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# Nonlinear model order reduction for parametrized hyperbolic conservation laws in a space-time domain

CMAM 2022 – Minisymposium on “Numerical methods for wave propagation problems”

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# Parametrized PDEs and model order reduction I

## PDEs depending on additional parameter $\mu$, for instance

$$-\nabla \cdot a(\mu)\nabla u_\mu = f(\mu) \quad \text{in } \Omega_\mu, \qquad u_\mu = g(\mu) \quad \text{on } \partial\Omega_\mu$$

$$\partial_t u_\mu + \nabla_x \cdot f(u_\mu;\mu) = 0 \quad \text{in } (0,T)\times\Omega, \qquad u_\mu(0) = u_0 \quad \text{in } \Omega$$

Examples for parameters:

- Diffusion coefficients
- Boundary conditions
- Domain shapes
- Sources and sinks
- Advection velocities

Applications:

- Many-query scenarios, i.e. solve for many parameters $\mu \in \mathcal{P}$.
- Real-time solution, i.e. solve in real-time for new parameter $\mu \in \mathcal{P}$.

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# Parametrized PDEs and model order reduction II

**Problem: Solving the high-fidelity model is too costly!**

Idea: Compute a *Reduced Order Model* in a (potentially costly) offline phase that can be cheaply evaluated in the online phase.

Standard approach for elliptic and parabolic PDEs:

- Compute (offline) a reduced space $V_N \subset V_h$ with $\dim V_N \ll \dim V_h$.
- Perform Galerkin projection of the full-order model onto $V_N$.
- Perform hyper-reduction to obtain a reduced order model independent of $\dim V_h$.
- Solve a small system with size depending only on $\dim V_N$ during the online phase.

$\longrightarrow$ **Search for solutions in the reduced space $V_N$!**

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# How good can linear approaches be?

## Kolmogorov $N$-width

The Kolmogorov $N$-width $d_N(\mathcal{M})$ of a set $\mathcal{M}$ is defined as

$$d_N(\mathcal{M}) = \inf_{\dim(V)=N} \operatorname{dist}(V, \mathcal{M}).$$

It measures the approximability of $\mathcal{M}$ by linear subspaces of dimension $N$.

Consider the Kolmogorov $N$-width of the solution manifold $\mathcal{M} = \{u_\mu : \mu \in \mathcal{P}\}$:

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# Transport dominated problems

- Transport problems or wave equations [Ohlberger/Rave’16,Greif/Urban’19]:

$$d_N(\mathcal{M}) \sim CN^{-1/2}$$

Simplest example: The linear transport equation

$$\partial_t u_\mu(t,x) + \mu \cdot \partial_x u_\mu(t,x) = 0,$$
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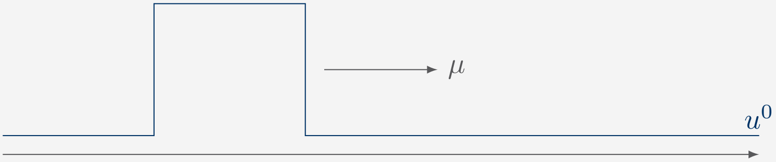


# Another difficulty: Shocks and their interaction

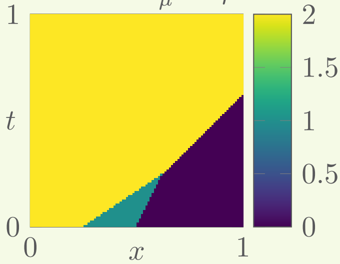
## Example – Burgers equation

$$\partial_t u_\mu + \frac{\mu}{2}\partial_x u_\mu^2 = 0, \qquad (t,x) \in [0,T] \times \Omega,$$

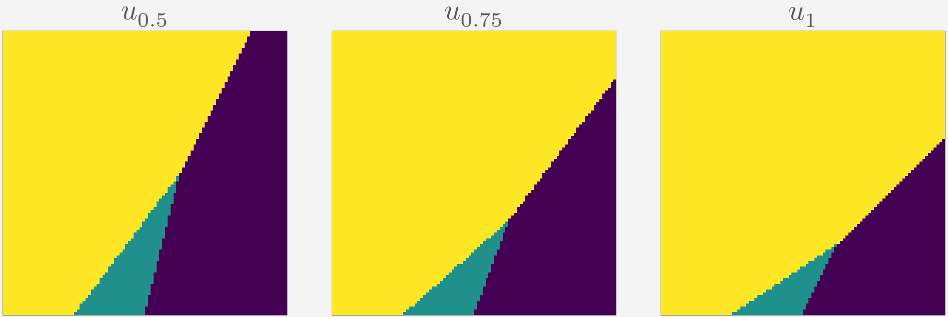
$$u_\mu(0) = u_0, \qquad x \in \Omega,$$

