On the periodic Wigner-Poisson-Fokker-Planck system

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AMS 2000 Subject Classification: 35Q40, 35K55, 35S10, 81S30

Key words: Wigner equation, Fokker-Planck equation, diffusive quantum systems, semigroup theory

Last update February 4, 2002

Abstract

This paper is concerned with the existence and uniqueness analysis of global classical solutions of a diffusive quantum evolution equation with non-linear coupling to the Poisson equation. The main technical difficulty in the existence proof is to show that the quantum Fokker-Planck term is a semigroup-generator in a weighted L^2–space. The potential term is then a Lipschitz-perturbation of it.

1 Introduction

The object of this paper is the analysis of the coupled Wigner-Poisson-Fokker-Planck (WPFP) system in one dimension with periodic boundary conditions. We focus on the existence and uniqueness of global-in-time solutions to this system.

Wigner functions provide a kinetic description of quantum mechanics (cf. [19]) and have recently become a valuable modeling and simulation tool in fields like semiconductor device modeling (cf. [12] and references therein), quantum Brownian motion, and quantum optics ([6, 8]). The real-valued Wigner function $w(x,v,t)$ is a probabilistic quasi-distribution function in the position-velocity (x,v) phase space for the considered quantum system at time t.

Its temporal evolution is governed by the Wigner-Fokker-Planck (WFP) equation:

$$w_t + vw_x + \Theta[V]w \;=\; \beta(vw)_v + \sigma w_{vv} + 2\gamma w_{xv} + \alpha w_{xx}, \qquad t>0, \tag{1.1}$$

on the phase space slab $x \in (0,2\pi),\ v \in \mathbb{R}$ with periodic boundary conditions in x:

$$w(0,v,t) \;=\; w(2\pi,v,t),$$

and the initial condition

$$w(x,v,t=0) \;=\; w^I(x,v).$$

With a vanishing right hand side Equation (1.1) would be the (diffusion-free) Wigner equation. It describes the reversible evolution of a quantum system under the action of a (possibly time-dependent) electrostatic potential $V = V(x,t)$. Its effect enters in the equation via the pseudo-differential operator $\Theta[V]$:

$$\begin{aligned}
(\Theta[V]w)(x,v,t) &= i[V(x+\frac{1}{2i}\nabla_v,t) - V(x-\frac{1}{2i}\nabla_v,t)]w(x,v,t) \\
&= \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}} \delta V(x,\eta,t)\mathcal{F}_v w(x,\eta,t)e^{iv\eta}\,d\eta \\
&= \frac{i}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \delta V(x,\eta,t)w(x,v',t)e^{i(v-v')\eta}\,dv'd\eta, \qquad (1.2)
\end{aligned}$$

where $\delta V(x,\eta,t) = V(x+\frac{\eta}{2},t) - V(x-\frac{\eta}{2},t)$ and $\mathcal{F}_v w$ denotes the Fourier transform of w with respect to v:

$$\mathcal{F}_v w(x,\eta,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} w(x,v',t)e^{-iv'\eta}\,dv'.$$

For simplicity of the notation we have here set the Planck constant, particle mass and charge equal to unity.

The right hand side of (1.1) is a Fokker-Planck type model for the non-reversible interaction of this quantum system with an environment, e.g. the interaction of electrons with a phonon bath (cf. [9, 18]). In (1.1) $\beta \geq 0$ is the friction parameter and the parameters α, $\gamma \geq 0$, $\sigma > 0$ constitute the phase-space diffusion matrix of the system. In the kinetic Fokker-Planck equation of classical mechanics (cf. [17, 7]) one would have $\alpha = \gamma = 0$. For the WFP equation (1.1) we have to assume

$$\begin{pmatrix} \alpha & \gamma + \frac{i}{4}\beta \\ \gamma - \frac{i}{4}\beta & \sigma \end{pmatrix} \;\geq\; 0,$$

which guarantees that the system is *quantum mechanically correct.* More precisely, it guarantees that the corresponding von Neumann equation is in Lindblad form and that the density matrix of the quantum system stays a positive operator under temporal evolution (see [4] for details).

In the sequel we shall hence assume

$$\alpha\sigma \geq \gamma^2 + \frac{\beta^2}{16}. \qquad (1.3)$$

However, the subsequent mathematical analysis will even hold for

$$\alpha\sigma \geq \gamma^2.$$

The WFP equation (1.1) is self-consistently coupled with the Poisson equation for the (real-valued) potential $V[w](x,t)$:

$$\begin{aligned}
V_{xx} &= n[w] - D, \qquad x \in (x,2\pi),\ t>0, \\
V(0,t) &= V(2\pi,t),
\end{aligned}$$

with the particle density

$$n[w](x,t) = \int_{\mathbb{R}} w(x,v,t)\,dv. \qquad (1.4)$$

$D = D(x)$ denotes the density of some fixed charges ("doping profile" in the context of semiconductor modeling), which is assumed to be given.

