The Wigner-Poisson-Fokker-Planck system: global-in-time solution and dispersive effects

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Abstract

This paper is concerned with the Wigner-Poisson-Fokker-Planck system, a kinetic evolution equation for an open quantum system with a non-linear Hartree potential. Existence, uniqueness and regularity of global solutions to the Cauchy problem in 3 dimensions are established. The analysis is carried out in a weighted L^2–space, such that the linear quantum Fokker-Planck operator generates a dissipative semigroup. The non-linear potential can be controled by using the parabolic regularization of the system.

The main technical difficulty for establishing global-in-time solutions is to derive a-priori estimates on the electric field: Inspired by a strategy for the classical Vlasov-Fokker-Planck equation, we exploit dispersive effects of the free transport operator. As a "by-product" we also derive a new a-priori estimate on the field in the Wigner-Poisson equation.

1 Introduction

The goal of this paper is to prove the existence and uniqueness of global-in-time solutions to the coupled Wigner-Poisson-Fokker-Planck (WPFP) system in three dimensions. This kinetic equation is the quantum mechanical analogue of the classical Vlasov-Poisson-Fokker-Planck (VPFP) system, which models the diffusive transport of charged particles (in plasmas e.g.).
Wigner functions provide a kinetic description of quantum mechanics (cf. [Wi]) and have recently become a valuable modeling and simulation tool in fields like semiconductor device modeling (cf. [MRS] and references therein), quantum Brownian motion, and quantum optics ([CL, Di]). The real-valued Wigner function $w(x, v, t)$ is a probabilistic quasi-distribution function in the position-velocity (x, v) phase space for the considered quantum system at time t.
Its temporal evolution is governed by the Wigner-Fokker-Planck (WFP) equation:

$$w_t + v \cdot \nabla_x w - \Theta[V]w \;=\; \beta \mathrm{div}_v(vw) + \sigma \Delta_v w + 2\gamma \mathrm{div}_v(\nabla_x w) + \alpha \Delta_x w, \quad t > 0, \tag{1.1}$$

on the phase space $x \in \mathbb{R}^3,\ v \in \mathbb{R}^3$, with the initial condition

$$w(x, v, t = 0) \;=\; w_0(x, v).$$

With a vanishing right hand side, equation (1.1) would be the (diffusion-free) Wigner equation. It describes the reversible evolution of a quantum system under the action of a (possibly time-dependent) electrostatic potential $V = V(x, t)$. The potential effect enters in the equation via the pseudo-differential operator $\Theta[V]$:

$$\begin{aligned}(\Theta[V]w)(x, v, t) \;&=\; i[V(t, x + \frac{1}{2i}\nabla_v) - V(t, x - \frac{1}{2i}\nabla_v)]w(x, v, t)\\ &=\; \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \delta V(x, \eta, t) \mathcal{F}_{v\to\eta} w(x, \eta, t) e^{iv\cdot\eta}\, d\eta\\ &=\; \frac{i}{(2\pi)^3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \delta V(x, \eta, t) w(x, v', t) e^{i(v-v')\cdot\eta}\, dv'\, d\eta,\end{aligned} \tag{1.2}$$

where $\delta V(x, \eta, t) = V(x + \frac{\eta}{2}, t) - V(x - \frac{\eta}{2}, t)$ and $\mathcal{F}_{v\to\eta} w$ denotes the Fourier transform of w with respect to v:

$$\mathcal{F}_{v\to\eta} w(t, x, \eta) \;=\; \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} w(t, x, v') e^{-iv'\cdot\eta}\, dv'.$$

For simplicity of the notation we have set here the Planck constant, particle mass and charge equal to unity.

The right hand side of (1.1) is a Fokker-Planck type model for the non-reversible interaction of this quantum system with an environment, e.g. the interaction of electrons with a phonon bath (cf. [CL, CEFM] for derivations from reversible quantum systems, and [GGKS, Sto] for applications in quantum transport). In (1.1), $\beta \geq 0$ is the friction parameter and the parameters $\alpha,\ \gamma \geq 0$, $\sigma > 0$ constitute the phase-space diffusion matrix of the system. In the kinetic Fokker-Planck equation of classical mechanics (cf. [Ri, CSV]) one would have $\alpha = \gamma = 0$. For the WFP equation (1.1) we have to assume

$$\begin{pmatrix} \alpha & \gamma + \frac{i}{4}\beta \\ \gamma - \frac{i}{4}\beta & \sigma \end{pmatrix} \;\geq\; 0,$$

which guarantees that the system is *quantum mechanically correct.* More precisely, it guarantees that the corresponding von Neumann equation is in Lindblad form and that the density matrix of the quantum system stays a positive operator under temporal evolution (see [ALMS] for details). In the sequel we shall therefore assume

$$\alpha\sigma \;\geq\; \gamma^2 + \frac{\beta^2}{16} \quad \text{and} \quad \alpha\sigma \;>\; \gamma^2. \tag{1.3}$$

Hence, the principle part of the Fokker-Planck term is uniformly elliptic. This makes the present work complementary to [ALMS], where the friction-free, hypoelliptic case (with $\alpha = \beta = \gamma = 0$) was analyzed.

The WFP equation (1.1) is self-consistently coupled with the Poisson equation for the (real-valued) potential $V = V[w](x,t)$:

$$-\Delta V \;=\; n[w], \quad x \in \mathbb{R}^3, \quad t > 0, \tag{1.4}$$

with the *particle density*

$$n[w](x,t) \;:=\; \int_{\mathbb{R}^3} w(x,v,t)\,dv. \tag{1.5}$$

This potential models the repulsive Coulomb interaction within the considered particle system in a mean-field description.

The main analytical challenge for tackling Wigner-Poisson systems is the proper definition of $n[w]$ in appropriate L^p spaces. Due to the definition of the operator Θ in Fourier space, $w \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ is the natural set-up. Without further assumptions, of course, this does not justify to define $n[w]$. We shall now summarize the existing literature of this field and the typical strategies to overcome the above problem:
a) The standard approach for the *Wigner-Poisson* equation is to reformulate it as a Schrödinger-Poisson system, where the particle density then appears in L^1 (cf. [BM, Ca1] for the 3D–whole space case).
b) In *one spatial dimension with periodic boundary conditions* in x the Wigner-Poisson system (and WPFP) can be dealt with directly on the kinetic level. For w in a weighted L^2-space, the nonlinear term $\Theta[V]w$ is then bounded and locally Lipschitz [AR, ACD]. The same strategy was also used in [Ma] for the Wigner-Poisson system on a bounded (spatial) domain in three dimensions (local-in-time solution).
c) By adapting L^1-techniques from the classical Vlasov-Fokker-Planck equation, the *3D Wigner-Poisson-Fokker-Planck* system was analyzed in [ALMS] (local-in-time solution for the friction-free problem) and [CLN] (global-in-time solution). The latter paper, however, is not a purely kinetic analysis as it requires to assume the positivity of the underlying density matrix. In both cases the dissipative structure of the system allows to control $n[w]$.
d) In [Ar1, AS] the 3D Wigner-Poisson and WPFP systems were reformulated as von-Neumann equations for the quantum mechanical *density matrix.* This implies $n \in L^1(\mathbb{R}^3)$. While this approach is the most natural, both physically and in its mathematical structure, it is restricted to whole space cases. Extensions to initial-boundary value problems (as needed for practical applications and numerical analysis) seem unfeasible.
e) For the classical *Vlasov-Poisson-Fokker-Planck* equation there exists a vast body of mathematical literatur from the 1990's (cf. [Bo1, Bo2, CSV, Car, Ca2]), and many of those tools will be closely related to the present work.