$$u_0(x) = \begin{cases} 2, & \text{if } x \leq 1/4, \\ 1, & \text{if } 1/4 < x \leq 1/2, \\ 0, & \text{if } 1/2 < x. \end{cases}$$

Full solution $u_\mu$ for $\mu = 1$



# How do solutions for different parameters look like?

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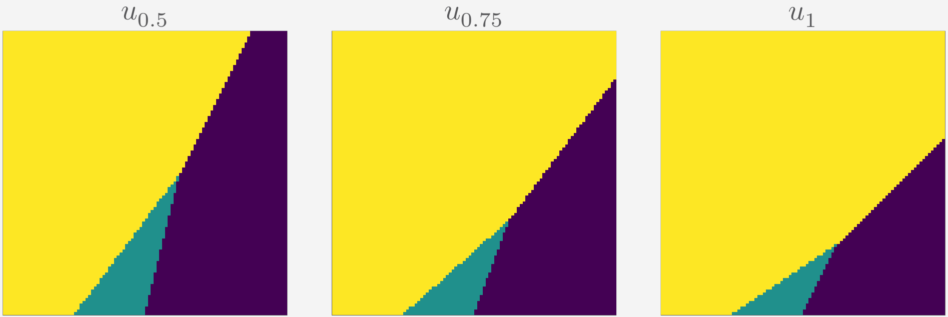


$\longrightarrow$ **Transform underlying space-time domain to match snapshots?**

# Transformations of space-time domains to align shocks and discontinuities

- Shock interaction already incorporated in space-time solutions (no need to treat them separately).
- Diffeomorphic transformations (that can be represented in a reduced space, see below) of the underlying space-time domain to match snapshots to each other.
- Choose a (fixed) reference snapshot that can be transformed into all other solutions.
- Apply ideas and concepts from (medical) image registration.

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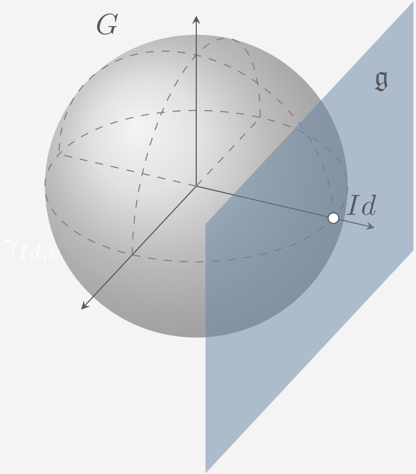
# Lie groups and Lie algebras

## Lie group

A group $G$ such that group multiplication and inversion are smooth maps, i.e. $G$ is a manifold.

## Lie algebra

The tangent space $\mathfrak{g} = T_{Id}G$ to a Lie group $G$ at the identity element $Id \in G$.



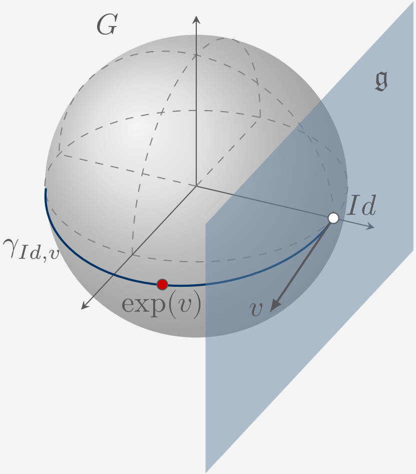
# Lie groups and Lie algebras

## Geodesic curve

Smooth curve $\gamma_{p,v}\colon I \to G$, $I$ an open interval, $\gamma_{p,v}(0) = p \in G$, $\gamma'_{p,v}(0) = v \in T_pG$, that (locally) minimizes lengths.

## Exponential map

Maps the Lie algebra into the Lie group, i.e. $\exp\colon \mathfrak{g} \to G$, by $\exp(v) = \gamma_{Id,v}(1)$.