Mathematical properties of the Wigner-Poisson equation and dissipative Wigner systems have been intensively studied in the last decade (see [12, 3] and references therein). The (friction-free) WPFP equation in 3 dimensions was first analyzed in [4], where unique local-in-time solutions were constructed. The main analytical challenge of Wigner-Poisson systems lies in controling the particle density (1.4) in appropriate L^p spaces. Usually this is achieved by either reformulating the Wigner equation as a Schrödinger system or a von Neumann equation ([12, 3]) or by exploiting the dissipative structure of the system ([4]). The 1-dimensional Wigner-Poisson equation, however, allows for a 'direct' analysis (cf. [5], §5). Hence our interest in this analytical framework.

2 Existence and Uniqueness of global-in-time Solution

In this section we shall establish existence and uniqueness of global mild and classical solutions to the WPFP system (1.1)-(1.4). This system will be considered as an evolution problem in the weighted (real-valued) L^2-space

$$X \;=\; L^2((0,2\pi)\times \mathbb{R}; (1+v^2)dxdv),$$

endowed with the scalar product

$$< u, w >_X = \int_0^{2\pi}\int_{\mathbb{R}} uw(1+v^2)dvdx.$$

This choice of the space X allows to define the particle density $n[w]$ of a Wigner function $w \in X$: a simple estimate (using Cauchy-Schwartz) yields

$$\|n[w]\|_{L^2(0,2\pi)} \;\leq\; C\|w\|_X. \tag{2.1}$$

Here and in the sequel C denotes generic, but not necessarily equal constants.

We shall use semigroup techniques to prove existence and uniqueness of a solution to the semi-linear WPFP system (1.1)-(1.4). To this end the quadratically nonlinear potential term $\Theta[V]w$ will be considered as a bounded perturbation in the kinetic Fokker-Planck equation $w_t + vw_x = \beta(vw)_v + \sigma w_{vv} + 2\gamma w_{xv} + \alpha w_{xx}$.

We first consider the unbounded linear operator $A : D(A) \longrightarrow X$,

$$Au \;=\; -v\partial_x u + \beta\partial_v(vu) + \sigma\partial_v^2 u + 2\gamma\partial_v\partial_x u + \alpha\partial_x^2 u, \tag{2.2}$$

defined on

$$\begin{aligned} D(A) \;=\; \{u \in X | vu_x, u_{vv}, vu_v, u_{xx}, u_{xv} \in X; u(0,v) = u(2\pi,v),\\ u_x(0,v) = u_x(2\pi,v)\, \forall v \in \mathbb{R}\}. \end{aligned}$$

Clearly, the restriction (to $(0,2\pi)\times\mathbb{R}$) of $C^\infty(\mathbb{R}^2)$-functions that are 2π-periodic in x and have a compact support in v are included in $D(A)$. Hence, $D(A)$ is dense in X. A simple calculation shows that for $u \in D(A), u_v$ is also in X.

A straightforward calculation using the periodicity in x and integrations by part yields

$$< Au, w >_X = < u, A_1^* w >_X + < u, A_2^* w >_X,\ \forall u, w \in D(A),$$

with

$$\begin{aligned} A_1^* w &= v\partial_x w - \beta v \partial_v w + \sigma \partial_v^2 w + 2\gamma \partial_v \partial_x w + \alpha \partial_x^2 w, \\ A_2^* w &= \frac{1}{1+v^2}[2\sigma(w + 2v\partial_v w) - 2\beta v^2 w + 4\gamma v w_x]. \end{aligned}$$

Hence, $A^* |_{D(A)}$ - the restriction of the adjoint of the operator A to $D(A)$ - is given by $A^* w = A_1^* w + A_2^* w$, $w \in D(A)$. A^* is densly defined on $D(A^*) \supseteq D(A)$, and hence A is a closable operator (cf. Theorem VIII.1.b of [15]). Its closure $\overline{A}$ satisfies $(\overline{A})^* = A^*$ (cf. [15], Theorem VIII.1.c).