In spite of the various existing well-posedness results for the WPFP problem, there is a need for a purely kinetic analysis, and this is our goal here. Such an approach could possibly allow

for an extension to boundary-value problems in the Wigner framework (where the positivity of the related density matrix is a touchy question).
Mathematically we shall develop the following new tools and estimates that could be important also for other quantum kinetic applications: In all of the existing literature on Wigner-Poisson problems (except [Ste]) the potential V is bounded, which makes it easy to estimate the operator $\Theta[V]$ in L^2. Our framework for the local in time analysis does not yield a bounded potential. However, the operator Θ only involves δV, a potential difference, which has better decay properties at infinity. This observation gives rise to new estimates that are crucial for our local-in-time analysis.
In order to establish global-in-time solutions we shall extend dispersive tools of Lions, Perthame and Castella (cf. [LP, Pe, CP] for applications to classical kinetic equation) to the WP and WPFP systems. The fact that the Wigner function w also takes negative values gives rise to an important difference between classical and quantum kinetic problems: In the latter case, the conservation of mass and energy or pseudo-conformal laws do *not* provide useful a-priori estimates on w. We shall hence assume that the initial state lies in a weighted L^2-space, but we shall *not* require that our system has finite mass or finite kinetic energy. Since the energy balance will not be used, this also implies that the sign of the interaction potential does *not* play a role in our analysis.

This paper is organized as follows: In Section 2 we introduce a weighted L^2-space for the Wigner function w that allows to define $n[w]$ and the nonlinear term $\Theta[V]w$. In §3 we obtain a local-in-time, mild solution for WPFP using a fixed point argument and the parabolic regularization of the Fokker-Planck term. In §4 we establish a-priori estimates to obtain global-in-time solutions. The key point is to derive first L^p-bounds for the electric field ∇V by exploiting dispersive effects of the free kinetic transport. "Bootstraping" then yields estimates on the Wigner function in a weighted L^2-space. Finally, we give regularity results on the solution. The technical proofs of several lemmata are defered to the Appendix.

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2 The functional setting

In this section we shall discuss the functional analytic preliminaries for studying the nonlinear problem (1.1)-(1.5). First we shall introduce an appropriate "state space" for the Wigner function w which allows to "control" the particle density $n[w]$ and the selfconsistent potential $V[w]$. Next, we shall discuss the linear Wigner-Fokker-Planck equation and the dissipativity of its (evolution) generator A.

2.1 State space and selfconsistent potential

Let us introduce the following weighted (real valued) L^2-space

$$X \;:=\; L^2(\mathbb{R}^6;(1+|v|^2)^2\,dx\,dv), \tag{2.1}$$

endowed with the scalar product

$$< u, w >_X \; = \; \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} u\, w(1+|v|^2)^2 \, dx\, dv. \tag{2.2}$$

The following proposition motivates the choice of X as the state space for our analysis.

Proposition 2.1 *For all $w \in X$, the function $n[w]$ defined by $n[w](x) := \int w(x,v)\, dv$, $x \in \mathbb{R}^3$, belongs to $L^2(\mathbb{R}^3)$ and satisfies*

$$\|n[w]\|_{L^2(\mathbb{R}^3)} \; \leq \; C\|w\|_X, \tag{2.3}$$

with a constant C independent of w.

Here and in the sequel C shall denote generic, but not necessarily equal, constants.
Proof. By using Hölder inequality in the v-integral, we get

$$\|n[w]\|^2_{L^2(\mathbb{R}^3)} \; \leq \; \int\!\Big(\int |w(x,v)|^2(1+|v|^2)^2\, dv\Big)\Big(\int \frac{dv}{(1+|v|^2)^2}\Big)\, dx \; = \; C\|w\|_X^2.$$

□

Remark 2.2 *The choice of the $|v|^2$ weight was already seen to be convenient to control the L^2-norm of the density on a bounded domain of $\mathbb{R}^3_x$ (cf. [Ma]).*
The subsequent analysis would hold also by including a symmetric weight in the x-variable (i.e. for $w \in L^2(\mathbb{R}^6; (1+|x|^2+|v|^2)^2\, dx\, dv)$), which would yield a L^p-bound with $p \in (3/2, 2]$ for the density. On the other hand, Lemmata 2.8, 2.10 would prevent us from introducing a non-symmetric weight in x.

In this framework the following estimates for the self-consistent potential hold.

Proposition 2.3 *For all $w \in X$, the (Newton potential) solution $V = V[w]$ of the equation $-\Delta_x V[w] = n[w]$, $x \in \mathbb{R}^3$, satisfies*

$$\|\nabla V[w]\|_{L^6(\mathbb{R}^3)} \; \leq \; C\|n[w]\|_{L^2(\mathbb{R}^3)}. \tag{2.4}$$

Proof. Since $V = -\frac{1}{4\pi|x|} * n$, we have $\nabla V = \frac{x}{4\pi|x|^3} * n$, and the estimate follows from the generalized Young inequality. □

Remark 2.4 *Note that $n \in L^2(\mathbb{R}^3)$ does not yield (via the generalized Young inequality) a control of V in any L^r-space (even $n \in L^p(\mathbb{R}^3)$ with $p \in (3/2, 2]$ would not "help"). However, the operator $\Theta[V]$ involves only the function δV, which is slightly "better behaved". We anticipate that we will later recover some information on the potential V via new* a-priori *estimates on the electric field $\nabla V[w]$ (see Corollary 4.17).*

Omitting the time-dependence we have

$$\begin{aligned}\delta V(x,\eta) \; &= \; V(x+\frac{\eta}{2}) - V(x-\frac{\eta}{2}) \; = \; \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{n[w](x-\frac{\eta}{2}-\xi) - n[w](x+\frac{\eta}{2}-\xi)}{|\xi|}\, d\xi \\ &= \; \frac{1}{4\pi}\int_{\mathbb{R}^3} f(y;\eta)\, n[w](x-y)\, dy,\end{aligned}$$

with the "dipole-kernel" $f(y;\eta) := \left(\frac{1}{|y-\frac{\eta}{2}|} - \frac{1}{|y+\frac{\eta}{2}|}\right)$.

Proposition 2.5 *For all $w \in X$ and fixed $\eta \in \mathbb{R}^3$, we have $\delta V[w](.,\eta) \in L^q(\mathbb{R}^3_x)$, $6 < q \leq \infty$. Moreover*

$$\|\delta V[w](.,\eta)\|_{L^\infty(\mathbb{R}^3_x)} \leq C|\eta|^{1/2}\|n[w]\|_{L^2(\mathbb{R}^3)}. \tag{2.5}$$

Proof. By using the triangle inequality,

$$|f(y;\eta)| \leq \frac{|\eta|}{|y-\frac{\eta}{2}||y+\frac{\eta}{2}|},$$

and the transformation $y = |\eta|x$, we estimate for $3/2 < p < 3$

$$\|f(.\,;\eta)\|^p_{L^p(\mathbb{R}^3)} = |\eta|^{3-p}\int\limits_{\mathbb{R}^3} \frac{dx}{(|x-\frac{e}{2}||x+\frac{e}{2}|)^p} < \infty,$$

where $e \in \mathbb{R}^3$ is some unit vector (due to the rotational symmetry of $\|f(.;\eta)\|^p_{L^p(\mathbb{R}^3)}$ with respect to η). Young inequality then gives $\delta V(.,\eta) \in L^q(\mathbb{R}^3)$, $6 < q \leq \infty$, and the assertion holds.

□

In most of the literature the *Wigner operator* Θ is defined on $L^2(\mathbb{R}^d_v)$ for bounded potentials V, cf. [MR, MB, ACD]. For our nonlinear problem (1.1)-(1.5), however, $V \in L^\infty(\mathbb{R}^3)$ does *not* hold. As a compensation we shall hence exploit the additional regularity of the Wigner function to define the quadratic term $\Theta[V[w]]w$ (cf. Prop. 2.8 in [Ma] for a similar strategy).

