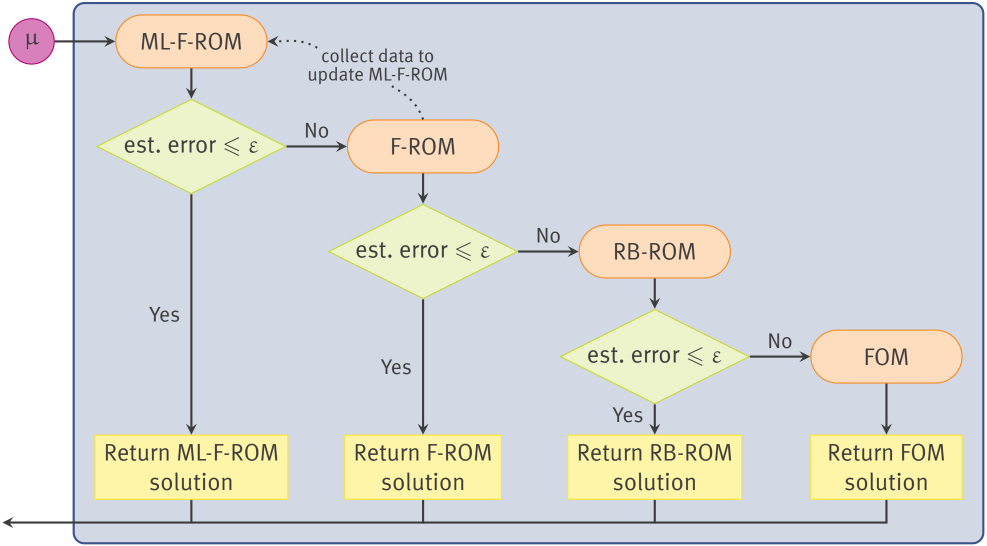

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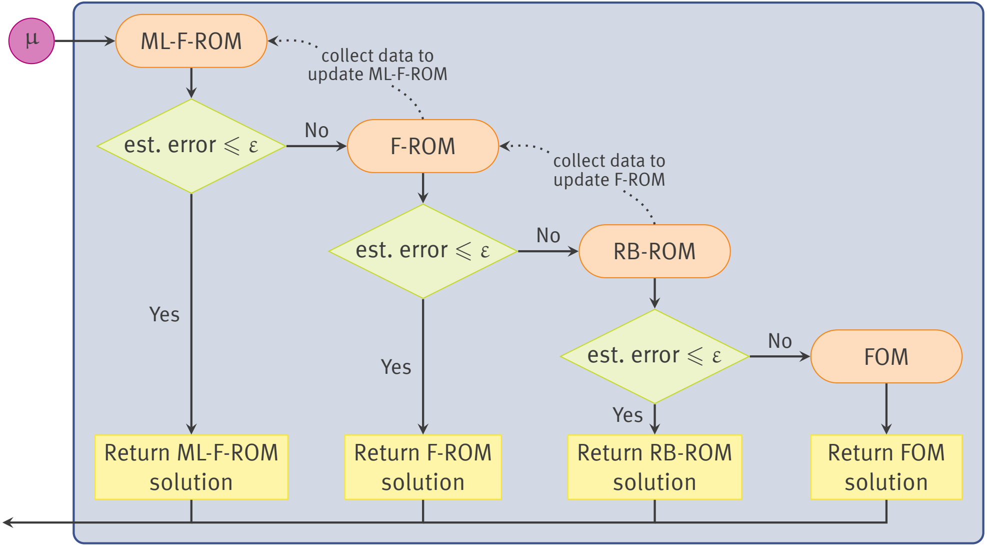

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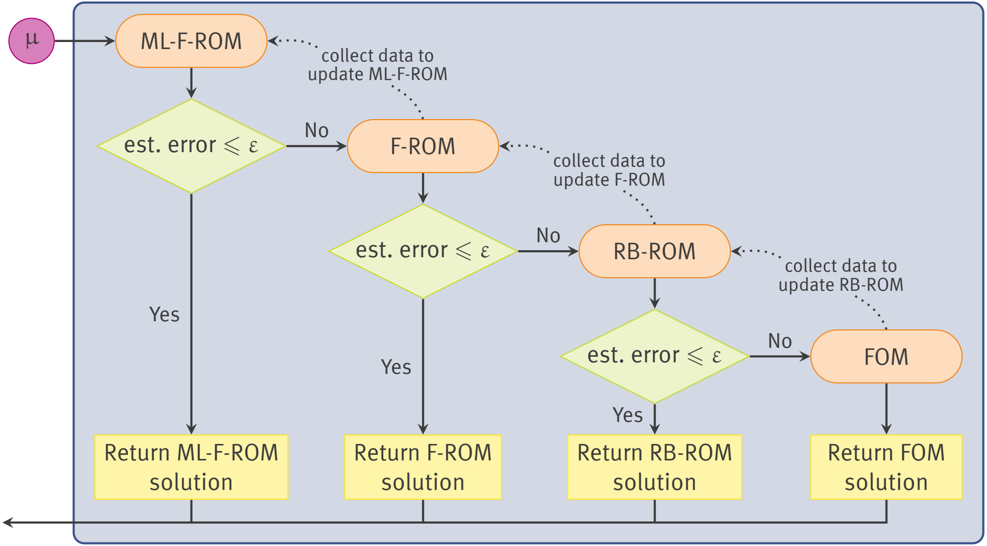