Next we study the dissipation property of the operator A, which is defined on the Hilbert space X (over $\mathbb{R}$) by:

$$< Au, u >_X \; \leq \; 0, \quad \forall u \in D(A).$$

Lemma 2.1 *Let the coefficients of the operator A satisfy $\alpha\sigma \geq \gamma^2$. Then $A - (\sigma + \frac{\beta}{2})I$ and its closure are dissipative.*

PROOF.- Using integrations by part we have for $u \in D(A)$:

$$\begin{aligned} < Au, u >_X &= -\iint v u_x u + \beta \iint (vu)_v u + \sigma \iint u_{vv} u + 2\gamma \iint u_{xv} u \\ &\quad + \alpha \iint u_{xx} u - \iint v^3 u_x u + \beta \iint v^2 (vu)_v u + \sigma \iint v^2 u_{vv} u \\ &\quad + 2\gamma \iint v^2 u_{xv} u + \alpha \iint v^2 u_{xx} u \\ &= -\beta \iint u v u_v - \sigma \iint u_v u_v + 2\gamma \iint u_{xv} u - \alpha \iint u_x u_x - \beta \iint (v^2 u)_v v u \\ &\quad - \sigma \iint (v^2 u)_v u_v + 2\gamma \iint (vu)_{xv} v u - 2\gamma \iint u_x v u - \alpha \iint v^2 u_x u_x \end{aligned}$$

where $\iint f$ denotes the integral $\int_0^{2\pi} \int_{\mathbb{R}} f(x, v)\, dv dx$.
For the two integrals of the right side that involve mixed x-v-derivatives we shall now use the interpolation inequality

$$\iint u_{xv} u \;\leq\; \frac{\epsilon}{2} \|u_x\|_2^2 + \frac{1}{2\epsilon} \|u_v\|_2^2, \quad \epsilon > 0, \tag{2.3}$$

which is immediately obtained by an integration by parts (in v) and Young's inequality. With $\epsilon = \frac{\gamma}{\sigma}$ we then obtain

$$\begin{aligned} < Au, u >_X &\leq \frac{\beta}{2} \|u\|_2^2 - \sigma \|u_v\|_2^2 + \epsilon\gamma \|u_x\|_2^2 + \frac{1}{\epsilon}\gamma \|u_v\|_2^2 \\ &\quad - \alpha \|u_x\|_2^2 - 2\beta \|vu\|_2^2 - \beta \iint v^3 u_v u - 2\sigma \iint v u u_v \\ &\quad - \sigma \|v u_v\|_2^2 + \epsilon\gamma \|v u_x\|_2^2 + \frac{1}{\epsilon}\gamma \|(vu)_v\|_2^2 - \alpha \|v u_x\|_2^2 \\ &= \frac{\beta}{2} \|u\|_2^2 + \frac{\gamma^2}{\sigma} \|u_x\|_2^2 - \alpha \|u_x\|_2^2 - 2\beta \|vu\|_2^2 + \frac{3}{2}\beta \|vu\|_2^2 \\ &\quad + \sigma \|u\|_2^2 + \frac{\gamma^2}{\sigma} \|v u_x\|_2^2 - \alpha \|v u_x\|_2^2 \\ &\leq (\sigma + \frac{\beta}{2}) \|u\|_2^2, \end{aligned}$$

Thus

$$< [A - (\sigma + \frac{\beta}{2})I]u, u >_X \;\leq\; -\sigma \|vu\|_2^2 - \frac{\beta}{2} \|vu\|_2^2 \;\leq\; 0 \tag{2.4}$$

and the operator $A-(\sigma+\frac{\beta}{2})I$ is dissipative. By Theorem 1.4.5b of [14] its closure, $\overline{A-(\sigma+\frac{\beta}{2})I}=\overline{A}-(\sigma+\frac{\beta}{2})I$ is also dissipative. $\Box$

It is easy to see that the operator $A-\frac{\beta}{2}I$ defined on $\widetilde{D(A)}=\{u\in L^2((0,2\pi)\times \mathbb{R})|vu_x,u_{vv},vu_v,u_{xx}\in L^2((0,2\pi)\times\mathbb{R});u(0,v)=u(2\pi,v),\ u_x(0,v)=u_x(2\pi,v),\ \forall v\in \mathbb{R}\}$ is dissipative in $L^2((0,2\pi)\times\mathbb{R})$ and the L^2-adjoint of A is $A^*=A_1^*$ on $\widetilde{D(A)}$.

Let us now study the dissipativity of the operator A^* restricted to $D(A)$. Analogously to Lemma 2.1 we have:

$$<A^*u,u>_X\quad\leq\quad(\sigma+\frac{\beta}{2})\|u\|_2^2,\quad\forall u\in D(A).$$

Hence the restriction of the operator $A^*-(\sigma+\frac{\beta}{2})I=[A-(\sigma+\frac{\beta}{2})I]^*$ to $D(A)$ is dissipative.

Next we consider the dissipativity of this operator on its proper domain $D(A^*)$, which, however, is not known explicitly. To this end we shall use the following technical lemma whose proof is defered to the appendix. Here we shall denote by $\tilde{u}$ the (in x) 2π-periodic extension of a function $u\in X$ to $\mathbb{R}^2$.