# Lie groups and Lie algebras in image registration

- Diffeomorphism group $G$ on $\mathbb{R}^n$ forms a Lie group.
- Lie algebra $\mathfrak{g}$ is the space of smooth vector fields.
- Diffeomorphism group acts on underlying space by transforming it.

- Attention: Diffeomorphism group is infinite dimensional! (Theory is much harder in general, e.g. exponential map and geodesics do not coincide.)

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# Diffeomorphisms and vector fields

- Diffeomorphism $\phi_t : \Omega \to \Omega$, $\Omega \subset \mathbb{R}^d$, given as flow of (time-dependent) velocity field $v_t : \Omega \to \mathbb{R}^d$, i.e.

$$\frac{\partial \phi_t}{\partial t} = v_t \circ \phi_t .$$

- Differential operator $L = (Id - \alpha\Delta)^s$, with inverse $K = L^{-1}$.
- Geodesic evolution of $v_t$ is given by EPDiff equation

$$\frac{\partial v_t}{\partial t} = -K\left[(Dv_t)^T \cdot Lv_t + D(Lv_t) \cdot v_t + Lv_t \cdot \operatorname{div} v_t\right].$$

- Knowledge of $v_0$ sufficient to compute $\phi_1$!
  $\longrightarrow$ Main idea of *geodesic shooting* [Miller/Trouvé/Younes'06].

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# Ideas and concepts from image registration

How to compute $v_0$ given a "template image" $u_0 : \Omega \to \mathbb{R}$ and a "target image" $u_1 : \Omega \to \mathbb{R}$?

Minimize energy

$$E_{u_0 \to u_1}(v_0) := \underbrace{(Lv_0, v_0)_{L^2(\Omega)}}_{\text{Regularization term}} + \frac{1}{\sigma^2} \underbrace{\|u_0 \circ \phi_1^{-1} - u_1\|^2_{L^2(\Omega)}}_{\text{Mismatch measurement}}$$

using descent methods (e.g. L-BFGS).

# (Linear) Model order reduction in the Lie algebra

- Lie algebra of smooth vector fields $\mathfrak{g}$ forms Hilbert space with inner product $\langle v, w \rangle_{\mathfrak{g}} := (Lv, w)_{L^2(\Omega)} = (v, Lw)_{L^2(\Omega)}$.
- We can apply well-known linear model order reduction methods in $\mathfrak{g}$, like POD [Wang/Xing/Kirby/Zhang'19] or Greedy algorithms!
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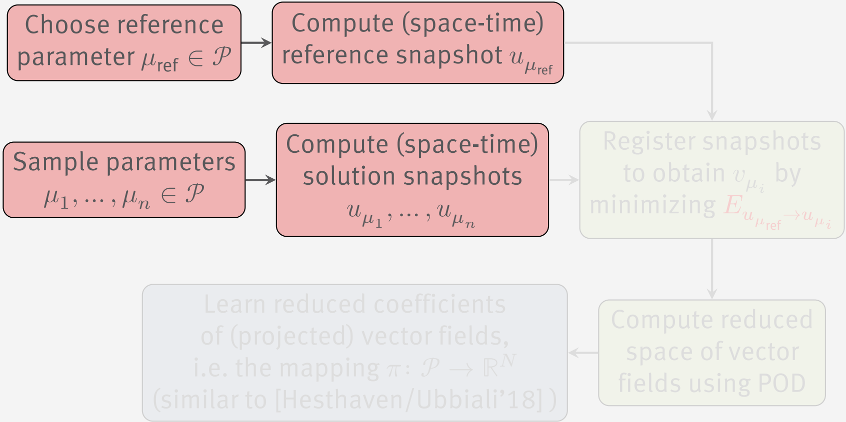

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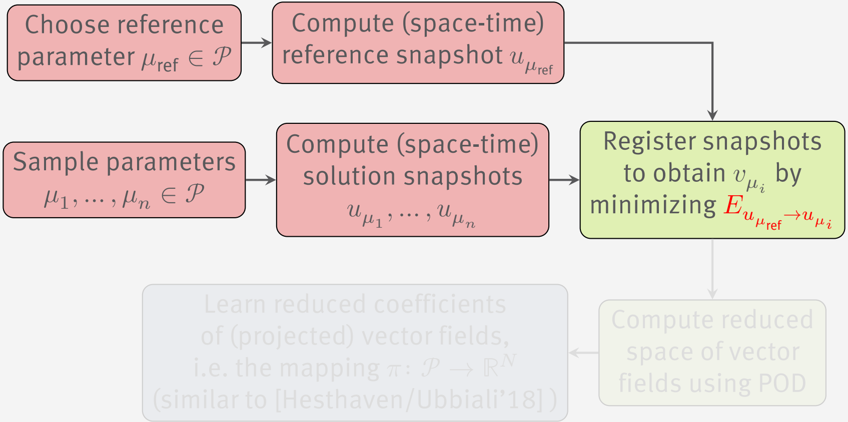