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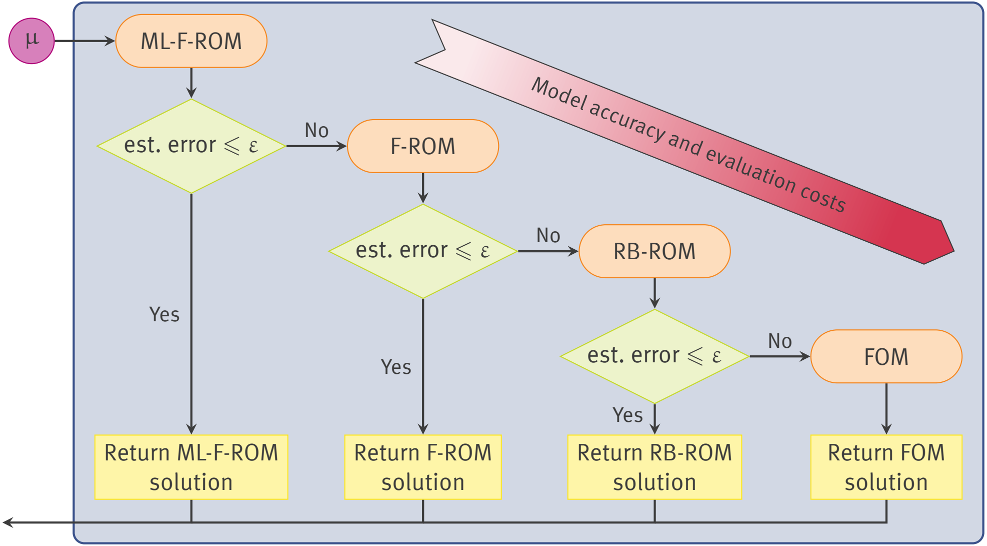
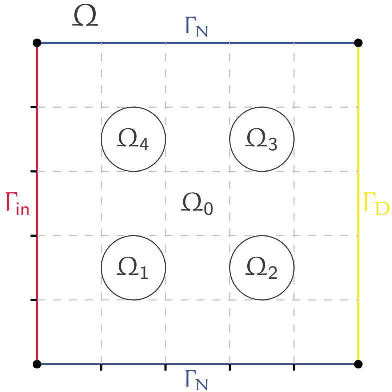


Numerical experiment



- Heat equation as state system:

$$
\begin{aligned}
\partial_t \theta(t;\mu) - \nabla \cdot (\sigma(t;\mu)\nabla\theta(t;\mu)) &= 0 && \text{in } \Omega,\\
\sigma(t;\mu)\nabla\theta(t;\mu)\cdot\vec{n} &= u(t) && \text{on } \Gamma_{in},
\end{aligned}
$$

  \+ homogeneous initial, Dirichlet and Neumann conditions

- Parametric and time-dependent diffusivity σ
- Output quantities: Average temperature in the four cookies
- Target state: Prescribed average temperature in all cookies

Numerical results for the four-stage hierarchy

Results for querying the hierarchy for 10,000 random parameters

Model	Number of solves	Number of error estimates	Total time for error est. and solving [s]	Average time for error est. and solving per solve [s]
FOM	4	—	304.95	76.24
RB-ROM	12	16	234.56	19.55
F-ROM	437	453	450.39	1.03
ML-F-ROM	9,547	10,000	5,136.95	0.54

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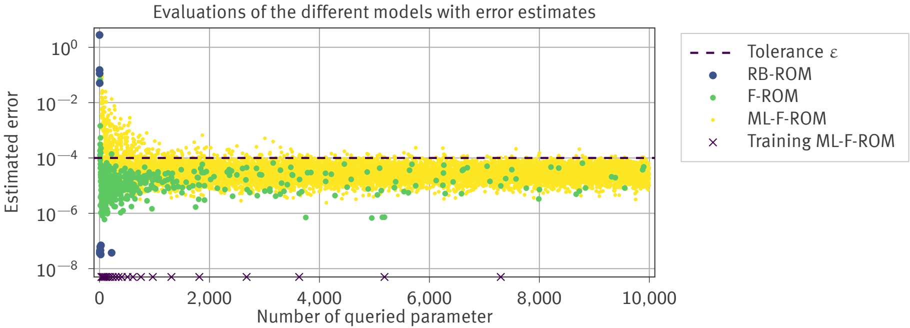

Thank you for your attention!

Papers and code:

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