Lemma 2.2 *Let $P:=p(v,\partial_x,\partial_v)$ be a linear operator in X, where p is a quadratic polynomial and*

$$D(P):=\{u\in X\,/\,\tilde{u}\in C^\infty(\mathbb{R}^2)\ \textit{with compact support in } v\}\subset X.$$

Then $\overline{P}$ is the maximum extension of P in the sense that

$$D(\overline{P}):=\{u\in X\,/\ \textit{the distribution } Pu\in X\}.$$

We now apply Lemma 2.2 to $P=A^*-(\sigma+\frac{\beta}{2})I$, which is dissipative on $D(P)\subset D(A)$. Since A^* is closed, we have $D(A^*)=D(\overline{P})=\{u\in X\,/\,A^*u\in X\}$ and $A^*-(\sigma+\frac{\beta}{2})I$ is dissipative on all of $D(A^*)$.

Applying Corollary 1.4.4 of [14] to $\overline{A}-(\sigma+\frac{\beta}{2})I$ (with $(\overline{A})^*=A^*$), then implies that $\overline{A}-(\sigma+\frac{\beta}{2})I$ generates a C_0 semigroup of contractions on X, and the C_0 semigroup generated by $\overline{A}$ satisfies

$$\|e^{t\overline{A}}u\|_X\leq e^{(\sigma+\frac{\beta}{2})t}\|u\|_X,\ u\in X,t\geq 0.$$

By the same arguments $\overline{A}-\frac{\beta}{2}I$ generates a C_0 semigroup of contractions on the space $L^2((0,2\pi)\times\mathbb{R})$.

Next we shall analyze the properties of the quadratically nonlinear term $\Theta[V]w$, which will later be considered as a perturbation of the generator $\overline{A}$.

For $V\in L^\infty(\mathbb{R})$ the pseudo-differential operator $\Theta[V]$ from (1.2) is defined by

$$(\mathcal{F}_v\Theta[V]u)(x,\eta)=i\delta V(x,\eta)\mathcal{F}_vu(x,\eta),\ u\in L^2((0,2\pi)\times\mathbb{R}_v).$$

Since $\delta V(x,\eta)\in\mathbb{R}$, the operator $\Theta[V]$ is skew-symmetric on $L^2((0,2\pi)\times\mathbb{R}_v)$ and it satisfies
(cf. [11], [5]):

$$\|\Theta[V]\|_{\mathcal{B}(L^2((0,2\pi)\times\mathbb{R}_v))}\leq 2\|V\|_\infty.$$

For $V\in L^\infty(\mathbb{R})$ we define the pseudo-differential operator $\Omega[V]$ on $L^2((0,2\pi)\times\mathbb{R}_v)$ by

$$\begin{aligned}(\Omega[V]u)(x,v)&=\frac{1}{2}[V(x+\frac{1}{2i}\nabla_v)+V(x-\frac{1}{2i}\nabla_v)]u(x,v)\\&=\frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}}[V(x+\frac{\eta}{2})+V(x-\frac{\eta}{2})]\mathcal{F}_vu(x,\eta)e^{iv\eta}\,d\eta.\end{aligned}\qquad(2.5)$$

As for the operator $\Theta[V]$ we obtain

$$\|\Omega[V]\|_{\mathcal{B}(L^2((0,2\pi)\times \mathbb{R}_v))} \leq \|V\|_\infty. \tag{2.6}$$

Proposition 2.3 *Let $V \in W^{1,\infty}(\mathbb{R})$. Then,*

$$\Theta[V](vw) = v\Theta[V]w + \Omega[V_x]w \tag{2.7}$$

holds for $w \in X$.

PROOF.- By partial integration we obtain

$$\begin{aligned}
\Theta[V](vw) &= \frac{i}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}(V(x+\frac{\eta}{2}) - V(x-\frac{\eta}{2}))v'w(x,v')e^{i(v-v')\eta}\,dv'd\eta \\
&= \frac{i}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[(V(x+\frac{\eta}{2}) - V(x-\frac{\eta}{2}))w(x,v')e^{iv\eta}\right]\left[v'e^{-iv'\eta}\right]\,d\eta dv' \\
&= \frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}(V_x(x+\frac{\eta}{2}) + V_x(x-\frac{\eta}{2}))w(x,v')e^{i(v-v')\eta}\,d\eta dv' \\
&\quad + \frac{i}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}v(V(x+\frac{\eta}{2}) - V(x-\frac{\eta}{2}))w(x,v')e^{i(v-v')\eta}\,d\eta dv' \\
&= \Omega[V_x]w + v\Theta[V]w.
\end{aligned}$$

□

Now, let us consider the nonlinear operator B defined on X by

$$u \longmapsto Bu := -\Theta[V[u]]u,$$

where $V[u]$ is the 2π-periodically extended solution of the Poisson equation

$$\begin{aligned}
V_{xx} &= n[u] - D, \quad x \in (0,2\pi), \\
V(0) &= V(2\pi),
\end{aligned} \tag{2.8}$$

with $n[u](x) = \int_{\mathbb{R}} u(x,v)dv$.