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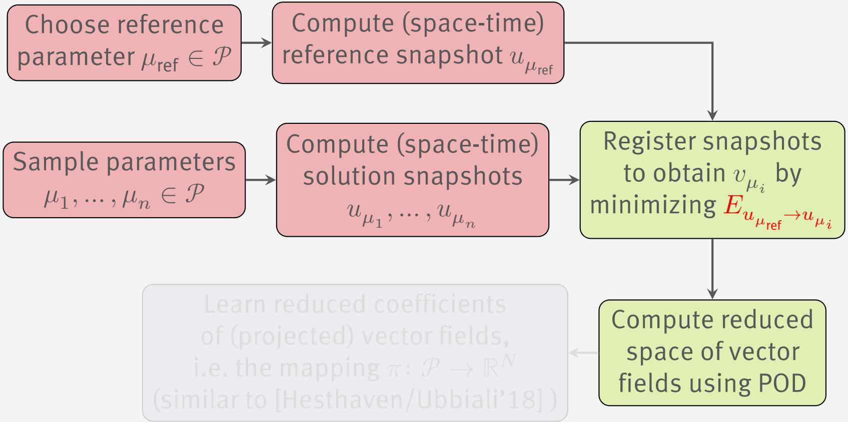

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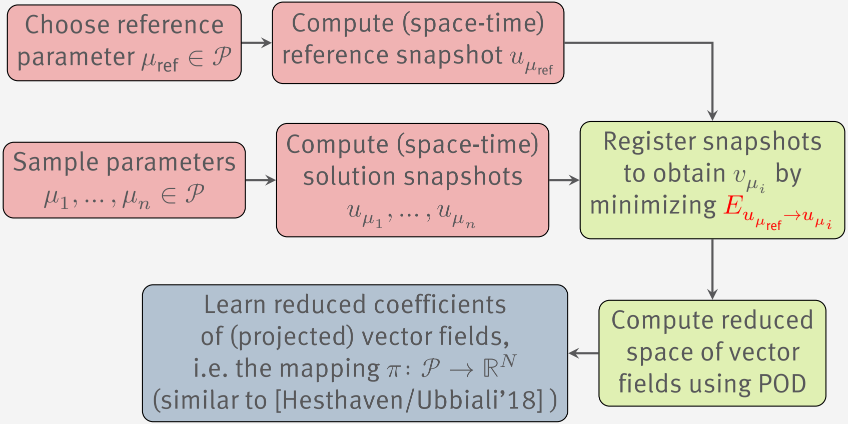

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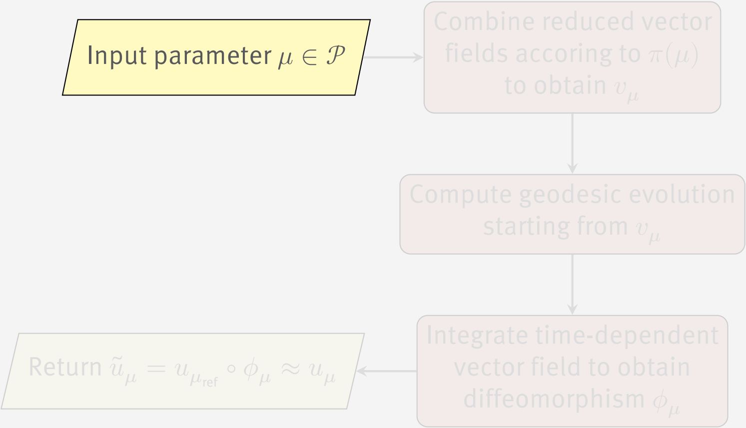