Proposition 2.6 *Let $u \in X$ and $\nabla_v u \in X$ be given. Then, the linear operator*

$$z \longmapsto \Theta[V[z]]u,$$

with the function $V[z] = -\frac{1}{4\pi|x|} * n[z]$, is bounded from the space X into itself and satisfies

$$\|\Theta[V[z]]u\|_X \leq C\{\|u\|_X + \|\nabla_v u\|_X\}\,\|z\|_X, \quad \forall z \in X. \tag{2.6}$$

Proof. To estimate $\|\Theta[V[z]]u\|_X$ we shall consider separately the two terms of the equivalent norm

$$\|u\|^2_{\tilde{X}} := \|u\|^2_2 + \sum_{i=1}^3 \|v_i^2 u\|^2_2. \tag{2.7}$$

First, by denoting $\hat{u} := \mathcal{F}_{v\to\eta}u$, we get

$$\begin{aligned}\|\Theta[V[z]]u\|^2_2 &= \iint |\delta(V[z])(x,\eta)\hat{u}(x,\eta)|^2 dx\,d\eta \leq \iint \|\delta(V[z])(\,.\,,\eta)\|^2_\infty |\hat{u}(x,\eta)|^2 d\eta\,dx \\ &\leq C\|z\|^2_X \iint \left(|\eta|^{1/2}|\hat{u}(x,\eta)|\right)^2 d\eta\,dx \leq C\|z\|^2_X\left(\|u\|^2_2 + \|\nabla_v u\|^2_2\right),\end{aligned} \tag{2.8}$$

by applying first the Plancherel Theorem, then Hölder's inequality in the x variable, the estimates (2.3), (2.5) for the function $\delta V[z]$, and finally, Young inequality and the Plancherel Theorem to the last integral.
For the second term of $\|\Theta[V[z]]u\|_{\tilde{X}}$ we shall use

$$v_i^2\Theta[V]w(x,v) = \frac{1}{4}\Theta[\partial_i^2 V]w(x,v) + \Omega[\partial_i V](v_i w)(x,v) + \Theta[V]v_i^2 w(x,v), \tag{2.9}$$

with the pseudo-differential operator

$$\Omega[V] := i(\delta_+ V)\left(x, \frac{\nabla_v}{i}\right), \qquad (\delta_+ V)(x,\eta) := V\left(x+\frac{\eta}{2}\right) + V\left(x-\frac{\eta}{2}\right). \tag{2.10}$$

Here and in the sequel we use the abreviation $\partial_i := \partial_{x_i}$. (2.9) is now estimated:

$$\|v_i^2\Theta[V[z]]u\|_2 \;\le\; \frac{1}{4}\|\delta(\partial_i^2 V[z])\hat{u}\|_2 + \|\delta_+(\partial_i V[z])\partial_{\eta_i}\hat{u}\|_2 + \|\delta V[z]\partial_{\eta_i}^2\hat{u}\|_2 \tag{2.11}$$

The first two terms of (2.11) can be estimated as follows:

$$\begin{aligned}\|\delta(\partial_i^2 V[z])\hat{u}\|_{L^2(\mathbb{R}^6)} &\le 2\|\partial_i^2 V[z]\|_{L^2(\mathbb{R}^3_x)}\|\hat{u}\|_{L^2(\mathbb{R}^3_x;L^\infty(\mathbb{R}^3_\eta))}\\ &\le C\|z\|_X\|(1+|v|^2)u\|_{L^2(\mathbb{R}^6)},\end{aligned}$$

by applying Hölder's inequality, (2.3) and the Sobolev imbedding $\hat{u}(x,.) \in H^2(\mathbb{R}^3_\eta) \hookrightarrow L^\infty(\mathbb{R}^3_\eta)$.

$$\begin{aligned}\|\delta_+(\partial_i V[z])\partial_{\eta_i}\hat{u}\|_{L^2(\mathbb{R}^6)} &\le C\|\partial_i V[z]\|_{L^4(\mathbb{R}^3_x)}\|\partial_{\eta_i}\hat{u}\|_{L^2(\mathbb{R}^3_x;L^4(\mathbb{R}^3_\eta))}\\ &\le C\|z\|_X\|(1+v_i^2)u\|_2,\end{aligned} \tag{2.12}$$

by the Sobolev imbedding and $\nabla_\eta\hat{u}(x,.) \in H^1(\mathbb{R}^3_\eta) \hookrightarrow L^4(\mathbb{R}^3_\eta)$, and by estimate (2.4) for $\nabla V[z]$ and (2.3). For the last term of (2.11) we estimate as in (2.8):

$$\begin{aligned}\|\delta V[z]\partial_{\eta_i}^2\hat{u}\|_2^2 &\le \iint \|\delta V[z](.,\eta)\|_\infty^2 |\partial_{\eta_i}^2\hat{u}(x,\eta)|^2 d\eta\, dx\\ &\le C\|z\|_X^2 \iint \left(|\eta|^{1/2}\partial_{\eta_i}^2\hat{u}(x,\eta)\right)^2 d\eta\, dx\\ &\le C\|z\|_X^2\left(\|\partial_{\eta_i}^2\hat{u}\|_2^2 + \|\eta\partial_{\eta_i}^2\hat{u}\|_2^2\right)\\ &\le C\|z\|_X^2\left(\|\partial_{\eta_i}^2\hat{u}\|_2^2 + \|\partial_{\eta_i}^2(\eta\hat{u})\|_2^2 + \|\partial_{\eta_i}\hat{u}\|_2^2\right)\\ &\le C\|z\|_X^2\left(\|(1+v_i^2)u\|_2^2 + \|v_i^2\nabla_v u\|_2^2\right),\end{aligned}$$

by interpolation and integration by parts.
This concludes the proof of estimate (2.6). □

Remark 2.7 *The previous proposition shows that the bilinear map*

$$(z,u) \longmapsto \Theta[V[z]]u$$

is well-defined for all $z,u \in X$, subject to $\nabla_v u \in X$. The unusual feature of the above proposition is the boundedness of this map with respect to the function z appearing in the self-consistent potential $V[z]$. This is in contrast to most of the existing literature ([ACD, MB, MR]), where the boundedness of the pseudo-differential operator $\Theta[V[z]]$ (with z fixed) is used. However, this can only hold for bounded potentials V.

2.2 Dissipativity of the linear equation

In our subsequent analysis we shall first consider the linear Wigner-Fokker-Planck equation, i.e. equation (1.1) with $V \equiv 0$. The generator of this evolution problem is the unbounded linear operator $A : D(A) \longrightarrow X$,

$$Au \;:=\; -v\cdot\nabla_x u + \beta\mathrm{div}_v(vu) + \sigma\Delta_v u + 2\gamma\mathrm{div}_v(\nabla_x u) + \alpha\Delta_x u, \tag{2.13}$$

defined on

$$D(A) \;=\; \{u \in X \mid v\cdot\nabla_x u, v\cdot\nabla_v u, \Delta_v u, \mathrm{div}_v\nabla_x u, \Delta_x u \in X\}. \tag{2.14}$$

Clearly, $C_0^\infty(\mathbb{R}^6) \subset D(A)$. Hence, $D(A)$ is dense in X. Next we study whether the operator A is dissipative on the (real) Hilbert space $\tilde{X}$, i.e. if

$$< Au, u >_{\tilde{X}} \;\leq\; 0, \quad \forall\, u \in D(A) \tag{2.15}$$

holds.

Lemma 2.8 *Let the coefficients of the operator A satisfy $\alpha\sigma \geq \gamma^2$. Then $A - \kappa I$ with*

$$\kappa \;:=\; \frac{3}{2}\beta + 9\sigma \tag{2.16}$$

is dissipative in $\tilde{X}$.

The proof is lengthy but straigthforward and deferred to the Appendix.