Lemma 2.4 *Let $D \in L^1(0,2\pi)$. Then*
(a) B maps X into itself.
(b) Moreover, the operator B is of class C^∞ in X, and satisfies

$$\|Bu_1 - Bu_2\|_X \leq C(\|u_1\|_X + \|u_2\|_X + \|D\|_{L^1(0,2\pi)})\|u_1 - u_2\|_X,$$

for $u_1, u_2 \in X$.

For the simple proof we refer the reader to [5].

Remark 2.5 *In the proof of Lemma 2.4 it is essential that $\|u\|_X$ controls $n[u]$ in $L^1(0,2\pi)$ (see (2.1)). Hence the solution of the Poisson equation (2.8) satisfies $V[u] \in W^{1,\infty}(0,2\pi)$, and $\|\Theta[V[u]]\|_{\mathcal{B}(X)} \leq C\|V[u]\|_{W^{1,\infty}(\mathbb{R})}$.*

We rewrite the WPFP system as

$$\begin{aligned}
w_t &= \overline{A}w + Bw,\ t > 0, \\
w(t=0) &= w^I \in X.
\end{aligned} \tag{2.9}$$

The main result of this paper is:

Theorem 2.6 *Let $D \in L^1(0,2\pi)$.*
a) For every $w^I \in X$, the WPFP problem (2.9) has a unique mild solution $w \in C([0,\infty),X)$.
b) If $w^I \in D(\overline{A})$, w is a classical solution, i.e. $w \in C^1([0,\infty),X)$, and $w(t) \in D(\overline{A})$ for $t \geq 0$.

PROOF.- We consider B as a bounded perturbation of the generator $\overline{A}$. Since B is locally Lipschitz continuous, Theorem 6.1.4 of [14] shows that (2.9) has a unique mild solution for every $w^I \in X$ on some time interval $[0,t_{max})$. Moreover, if $t_{max} = t_{max}(w^I) < \infty$ then $\lim_{t\nearrow t_{max}} \|w\|_X = \infty$. Since B is of class C^∞ in X, Theorem 6.1.5 in [14] proves that w is a classical solution on $[0,t_{max})$ for $w^I \in D(\overline{A})$.

To prove $t_{max} = \infty$ we shall now derive an a-priori estimate for $\|w(t)\|_X$.

Step 1. Here we shall derive this a-priori estimate under the assumption $w^I \in D(\overline{A})$. To this end we consider the evolution equation for $\|w\|_X^2$. By computing its time derivative and taking into account (2.9), we deduce

$$\frac{1}{2}\frac{d}{dt}\|w\|_X^2 = <\overline{A}w,w>_X + <Bw,w>_X .$$

Using the dissipativity of $\overline{A} - (\sigma + \frac{\beta}{2})I$ (cf. (2.4)) we conclude that

$$\frac{1}{2}\frac{d}{dt}\|w\|_X^2 \leq \left(\sigma + \frac{\beta}{2}\right)\|w\|_X^2 + <Bw,w>_X .$$

The skew-symmetry of the operator $\Theta[V]$ implies finally that

$$\frac{1}{2}\frac{d}{dt}\|w\|_X^2 \leq \left(\sigma + \frac{\beta}{2}\right)\|w\|_X^2 + \iint vw\Omega[V_x(t)]w. \tag{2.10}$$

On the other hand, since $\overline{A} - \frac{\beta}{2}I$ is dissipative on the space $L^2((0,2\pi)\times\mathbb{R})$, the estimates

$$\frac{d}{dt}\|w\|_2^2 \leq \beta\|w\|_2^2 \quad \text{and} \quad \|w\|_2^2 \leq \|w^I\|_2^2 e^{\beta t} \tag{2.11}$$

follow. From the proof of Lemma 2.4 in [5] we have for the solution of (2.8):

$$\|V[w]\|_{W^{1,\infty}(0,2\pi)} \leq C(\|w\|_X + \|D\|_{L^1(0,2\pi)}).$$

Using (2.6), (2.10) and (2.11) we hence obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|_X^2 - \left(\sigma + \frac{\beta}{2}\right)\|w\|_X^2 &\leq \iint vw\Omega[V_x(t)]w \\
&\leq \|V_x(t)\|_\infty \|vw\|_2 \|w\|_2 \\
&\leq C(\|w\|_X + \|D\|_1)\|vw\|_2 \|w^I\|_2 e^{\frac{\beta}{2}t} \\
&\leq C\|w^I\|_2 e^{\frac{\beta}{2}t}(\|w\|_X^2 + \|w\|_X\|D\|_1) \\
&\leq C\|w^I\|_2 e^{\frac{\beta}{2}t}(\|w\|_X^2 + \|D\|_1^2).
\end{aligned}$$