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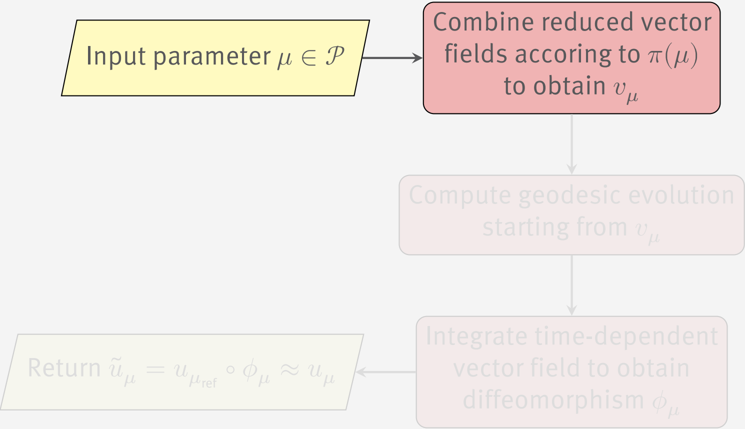

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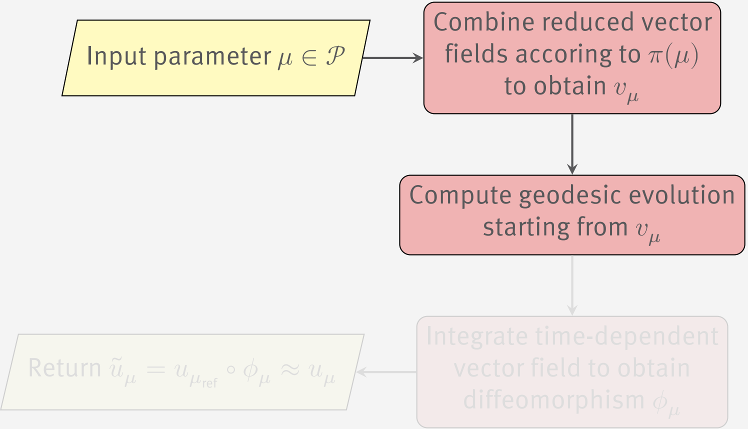

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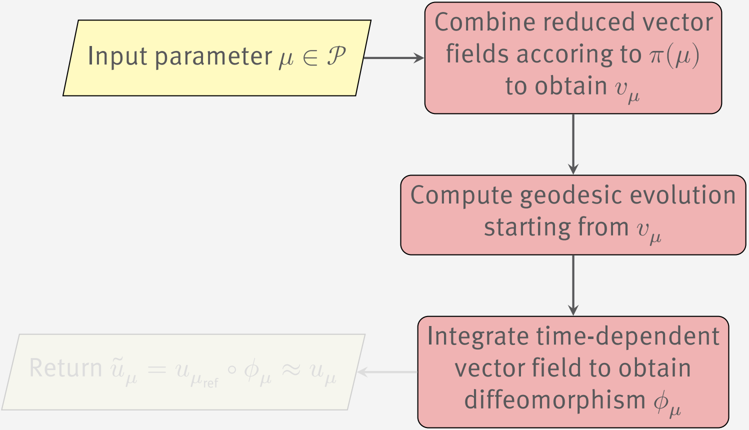

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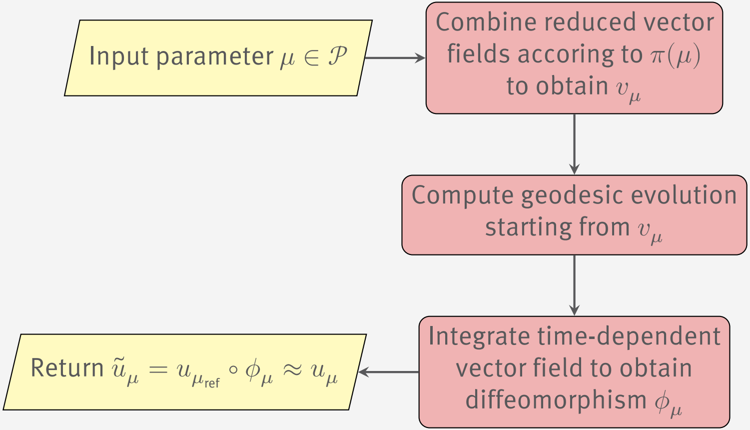