By Theorem 1.4.5b of [Pa] its closure, $\overline{A - \kappa I} = \overline{A} - \kappa I$ is also dissipative.
A straightforward calculation using integrations by parts yields

$$< Au, w >_{\tilde{X}} \;=\; < u, A_1^* w >_{\tilde{X}} \;+\; < u, A_2^* w >_{\tilde{X}}, \quad \forall\, u, w \in D(A),$$

with

$$\begin{aligned} A_1^* w \;&=\; v \cdot \nabla_x w - \beta v \cdot \nabla_v w + \sigma \Delta_v w + 2\gamma \mathrm{div}_v(\nabla_x w) + \alpha \Delta_x w, \\ < u, A_2^* w >_{\tilde{X}} \;&=\; \sum_{i=1}^{3} \Big(-\frac{4}{3}\beta \iint v_i^4 w u + \frac{8}{3}\sigma \iint v_i^3 w_{v_i} u + \frac{12}{3}\sigma \iint v_i^2 w u + \frac{8}{3}\gamma \iint v_i^3 w_{x_i} u \Big). \end{aligned}$$

Hence, $A^*|_{D(A)}$ – the restriction of the adjoint of the operator A to $D(A)$ – is given by $A^* = A_1^* + A_2^*$. A^* is densly defined on $D(A^*) \supseteq D(A)$, and hence A is a closable operator (cf. Theorem VIII.1.b of [RS1]). Its closure $\overline{A}$ satisfies $(\overline{A})^* = A^*$ (cf. [RS1], Theorem VIII.1.c).
Since $< A^* u, u > = < Au, u >$ the following lemma on the dissipativity of the operator A^* restricted to $D(A)$ holds.

Lemma 2.9 *Let the coefficients of the operator A satisfy $\alpha\sigma \geq \gamma^2$. Then $A^*|_{D(A)} - \kappa I$ is dissipative (with κ as in (2.16)).*

Next we consider the dissipativity of this operator on its proper domain $D(A^*)$, which, however, is not known explicitly. To this end we shall use the following technical lemma, which, for a matter of generality, is stated in the space with the symmetric x, v-weight. The proof is defered to the appendix: the arguments are inspired by [ACD], [AS], but there are also similar results for FP-type operators in [HelN, HerN], e.g.

Lemma 2.10 *Let $P = p(x, v, \nabla_x, \nabla_v)$ where p is a quadratic polynomial and*

$$D(P) \;:=\; C_0^\infty(\mathbb{R}^6) \subset Z \;:=\; L^2(\mathbb{R}^6; (1 + |x|^2 + |v|^2)^2\, dx\, dv).$$

Then $\bar{P}$ is the maximum extension of P in the sense that

$$D(\bar{P}) \;:=\; \{u \in Z \mid \textit{the distribution } Pu \in Z\}\,.$$

We now apply Lemma 2.10 to $P = A^* - \kappa I$, which is dissipative on $D(P) \subset D(A)$. Since A^* is closed, we have $D(A^*) = D(\overline{P}) = \{u \in X \,|\, A^* u \in X\}$ and $A^* - \kappa I$ is dissipative on all of $D(A^*)$.
Applying Corollary 1.4.4 of [Pa] to $\overline{A} - \kappa I$ (with $(\overline{A})^* = A^*$), then implies that $\overline{A} - \kappa I$ generates a C_0 semigroup of contractions on X, and the C_0 semigroup generated by $\overline{A}$ satisfies

$$\|e^{t\overline{A}} u\|_{\tilde{X}} \;\leq\; e^{\kappa t} \|u\|_{\tilde{X}}, \quad u \in X, \;\; t \geq 0.$$

Since $\|.\|_X$ and $\|.\|_{\tilde{X}}$ are equivalent norms in X with

$$\|u\|_{\tilde{X}} \;\leq\; \|u\|_X \;\leq\; 4\|u\|_{\tilde{X}},$$

we have

$$\|e^{t\overline{A}} u\|_X \;\leq\; 4 e^{\kappa t} \|u\|_X, \quad u \in X, \;\; t \geq 0. \tag{2.17}$$

3 Existence of the local-in-time solution

In this section we shall use a contractive fixed point map to establish a local solution of the WPFP system. To this end the parabolic regularization of the linear WFP equation will be crucial to define the self-consistent potential term.

3.1 The linear equation

First let us consider the linear equation

$$w_t \;=\; \overline{A} w(t), \quad t > 0, \qquad w(t=0) \;=\; w_0 \in X. \tag{3.1}$$

By the discussion in Subsection 2.2, its unique solution $w(t) = e^{t\overline{A}} w_0$ satisfies

$$\|w(t)\|_X \;\leq\; 4 e^{\kappa t} \|w_0\|_X, \quad \forall\, t \geq 0. \tag{3.2}$$

Actually, the solution of the equation can be expressed as

$$w(x,v,t) \;=\; \iint w_0(x_0,v_0) G(t,x,v,x_0,v_0)\, dx_0\, dv_0, \quad \forall\, (x,v) \in \mathbb{R}^6, \tag{3.3}$$

where the Green's function G satisfies (in a weak sense) the equation (3.1) and the initial condition

$$\lim_{t \to 0} G(t,x,v,x_0,v_0) \;=\; \delta(x - x_0, v - v_0),$$

for any fixed $(x_0, v_0) \in \mathbb{R}^6$ (cf. Def. 2.1 and Prop. 3.1 in [SCDM]).
The Green's function reads

$$G(t,x,v,x_0,v_0) \;=\; e^{3\beta t} F(t, X_{-t}(x,v) - x_0, \dot{X}_{-t}(x,v) - v_0), \tag{3.4}$$

with

$$F(t,x,v) \;=\; \frac{1}{(2\pi)^3 \left(4\lambda(t)\nu(t) - \mu^2(t)\right)^{3/2}} \cdot \exp\left\{ -\frac{\nu(t)|x|^2 + \lambda(t)|v|^2 + \mu(t)(x \cdot v)}{4\lambda(t)\nu(t) - \mu^2(t)} \right\}.$$

The characteristic flow $\Phi_t(x,v) = [X_t(x,v), \dot{X}_t(x,v)]$ of the first order part of (2.13), is given for $\beta > 0$ by

$$X_t(x,v) \;=\; x + v\left(\frac{1 - e^{-\beta t}}{\beta}\right),$$
$$\dot{X}_t(x,v) \;=\; v e^{-\beta t},$$

and $\Phi_t(x,v) = [x+vt, v]$, for $\beta = 0$. The asymptotic behaviour of the functions $\lambda(t)$, $\nu(t)$, $\mu(t)$ for small t is described (also for $\beta = 0$) by

$$\begin{aligned}
\lambda(t) &= \alpha t + \sigma\left[\tfrac{e^{2\beta t}-4e^{\beta t}+3}{2\beta^3} + \tfrac{1}{\beta^2}\, t\right] + \gamma\left[\tfrac{2}{\beta} t - \tfrac{2}{\beta^2}(e^{\beta t}-1)\right] &&\sim \quad \alpha t, \quad t \to 0,\\
\nu(t) &= \sigma\, \tfrac{e^{2\beta t}-1}{2\beta} &&\sim \quad \sigma t, \quad t \to 0,\\
\mu(t) &= \sigma\left(\tfrac{1-e^{\beta t}}{\beta}\right)^2 + \gamma\, \tfrac{2(1-e^{\beta t})}{\beta} &&\sim \quad -2\gamma t, \quad t \to 0.
\end{aligned}$$

And hence:

$$f(t) \; := \; 4\lambda(t)\nu(t) - \mu^2(t) \quad \sim \quad 4(\alpha\sigma - \gamma^2)t^2 > 0.$$

With these preliminaries, the following parabolic reguralization result can be deduced.

Proposition 3.1 *For each parameter set $\{\alpha, \beta, \gamma, \sigma\}$, there exist two constants $B = B(\alpha, \beta, \gamma, \sigma)$ and $T_0 = T_0(\alpha, \beta, \gamma, \sigma)$, such that the solution of the linear equation (3.1) satisfies*

$$\|\nabla_v w(t)\|_X \;\le\; B t^{-1/2} e^{\kappa t} \|w_0\|_X, \quad \forall\, 0 < t \le T_0, \tag{3.5}$$

$$\|\nabla_x w(t)\|_X \;\le\; B t^{-1/2} e^{\kappa t} \|w_0\|_X, \quad \forall\, 0 < t \le T_0, \tag{3.6}$$

for all $w_0 \in X$.

The proof is similar to [Car] and it will be defered to the Appendix.