Thus

$$\frac{d}{dt}\|w\|_X^2 \leq a(t)\|w\|_X^2 + b(t),$$

where

$$\begin{aligned}
a(t) &= C\|w^I\|_2 e^{\frac{\beta}{2}t} + \beta + 2\sigma, \\
b(t) &= C\|w^I\|_2 e^{\frac{\beta}{2}t}\|D\|_1^2.
\end{aligned}$$

Finally, applying Gronwall's inequality yields:

$$\|w(t)\|_X^2 \;\le\; \|w^I\|_X^2 e^{\int_0^t a(s)ds} + \int_0^t b(s) e^{\int_s^t a(\tau)d\tau} ds, \quad t \ge 0. \tag{2.12}$$

Hence $t_{max} = \infty$ holds.

Step 2. Since (2.12) only involves $\|w^I\|_X$ this result carries over to w^I only in X by the following density argument.
For $w^I \in X$ let (w_n^I) be a sequence in $D(\overline{A})$ such that $w_n^I \longrightarrow w^I$ in X. Using (2.12) we have for every w_n^I an a-priori estimate for the corresponding classical solution:

$$\|w_n(t)\|_X \;\le\; h(t), \quad \forall t \ge 0,\, n \in \mathbb{N}$$

with $h \in C[0,\infty)$ independent of n.
Let $w \in C([0, t_{max}(w^I)), X)$ be the unique mild solution for w^I, which exists according to the first part of this theorem.
Next we assume $t_{max}(w^I) < \infty$. Thus $\lim_{t \nearrow t_{max}(w^I)} \|w(t)\|_X = \infty$. For the continuous, monotonously increasing function $g(t) := \max\{\|w(\tau)\|_X,\ 0 \le \tau \le\ t\}$ we also have $\lim_{t \nearrow t_{max}(w^I)} g(t) = \infty$.
Choose $N \in \mathbb{N}$ with $N \ge 2\max\left\{h(t),\ t \in [0, t_{max}(w^I))\right\}$. Then there exists a $t_N < t_{max}(w^I)$ such that

$$\begin{aligned} g(t_N) &= N, \\ g(t) &\le N, \quad t \le t_N, \\ g(t) &\ge N, \quad t_N \le t < t_{max}(w^I). \end{aligned} \tag{2.13}$$

We denote by L_N the Lipschitz constant of the operator B on

$$B_N := \{u \in X,\ \|u\|_X < N\}.$$

Let $\hat{B}$ be a (globally) Lipschitz extension of B outside of B_N. Thus, applying Theorem 6.1.2 in [14] on $[0, t_N]$ we obtain a Lipschitz dependence of the solutions on their initial values,

$$\|w - w_n\|_{C([0,t_N],X)} \;\le\; C(L_N)\|w^I - w_n^I\|_X.$$

Thus, $w_n \longrightarrow w$ in $C([0,t_N], X)$, and $\|w(t)\|_X \le h(t) \le \frac{N}{2}$, for $0 \le t \le t_N$ follows. This contradicts the assumption (2.13). □

3 Appendix: Proof of Lemma 2.2

To prove the assertion we shall construct for each $f \in D(\overline{P}) \subset L^2((0,2\pi) \times \mathbb{R})$ a sequence $\{f_n\} \subset D(P)$ such that $f_n \to f$ in the graph norm $\|f\|_P = \|f\|_{L^2} + \|vf\|_{L^2} + \|Pf\|_{L^2} + \|vPf\|_{L^2}$.

To shorten the proof we shall consider here only the case

$$P = \mu + \nu v \partial_x + \beta v \partial_v + \sigma \partial_v^2 + 2\gamma \partial_v \partial_x + \alpha \partial_x^2$$

(cf. the definition of the operator A in (2.2)), but exactly the same strategy extends to the general case.