# Numerical results – Burgers' equation with two shocks

$$\begin{aligned} \partial_t u_\mu + \mu \partial_x u_\mu^2 &= 0, && (t,x) \in [0,T] \times \Omega, \\ u_\mu(0) &= u_0, && x \in \Omega, \end{aligned}$$

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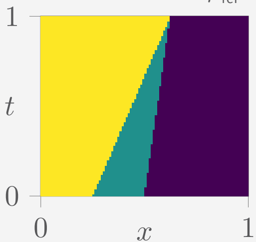
| | |
|---|---|
| Parameter domain | $\mathcal{P} = [0.25, 1.5]$ |
| Discretization | $N_x = N_t = 100$ |
| Training parameters | $n = 50$ |
| Reference parameter | $\mu_{\text{ref}} = 0.25$ |
| Reduced dimension | $N = 10$ |

Reference solution $u_{\mu_{\text{ref}}} = u_{0.25}$




Geodesic shooting implementation: `https://github.com/HenKlei/geodesic-shooting`

# Numerical results – Burgers' equation with two shocks

## Offline phase

Average relative $L^2$-error on the $n$ training snapshots: $5.1\%$.

## Online phase

Average relative $L^2$-error for parameter $\mu \in \{0.5, 0.75, 1, 1.25\}$: $5.5\%$.

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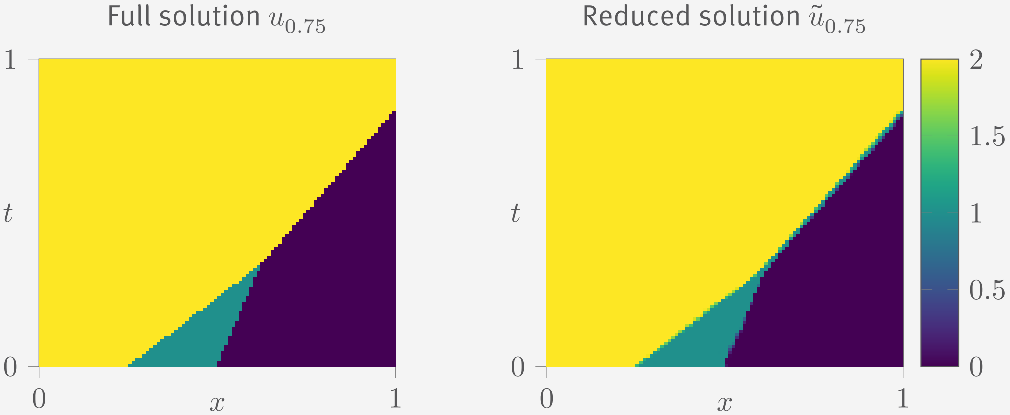

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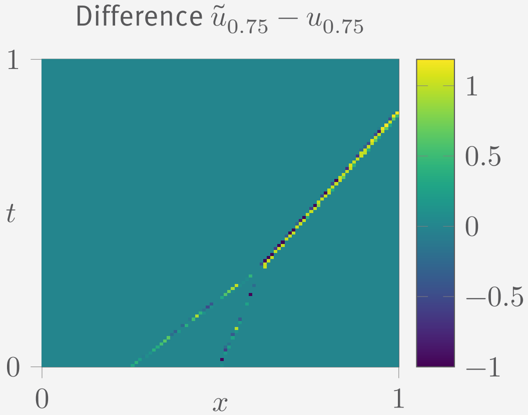


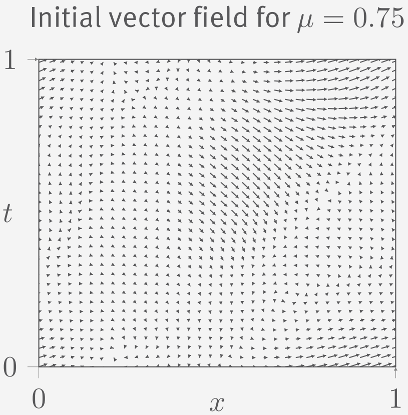

# Outlook and further research perspectives

- Theoretical investigation of reduced subspace of the Lie algebra.
- Greedy procedure instead of POD to extract vector fields.
- (Efficient) residual minimization during online phase instead of learning the coefficients.
- Compute mapping only for a small set of landmarks (e.g. empirical quadrature points) using Hamiltonian formulation of landmark matching problem.
- Localize the approach to be able to tackle the evolution of complex shock fronts in higher dimensions.

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Thank you for your attention!

# References I


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