Remark 3.2 *(a) Observe that the functions $\nabla_x w, \nabla_v w \in \mathcal{C}((0,\infty); X)$. The local boundedness of $\nabla_x w, \nabla_v w$ on any interval $(\tau, \tau + T_0]$ follows from (3.2) and Prop. 3.1.*
(b) Note that the strategy of the next section will not *work in the degenerated parabolic case $\alpha\sigma - \gamma^2 = 0$, since the decay rates of Prop. 3.1 would then be $t^{-3/2}$, which is not integrable at $t = 0$. Alternative strategies for this degenerate case were studied in [ALMS].*

3.2 The non-linear equation: local solution

Our aim is to solve the following non-linear initial value problem

$$w_t(t) \;=\; \overline{A} w(t) + \Theta[V[w(t)]]w(t), \quad \forall\, t > 0, \qquad w(t=0) \;=\; w_0 \in X, \tag{3.7}$$

where the pseudo-differential operator Θ is formally defined by (1.2) and the potential $V[w(t)]$ is the (Newton potential) solution of the Poisson equation

$$-\Delta_x V(t,x) \;=\; n[w(t)](x) \;=\; \int_{\mathbb{R}^3} w(t,x,v)\, dv, \qquad x \in \mathbb{R}^3, \tag{3.8}$$

for all $t > 0$. Actually, if we assume $w(t) \in X$ for all $t \ge 0$, then the function $n[w(t)]$ is well-defined for all $t \ge 0$ (cf. Prop. 2.1), and the solution $V[w(t)]$ satisfies the estimates of Propositions 2.3, 2.5 for all $t \ge 0$.

The Propositions 2.6 and 3.1 motivate the definition of the Banach space

$$Y_T \;:=\; \Big\{ z \in \mathcal{C}([0,T]; X) \,|\, \nabla_v z \in \mathcal{C}((0,T]; X) \text{ with } \|\nabla_v z(t)\|_X \le C t^{-1/2} \text{ for } t \in (0,T) \Big\},$$

endowed with the norm

$$\|z\|_{Y_T} \;:=\; \sup_{t\in[0,T]} \|z(t)\|_X + \sup_{t\in[0,T]} \|t^{1/2} \nabla_v z(t)\|_X,$$

for every fixed $0 < T < \infty$. We shall obtain the (local-in-time) well-posedness result for the problem (3.7) by introducing a non-linear iteration in the space Y_T, with an appropriate (small enough) T.

For a given $w \in Y_T$ we shall now consider the linear Cauchy problem for the function z,

$$z_t \;=\; \overline{A}z(t) + \Theta[V[z(t)]]w(t), \quad \forall\, t \in (0,T], \qquad z(t=0) \;=\; w_0 \in X, \tag{3.9}$$

with $0 < T \le T_0$ and T_0 is defined in Prop. 3.1. According to Prop. 2.6 the (time-dependent) operator $\Theta[V[.]]w(t)$ is, for each fixed $t \in (0, T_0]$, a well-defined, linear and bounded map on X, which we shall consider as a perturbation of the operator $\overline{A}$.

Lemma 3.3 *For all $w_0 \in X$ and $w \in Y_T$, with $T \le T_0$, the initial value problem*

$$z_t \;=\; \overline{A}z(t) + \Theta[V[z(t)]]w(t), \quad \forall\, t \in (0,T], \qquad z(t=0) \;=\; w_0,$$

has a unique mild solution $z \in \mathcal{C}([0,T];X)$, which satisfies

$$z(t) \;=\; \mathrm{e}^{t\overline{A}}w_0 + \int_0^t \mathrm{e}^{(t-s)\overline{A}}\Theta[V[z(s)]]w(s)\,ds, \quad \forall\, t \in [0,T]. \tag{3.10}$$

Moreover, the solution z belongs to the space Y_T.

Proof. The first assertion follows directly by applying (a trivial extension of) Thm. 6.1.2 in [Pa]:
For any fixed $w \in Y_T$, the function $g(t,.) := \Theta[V[.]]w(t)$ is a bounded linear operator on X for all $t \in (0,T)$, and it satisfies $g \in L^1((0,T);\mathcal{B}(X)) \cap C((0,T];\mathcal{B}(X))$ (by Prop. 2.6). Moreover, by estimates (2.17), (2.6), the following inequalities hold

$$\begin{aligned} \|z(t)\|_X \;&\le\; 4\mathrm{e}^{\kappa t}\|w_0\|_X + 4\int_0^t \mathrm{e}^{\kappa(t-s)}C\{\,\|w(s)\|_X + \|\nabla_v w(s)\|_X\}\|z(s)\|_X\,ds && (3.11)\\ &\le\; 4\mathrm{e}^{\kappa t}\|w_0\|_X + 4C\mathrm{e}^{\kappa T}\|w\|_{Y_T}\int_0^t (1+s^{-1/2})\|z(s)\|_X\,ds, && (3.12)\end{aligned}$$

for all $t \in [0,T]$. Then, by Gronwall's Lemma,

$$\|z(t)\|_X \;\le\; 4e^{\kappa T}\|w_0\|_X\left[1 + 4C\|w\|_{Y_T}e^{\left(\kappa T + 4Ce^{\kappa T}\|w\|_{Y_T}(T+2T^{1/2})\right)}(t+2t^{1/2})\right], \tag{3.13}$$

for all $t \in [0,T]$. By differentiating equation (3.10) in the v-direction, we obtain

$$\nabla_v z(t) \;=\; \nabla_v e^{t\overline{A}}w_0 + \int_0^t \nabla_v e^{(t-s)\overline{A}}g(s,z(s))\,ds, \quad \forall\, t \in [0,T]. \tag{3.14}$$

Using the estimates (3.5), (2.6), and (3.13) then yields

$$\begin{aligned} \|\nabla_v z(t)\|_X \;&\le\; Bt^{-1/2}\mathrm{e}^{\kappa t}\|w_0\|_X \\ &\quad + B\|w\|_{Y_T}\int_0^t (t-s)^{-1/2}\mathrm{e}^{\kappa(t-s)}C\{\,1+s^{-1/2}\}\,\|z(s)\|_X\,ds \\ &\le\; Bt^{-1/2}\mathrm{e}^{\kappa t}\|w_0\|_X + 4BCe^{2\kappa T}\|w_0\|_X\|w\|_{Y_T}\Big[\,\pi + 2t^{1/2} \\ &\quad + 4C\|w\|_{Y_T}e^{\left(\kappa T + 4Ce^{\kappa T}\|w\|_{Y_T}(T+2T^{1/2})\right)}\left(4t^{1/2} + \frac{3}{2}\pi t + \frac{4}{3}t^{3/2}\right)\Big], && (3.15)\end{aligned}$$

for all $t \in [0,T]$. The continuity in time of $\nabla_v z$ can be derived from (3.14) by using Remark 3.2 and the fact that $g(t,z(t)) \in \mathcal{C}((0,T];X)$. Hence, the function z belongs to the space Y_T. □

We now define the (affine) linear map M on Y_T (for any fixed $0 < T \leq T_0$):

$$w \longmapsto Mw := z,$$

where z is the unique mild solution of the initial value problem (3.9). According to Lemma 3.3, $z \in Y_T$. Next we shall show that M is a strict contraction on a closed subset of Y_T, for T sufficiently small. This will yield the local-in-time solution of the non-linear equation (3.7).

Proposition 3.4 *For any fixed $w_0 \in X$, let $R > \max\{4, B\}e^{\kappa}\|w_0\|_X$, with the constant B defined in Prop. 3.1. Then there exists a $\tau := \tau(\|w_0\|_X, B) > 0$ such that the map M,*

$$(Mw)(t) \;=\; e^{t\overline{A}}w_0 + \int_0^t e^{(t-s)\overline{A}}\Theta[V[Mw(s)]]w(s)\,ds, \quad \forall\, t \in [0,\tau], \tag{3.16}$$

is a strict contraction from the ball of radius R of Y_τ into itself.