First we define the molifying delta sequence

$$\varphi_n(x,v) := n^2 \varphi(nx, nv); \qquad n \in \mathbb{N};\ x, v \in \mathbb{R},$$

with the properties:

$$\begin{aligned}
&\varphi \in C_0^\infty(\mathbb{R}^2),\\
&\varphi(x,v) \geq 0,\\
&\int\int \varphi(x,v)dxdv = 1,\\
&\text{supp}\varphi \subset \{|x|^2+|v|^2 \leq 1\}.
\end{aligned}$$

The velocity-cutoff function

$$\psi_n(v) := \psi(\frac{v}{n}); \qquad n \in \mathbb{N};\ v \in \mathbb{R}$$

is assumed to have the properties

$$\begin{aligned}
&\psi \in C_0^\infty(\mathbb{R}),\\
&0 \leq \psi(v) \leq 1,\\
&|\psi^{(j)}(v)| \leq C_j \qquad \forall v \in \mathbb{R};\ j=1,2,\\
&\text{supp}\psi \subset [-1,1],\\
&\psi|_{[-\frac{1}{2},\frac{1}{2}]} \equiv 1.
\end{aligned}$$

We now define the approximating sequence

$$\tilde{f}_n(x,v) := (\tilde{f} * \varphi_n)(x,v)\cdot\psi_n(v), \quad n \in \mathbb{N},$$

where '$*$' denotes the convolution in x and v. Remember that $\tilde{f}$ denotes the (in x) 2π-periodic extension of the function $f \in X$ to $\mathbb{R}^2$. By construction we have $\tilde{f}_n \in C^\infty(\mathbb{R}^2)$ and $\tilde{f}_n$ is 2π-periodic in x with compact support in v. Now, let R denote the restriction operator of (in x) 2π-periodic functions to $(0,2\pi)\times\mathbb{R}$. Then, $f_n := R\tilde{f}_n \in D(P)$. According to the 4 terms of the graph norm we split the proof into 4 steps:

Step 1: Since $\varphi_n \to \delta$ in $\mathcal{D}'(\mathbb{R}^2)$ and $\psi_n(v) \to 1$ pointwise, we have $\tilde{f}_n \to \tilde{f}$ in $L^2_{loc}(\mathbb{R}_x)\times L^2(\mathbb{R}_v)$ and

$$f_n \to f \quad \text{in} \quad L^2((0,2\pi)\times\mathbb{R}).$$

Step 2: For the second term of the graph norm we write

$$v\tilde{f}_n = (v\tilde{f} * \varphi_n)\psi_n + (\tilde{f} * v\varphi_n)\psi_n.$$

The restriction of the first summand converges to vf in $L^2((0,2\pi)\times\mathbb{R})$ and the second term converges to 0 since $v\varphi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Hence we have

$$f_n \to f \qquad \text{in } X.$$

Step 3: To prove that $Pf_n \to Pf$ in $L^2((0,2\pi)\times\mathbb{R})$ we write:

$$\begin{aligned}
P\tilde{f}_n &= \mu(\tilde{f} * \varphi_n)\psi_n + \nu(v\tilde{f}_x * \varphi_n)\psi_n + \beta(v\tilde{f}_v * \varphi_n)\psi_n\\
&\quad +\sigma(\tilde{f}_{vv} * \varphi_n)\psi_n + 2\gamma(\tilde{f}_{xv} * \varphi_n)\psi_n + \alpha(\tilde{f}_{xx} * \varphi_n)\psi_n\\
&\quad +r_n^1(x,v)\\
&= (P\tilde{f} * \varphi_n)\psi_n + r_n^1(x,v).
\end{aligned}$$

As we shall show, the restriction of all six terms of the remainder

$$\begin{aligned}
r_n^1 &= \nu(\tilde{f} * v\partial_x\varphi_n)\psi_n + \beta(\tilde{f} * \varphi_n)(v\partial_v\psi_n) \\
&\quad +\beta(\tilde{f} * \partial_v(v\varphi_n))\psi_n + 2\sigma(\tilde{f} * (\frac{1}{n}\partial_v\varphi_n))(n\partial_v\psi_n) \\
&\quad +\sigma(\tilde{f} * \varphi_n)\partial_v^2\psi_n + 2\gamma(\tilde{f} * (\frac{1}{n}\partial_x\varphi_n))(n\partial_v\psi_n)
\end{aligned}$$

converge to 0 in $L^2((0,2\pi)\times \mathbb{R})$:

In the first term $v\partial_x\varphi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Hence we have

$$R\left(\tilde{f} * v\partial_x\varphi_n\right) \to 0 \qquad \text{in} \quad L^2((0,2\pi)\times \mathbb{R}),$$

and the same argument holds for the third term.