Proof. By (the proof of) Lemma 3.3, the function $z = Mw \in Y_\tau$ satisfies (3.13). Under the assumption $\|w\|_{Y_\tau} \leq R$, this estimate reads

$$\|Mw(t)\|_X \;\leq\; 4e^{\kappa\tau}\|w_0\|_X\left[1 + 4CRe^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau+2\tau^{1/2})\right)}(t + 2t^{1/2})\right], \quad \forall t \in [0,\tau].$$

If we assume

$$4e^{\kappa\tau}\|w_0\|_X\left[1 + 4CRe^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau+2\tau^{1/2})\right)}(\tau + 2\tau^{1/2})\right] \;\leq\; \frac{R}{3}, \tag{3.17}$$

then $\|Mw(t)\|_X \;\leq\; \frac{R}{3}$. Similar to (3.15) we have

$$\begin{aligned}\|\nabla_v Mw(t)\|_X \;\leq\;& Bt^{-1/2}e^{\kappa t}\|w_0\|_X + 4BCRe^{2\kappa\tau}\|w_0\|_X\Bigg[\pi + 2t^{1/2}\\ &+ 4CRe^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau+2\tau^{1/2})\right)}\left(4t^{1/2} + \frac{3}{2}\pi t + \frac{4}{3}t^{3/2}\right)\Bigg].\end{aligned}$$

If we assume

$$\begin{aligned}Be^{\kappa\tau}\|w_0\|_X \;+\; 4BCRe^{2\kappa\tau}\|w_0\|_X\Bigg[\pi\tau^{1/2} + 2\tau +&\\ +\; 4CRe^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau+2\tau^{1/2})\right)}\left(4\tau + \frac{3}{2}\pi\tau^{3/2} + \frac{4}{3}\tau^2\right)\Bigg] \;\leq\;& \frac{R}{3},\end{aligned} \tag{3.18}$$

then

$$t^{1/2}\|\nabla_v Mw(t)\|_X \;\leq\; \frac{R}{3}, \quad \forall\, t \in [0,\tau].$$

Let us now choose

$$\begin{aligned}\tau \;:=\; \min\Bigg\{&1, \left(\frac{R/3 - 4e^{\kappa}\|w_0\|_X}{48CR\|w_0\|_X e^{2\kappa + 12CRe^{\kappa}}}\right)^2,\\ &\left(\frac{R/3 - Be^{\kappa}\|w_0\|_X}{4BCRe^{2\kappa}\|w_0\|_X\left[\pi + 2 + 4CRe^{(\kappa+12CRe^{\kappa})}(\frac{3}{2}\pi + \frac{16}{3})\right]}\right)^2\Bigg\},\end{aligned} \tag{3.19}$$

which is positive since $\max\{4, B\}e^{\kappa}\|w_0\|_X < R$. Then, the estimates (3.17) and (3.18) hold, and hence the operator M maps the ball of radius R of Y_τ into itself.

To prove contractivity we shall estimate $\|Mu-Mw\|_{Y_\tau}$ for all $u,w\in Y_\tau$ with $\|u\|_{Y_\tau},\|w\|_{Y_\tau}\le R$. Since

$$\begin{aligned} Mu(t)-Mw(t) &= \int_0^t e^{(t-s)\overline{A}}\Theta[V[(Mu-Mw)(s)]]u(s)\,ds \\ &\quad+\int_0^t e^{(t-s)\overline{A}}\Theta[V[Mw(s)]](u-w)(s)\,ds, \quad \forall\, t\in[0,\tau], \end{aligned}$$

by analogous estimates,

$$\begin{aligned} \|Mu(t)-Mw(t)\|_X &\le 4CR\,e^{\kappa\tau}\left\{\int_0^t(1+s^{-1/2})\,\|(Mu-Mw)(s)\|_X\,ds \right.\\ &\quad \left. +\|u-w\|_{Y_\tau}\int_0^t(1+s^{-1/2})\,ds\right\}, \end{aligned}$$

and, by applying Gronwall's Lemma:

$$\begin{aligned} \|Mu(t)-Mw(t)\|_X &\le 4CRe^{\kappa\tau}\left[t+2t^{1/2}+\right.\\ &\quad \left.+\,4CRe^{\left(\kappa\tau+4CRe^{\kappa\tau}(\tau+2\tau^{1/2})\right)}\left(2t+2t^{3/2}+\frac{1}{2}t^2\right)\right]\|u-w\|_{Y_\tau}, \quad \forall\, t\in[0,\tau]. \end{aligned}$$

By using $0\le t\le\tau\le 1$, we obtain

$$\|Mu(t)-Mw(t)\|_X \le 4CR\,e^{\kappa}\left[3+18CRe^{(\kappa+12CRe^{\kappa})}\right]\tau^{1/2}\|u-w\|_{Y_\tau}. \tag{3.20}$$

Similarly,

$$\begin{aligned} \|\nabla_v Mu(t)-\nabla_v Mw(t)\|_X &\le CBRe^{k\tau}\left\{\int_0^t(t-s)^{-1/2}(1+s^{-1/2})\,\|(Mu-Mw)(s)\|_X\,ds\right.\\ &\quad\left.+\int_0^t(t-s)^{-1/2}(1+s^{-1/2})\,ds\|u-w\|_{Y_\tau}\right\}, \end{aligned}$$

and, by using estimate (3.20),

$$\begin{aligned} \|\nabla_v Mu(t)-\nabla_v Mw(t)\|_X &\le CBRe^{\kappa\tau}\left[1+4CR\,e^{\kappa}\left(3+18CRe^{(\kappa+12CRe^{\kappa})}\right)\tau^{1/2}\right]\\ &\quad\cdot\left(\pi+2t^{1/2}\right)\|u-w\|_{Y_\tau}, \quad \forall\, t\in[0,\tau]. \end{aligned}$$

Then, by exploiting $0<\tau\le 1$,

$$\begin{aligned} t^{1/2}\|\nabla_v Mu(t)-\nabla_v Mw(t)\|_X &\le CBRe^{\kappa}(\pi+2)\Bigg[1+4CR\,e^{\kappa}\\ &\quad\cdot\left(3+18CRe^{(\kappa+12CRe^{\kappa})}\right)\Bigg]\tau^{1/2}\|u-w\|_{Y_\tau}. \end{aligned} \tag{3.21}$$

When choosing $\tau>0$ small enough, estimates (3.20), (3.21) imply

$$\|Mu-Mw\|_{\mathcal{C}([0,\tau];X)} \le C\|u-w\|_{\mathcal{C}([0,\tau];X)},$$

for some $C<1$, and the assertion is proved. □

Corollary 3.5 *There exists a* $t_{\max}\le\infty$ *such that the initial value problem (3.7) has a unique* mild *solution* w *in* Y_T, $\forall\, T<t_{\max}$, *which satisfies*

$$w(t) = e^{t\overline{A}}w_0+\int_0^t e^{(t-s)\overline{A}}\Theta[V[w(s)]]w(s)\,ds, \quad \forall\, t\in[0,T]. \tag{3.22}$$

Moreover, if $t_{\max}<\infty$, *then*

$$\lim_{t\nearrow t_{\max}}\|w(t)\|_X = \infty.$$

PROOF. The solution of the problem is the fixed point of the map M previously introduced. By Prop. 3.4 this solution exists for a time interval of length τ (depending only on $\|w_0\|_X$) and it belongs to the space Y_τ. Since, in particular, $w(\tau) \in X$, the solution can be repeatedly continued up to the maximal time $t_{\max}$. It will then belong to Y_T, $\forall T < t_{\max}$.
If the second assertion of the corollary would not hold, there would be a sequence of times $t_n \uparrow t_{\max}$ such that $\|w(t_n)\|_X \le C$ for all n. Then, by solving a problem with the initial value $w(t_n)$, with t_n sufficiently close to $t_{\max}$, we would extend the solution up to a certain time $t_n + \tau(\|w(t_n)\|_X) > t_{\max}$. This construction would contradict our definition of $t_{\max}$.
The uniqueness of the mild solution follows by arguments analogous to those in the proof of Thm. 6.1.4 in [Pa].
□

Remark 3.6 *Note that the last statement in the thesis of the Corollary 3.5 differs from the standard setting (cf. Thm. 6.1.4 in [Pa]). For $t_{\max} < \infty$ we conclude the 'explosion' of $w(t)$, $t \to t_{max}$ in X and* not only *in Y_t. This is due to the parabolic regularization of the problem (3.7).*

4 Global-in-time solution, a-priori estimates

In this section we shall exploit dispersive effects of the free transport equation to derive an a-priori estimate on the electric field. This is the key ingredient for proving the main result of the paper, the global solution for the WPFP system:

Theorem 4.1 *Let $w_0 \in X$ satisfy for some $\omega \in [0,1)$*

$$\Big\| \int w_0(x - \vartheta(t)v, v)\, dv \Big\|_{L^{6/5}(\mathbb{R}^3_x)} \;\le\; C_T \vartheta(t)^{-\omega}, \quad \forall\, t \in (0,T], \quad \forall\, T > 0, \tag{$\mathbf{A}$}$$

with $\vartheta(t) := \frac{1-e^{-\beta t}}{\beta}$ for $\beta > 0$, and $\vartheta(t) = t$ for $\beta = 0$. Then the WPFP equation (3.7) admits a unique global-in-time mild solution $w \in Y_T$, $\forall\, 0 < T < \infty$.

In order to prove that $t_{\max} = \infty$, we have to show that $\|w(t)\|_X$ is finite for all $t \ge 0$ (cf. Corollary 3.5). To this end, we shall derive a-priori estimates for $\|w(t)\|_2$ and $\||v|^2 w(t)\|_2$. Thus, the proof of Thm. 4.1 will be a consequence of a series of Lemmata, in particular of Lemma 4.2 and Lemma 4.19. In the sequel, $w(t)$ denotes the unique mild solution for $0 \le t \le T$, for an arbitrary $0 < T < t_{\max}$.

Lemma 4.2 *For all $w_0 \in X$, the mild solution of the WPFP equation (3.7) satisfies*

$$\|w(t)\|_2^2 \;\le\; e^{3\beta t}\|w_0\|_2^2, \quad \forall\, t \in [0,T]. \tag{4.1}$$

PROOF. Roughly speaking, this follows from the dissipativity of the operator $\overline{A} - \frac{3\beta}{2}$ in $L^2(\mathbb{R}^6)$ (cf. (5.2)) and the skew-symmetry of the pseudo-differential operator. However, since we are dealing only with the *mild* solution of the equation, the proof requires an approximation of w by classical solutions.
Since the solution satisfies $w \in Y_T$, $\forall\, T < t_{\max}$, Prop. 2.6 shows that the function $f(t) := \Theta[V[w(t)]]w(t)$, $t \in (0, t_{\max})$ is well defined and it is in $\mathcal{C}((0,t_{\max}); X) \cap L^1((0,T); X)$, $\forall\, 0 < T < t_{\max}$.
For $0 < T < t_{\max}$ fixed, let us consider the following linear inhomogeneous problem:

$$\frac{d}{dt} y(t) \;=\; \overline{A} y(t) + f(t), \quad t \in [0,T], \qquad y(t=0) \;=\; w_0 \in X. \tag{4.2}$$

Its mild solution in $[0,T]$ is the function w, due to the uniqueness of the mild solution of problem (3.7). For this linear problem, we can apply Thm. 4.2.7 of [Pa]: The mild solution w is the uniform limit (on $[0,T]$) of *classical* solutions of problem (4.2). More precisely, there is a sequence $\{w_0^n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\overline{A})$, $w_0^n \to w_0$ in X, and a sequence $\{f_n(t)\} \subset \mathcal{C}^1([0,T];X)$, $f_n(t) \to f(t)$ in $L^1((0,T);X)$. And the classical solutions $y_n \in \mathcal{C}^1([0,T];X)$ of the corresponding problems

$$\frac{d}{dt}y_n(t) \;=\; \overline{A}y_n(t) + f_n(t), \quad t \in [0,T], \qquad y_n(t=0) \;=\; w_0^n, \tag{4.3}$$

converge in $C([0,T];X)$ to the solution w of problem (4.2).
We shall need these approximating classical solutions y_n in order to justify the derivation of the a-priori estimate: Multiplying both sides of (4.3) by $y_n(t)$ and integrating yields

$$\frac{1}{2}\frac{d}{dt}\|y_n(t)\|_2^2 \;\leq\; \frac{3\beta}{2}\,\|y_n(t)\|_2^2 + \iint y_n(t) f_n(t)\,dx\,dv,$$

since the operator $\overline{A} - \frac{3\beta}{2}$ is dissipative in $L^2(\mathbb{R}^6)$ (cf. (5.2)). By integrating in t and letting $n \to \infty$, we have

$$\|w(t)\|_2^2 \;\leq\; \|w_0\|_2^2 + 3\beta\int_0^t \|w(s)\|_2^2\,ds + 2\int_0^t\iint w(s)f(s)\,dx\,dv\,ds, \quad \forall\, t \in [0,T].$$

The second integral is equal to zero since the pseudo-differential operator Θ is skew-symmetric. Hence, applying Gronwall's Lemma yields

$$\|w(t)\|_2^2 \;\leq\; e^{3\beta t}\|w_0\|_2^2, \quad \forall\, t \in [0,T]. \tag{4.4}$$

$\square$

In order to recover similar estimates for $\||v|^2 w(t)\|_2$, we first need a-priori bounds for the self-consistent field $E = \nabla V$. To this end, we are going to exploit dispersive effects of the free streaming operator. We shall adapt to the Wigner-Poisson and Wigner-Poisson-Fokker-Planck problems the strategies introduced for the (classical) Vlasov-Poisson problem ([LP, Pe]), and for the Vlasov-Poisson-Fokker-Planck problem ([Bo1, Bo2, Ca2]).

4.1 A-priori estimates for the electric field: the Wigner-Poisson case

To explain the strategy, we first consider the (simpler) WP problem: Let us assume that w^{wp} is a "regular" solution of the WP problem (e.g., let $w^{\mathrm{wp}}(t) \in L_x^2(H_v^1), \nabla_x V[w^{\mathrm{wp}}](t) \in \mathcal{C}_B(\mathbb{R}^3)$, uniformly on bounded t-intervals) for which the Duhamel formula holds:

$$w^{\mathrm{wp}}(x,v,t) \;=\; w_0^{\mathrm{wp}}(x-tv,v) + \int_0^t \Big(\Theta[V[w^{\mathrm{wp}}]]w^{\mathrm{wp}}\Big)(x-sv,v,t-s)\,ds.$$

We formally integrate in v:

$$\begin{aligned} n[w^{\mathrm{wp}}](x,t) \;&=\; \int_{\mathbb{R}^3} w_0^{\mathrm{wp}}(x-tv,v)\,dv + \int_0^t\int_{\mathbb{R}^3}\Big(\Theta[V[w^{\mathrm{wp}}]]w^{\mathrm{wp}}\Big)(x-sv,v,t-s)\,dv\,ds \\ &=: \; n_0^{\mathrm{wp}}(x,t) + n_1^{\mathrm{wp}}(x,t), \end{aligned}$$

and split the self-consistent field accordingly:

$$E_0^{\mathrm{wp}}(x,t) \;:=\; \lambda\frac{x}{|x|^3} *_x n_0^{\mathrm{wp}}(x,t) \;=\; \lambda\frac{x}{|x|^3} *_x \int w_0^{\mathrm{wp}}(x-tv,v)\,dv, \tag{4.5}$$

$$E_1^{\mathrm{wp}}(x,t) \;:=\; \lambda\frac{x}{|x|^3} *_x \int_0^t\int \Big(\Theta[V[w^{\mathrm{wp}}]]w^{\mathrm{wp}}\Big)(x-sv,v,t-s)\,dv\,ds, \tag{4.6}$$

with $\lambda = \frac{1}{4\pi}$.
Then, we can estimate separately the two terms $E_0^{\rm wp}(t)$, $E_1^{\rm wp}(t)$ by exploting the properties of the convolution kernel $1/|x|$ (cf. [LP, Pe] for VP, [Bo1, Bo2, Ca2] for VPFP, [ALMS] for WPFP). To this end, we need an appropriate redefinition of the pseudo-differential operator $\Theta[V]$ in (1.2). It is inspired by the operator $\nabla_x V\cdot\nabla_v w$ in the VP equation that can be recovered from $\Theta[V]w$ in the semiclassical limit (cf. Remark 4.5).