For the second term we have

$$v\partial_v\psi_n = \frac{v}{n}\psi'(\frac{v}{n}),$$

which is in $L^\infty(\mathbb{R})$, uniformly for $n\in\mathbb{N}$ and with support in $[-n,-\frac{n}{2}]\cup[\frac{n}{2},n]$. Hence, the second term converges to 0 in $L^2((0,2\pi)\times \mathbb{R})$.

In the fourth term $\frac{1}{n}\partial_v\varphi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^2)$, and hence

$$R\left(\tilde{f} * (\frac{1}{n}\partial_v\varphi_n)\right) \to 0 \quad \text{in} \quad L^2((0,2\pi)\times \mathbb{R}).$$

Furthermore, $n\partial_v\psi_n = \psi'(\frac{v}{n})$ with $|\psi'|\le C_1$. By the same argument also the sixth term converges to 0 in $L^2((0,2\pi)\times \mathbb{R})$.

Finally, the fifth term converges to 0 since $\partial_v^2\psi_n = \frac{1}{n^2}\psi''(\frac{v}{n})$ with $|\psi''|\le C_2$.

Step 4: To prove that $vPf_n \to vPf$ in $L^2((0,2\pi)\times \mathbb{R})$ we write:

$$\begin{aligned}
vP\tilde{f}_n &= \mu(v\tilde{f} * \varphi_n)\psi_n + \nu(v^2\tilde{f}_x * \varphi_n)\psi_n + \beta(v^2\tilde{f}_v * \varphi_n)\psi_n \\
&\quad +\sigma(v\tilde{f}_{vv} * \varphi_n)\psi_n + 2\gamma(v\tilde{f}_{xv} * \varphi_n)\psi_n + \alpha(v\tilde{f}_{xx} * \varphi_n)\psi_n \\
&\quad +r_n^2(x,v) \\
&= ((vP\tilde{f}) * \varphi_n)\psi_n + r_n^2(x,v),
\end{aligned}$$

with the remainder

$$\begin{aligned}
r_n^2 &= \mu(\tilde{f} * v\varphi_n)\psi_n + 2\nu(v\tilde{f} * v\partial_x\varphi_n)\psi_n + \nu(\tilde{f} * v^2\partial_x\varphi_n)\psi_n \\
&\quad +\beta(v\tilde{f} * \varphi_n + \tilde{f} * v\varphi_n)v\partial_v\psi_n + 2\beta(v\tilde{f} * \partial_v(v\varphi_n))\psi_n \\
&\quad +\beta(\tilde{f} * v^2\partial_v\varphi_n)\psi_n + \sigma(\tilde{f} * \partial_{vv}(v\varphi_n))\psi_n \\
&\quad +2\sigma\left(v\tilde{f} * \frac{\partial_v\varphi_n}{n} + \tilde{f} * \frac{v\partial_v\varphi_n}{n}\right)\psi'\left(\frac{v}{n}\right) \\
&\quad +\sigma(\tilde{f} * \varphi_n)v\partial_v^2\psi_n + 2\gamma(\tilde{f} * \partial_{xv}(v\varphi_n))\psi_n \\
&\quad +2\gamma\left(v\tilde{f} * \frac{\partial_x\varphi_n}{n} + \tilde{f} * \frac{v\partial_x\varphi_n}{n}\right)\psi'\left(\frac{v}{n}\right) + \alpha(\tilde{f} * v\partial_{xx}\varphi_n)\psi_n.
\end{aligned}$$

For proving that the restriction of all terms of r_n^2 converge to 0 in $L^2((0,2\pi)\times \mathbb{R})$ we recall that both $f, vf \in L^2((0,2\pi)\times \mathbb{R})$. Since the strategy of the proof is the same as in Step 3 we shall only give the key points:

The distributions $v\varphi_n$, $v\partial_x\varphi_n$, $v^2\partial_x\varphi_n$, $\partial_v(v\varphi_n)$, $v^2\partial_v\varphi_n$, $\partial_{vv}(v\varphi_n)$, $\frac{\partial_v\varphi_n}{n}$, $\frac{v\partial_v\varphi_n}{n}$, $\partial_{xv}(v\varphi_n)$, $\frac{\partial_x\varphi_n}{n}$, $\frac{v\partial_x\varphi_n}{n}$, and $v\partial_{xx}\varphi_n$ all converge to 0 in $\mathcal{D}'(\mathbb{R}^2)$. Further, $v\partial_v^2\psi_n \to 0$ in $L^\infty(\mathbb{R})$ and the term $v\partial_v\psi_n$ was already discussed in Step 3. $\square$

Acknowledgements.- Partially supported by the grants ERBFMRXCT970157 (TMR-Network) from the EU, the bilateral DAAD-Acciones Integradas Program, and DGES (Spain) PB98-1281. The third author acknowledges support from the DFG under Grant-No. AR 277/3-1.

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