Let us recall that $\Theta[V]w(x,v) = \mathcal{F}^{-1}_{\eta\to v}\Big(i\,\delta V(x,\eta)\mathcal{F}_{v\to\eta}w(x,\eta)\Big)$. We can rewrite

$$\delta V(x,\eta) \;=\; \int_{x-\eta/2}^{x+\eta/2} \nabla_x V(z)\cdot dz \;=\; \int_{-1/2}^{1/2} \eta\cdot\nabla_x V(x-r\eta)\,dr \;=\; \eta\cdot W(x,\eta)\,, \qquad (4.7)$$

with the vector-valued function

$$W(x,\eta) \;:=\; \int_{-1/2}^{1/2} \nabla_x V(x-r\eta)\,dr\,, \quad \forall\,(x,\eta)\in\mathbb{R}^6\,.$$

Then, we define the vector-valued operator

$$\Gamma[\nabla_x V]u(x,v) \;:=\; \mathcal{F}^{-1}_{\eta\to v}\Big(W(x,\eta)\mathcal{F}_{v\to\eta}u(x,\eta)\Big). \qquad (4.8)$$

It holds:

Lemma 4.3 *Let $\nabla_x V\in\mathcal{C}_B(\mathbb{R}^3)$. Then*

1. $W(x,\eta)\in\mathcal{C}_B(\mathbb{R}^6)$, $\|W\|_\infty \;\le\; \|\nabla_x V\|_\infty$;
2. $\Gamma[\nabla_x V] : L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)$ *and, for all* $u\in L^2(\mathbb{R}^6)$,
$$\|\Gamma[\nabla_x V]u\|_{L^2(\mathbb{R}^6)} \;\le\; \|\nabla_x V\|_\infty\|u\|_{L^2(\mathbb{R}^6)};$$
3. $\Gamma[\nabla_x V] : L^2_x(H^1_v)\to L^2_x(H^1_v)$ *and, for all* $u\in L^2_x(H^1_v)$,
$$\|\Gamma[\nabla_x V]u\|_{L^2_x(H^1_v)} \;\le\; \|\nabla_x V\|_\infty\|u\|_{L^2_x(H^1_v)}. \qquad (4.9)$$

Proof. The first and the second assertion are obvious. For (4.9) we use

$$\partial_{v_j}\Gamma[\nabla_x V]u(x,v) \;=\; i\mathcal{F}^{-1}_{\eta\to v}\Big(\eta_j W(x,\eta)\mathcal{F}_{v\to\eta}u(x,\eta)\Big) \;=\; \Gamma[\nabla_x V]\partial_{v_j}u;\;\; j=1,2,3. \qquad (4.10)$$

$\square$

Lemma 4.4 *Let $\nabla_x V\in\mathcal{C}_B(\mathbb{R}^3)$ and $u\in L^2_x(H^1_v)$. Then*

$$\Theta[V]u(x,v) \;=\; \mathrm{div}_v\left(\Gamma[\nabla_x V]u\right)(x,v) \qquad (4.11)$$

Proof. By the definition (4.7) and Lemma 4.3,

$$\|\delta V(\,.\,,\eta)\|_\infty \;\le\; |\eta|\,\|W(\,.\,,\eta)\|_\infty \;\le\; |\eta|\,\|\nabla_x V\|_\infty.$$

Thus, $\|\Theta[V]u\|_{L^2(\mathbb{R}^6)} \le \|\nabla_x V\|_\infty\|u\|_{L^2_x(H^1_v)}$; the right hand side of equation (4.11) is also well-defined in $L^2(\mathbb{R}^6)$ by estimate (4.9). Equality then follows by equation (4.7) and

$$i\mathcal{F}^{-1}_{\eta\to v}\Big(\eta\cdot W(x,\eta)\mathcal{F}_{v\to\eta}u(x,\eta)\Big) \;=\; \sum_{j=1}^{3}\partial_{v_j}\Gamma_j[\nabla_x V]u(x,v) \;=\; \mathrm{div}_v\left(\Gamma[\nabla_x V]u\right)(x,v).$$

$\square$

Remark 4.5 (The semiclassical limit) *The correctly scaled version of the pseudo-differential operator with the reduced Planck constant* $\hbar = \frac{h}{2\pi}$ *reads*

$$\Theta_\hbar[V]w(x,v) \;=\; \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{V(x+\frac{\hbar}{2}\eta) - V(x-\frac{\hbar}{2}\eta)}{\hbar} \mathcal{F}_{v\to\eta} w(x,\eta) e^{iv\cdot\eta}\, d\eta.$$

Under the assumptions of Lemma 4.4, we thus have

$$\begin{aligned}\mathcal{F}_{v\to\eta}(\Theta_\hbar[V]w(x,v)) &= \frac{i}{\hbar}\,\delta V(x,\hbar\eta)\mathcal{F}_{v\to\eta}w(x,\eta)\\ &= i\,W(x,\hbar\eta)\,\cdot\,\eta\,\mathcal{F}_{v\to\eta}w(x,\eta).\end{aligned}$$

The limit $\hbar \to 0$ *then yields:*

$$i\,W(x,\hbar\eta)\,\cdot\,\eta\,\mathcal{F}_{v\to\eta}w(x,\eta) \;\longrightarrow\; i\nabla_x V(x)\cdot\eta\,\mathcal{F}_{v\to\eta}w(x,\eta) \;=\; \mathcal{F}^{-1}_{\eta\to v}\Big(\nabla_x V(x)\cdot\nabla_v w(x,v)\Big);$$

and hence

$$\Theta_\hbar[V]w(x,v) \;\longrightarrow\; \nabla_x V(x)\cdot\nabla_v w(x,v) \quad \textit{in}\;\; L^2(\mathbb{R}^6),$$

which is the non-linear term in the VP equation.

Using the redefinition (4.11) of the pseudo-differential operator, and under the additional assumptions $w^{\mathrm{wp}} \in H^1_x(L^2_v)$, $\Delta V[w^{\mathrm{wp}}] \in \mathcal{C}_B(\mathbb{R}^3)$, we have for $s \in \mathbb{R}$

$$\begin{aligned}\Big(\Theta[V[w^{\mathrm{wp}}]]w^{\mathrm{wp}}\Big)(x-sv,v) &= \mathrm{div}_v\Big(\Gamma[\nabla_x V[w^{\mathrm{wp}}]]w^{\mathrm{wp}}(x-sv,v)\Big)\\ &\quad + s\,\mathrm{div}_x\Big(\Gamma[\nabla_x V[w^{\mathrm{wp}}]]w^{\mathrm{wp}}\Big)(x-sv,v). \qquad (4.12)\end{aligned}$$

Thus, the field E_1^{wp} in (4.6) can be rewritten as ($j = 1,2,3$)

$$\begin{aligned}(E_1^{\mathrm{wp}})_j(x,t) &:= \lambda\frac{x_j}{|x|^3} *_x \mathrm{div}_x \int_0^t s \int \Big(\Gamma[\nabla_x V[w^{\mathrm{wp}}]]w^{\mathrm{wp}}\Big)(x-sv,v,t-s)\,dv\,ds \qquad (4.13)\\ &= \lambda\sum_{k=1}^3 \frac{-3x_jx_k + \delta_{jk}|x|^2}{|x|^5} *_x \int_0^t s\int \Big(\Gamma_k[\nabla_x V[w^{\mathrm{wp}}]]w^{\mathrm{wp}}\Big)(x-sv,v,t-s)\,dv\,ds.\end{aligned}$$

The following two lemmata are concerned with giving a meaning to the definition (4.13) of the field E_1, independently of the previous derivation.

Lemma 4.6 *For all* $u \in L^2(\mathbb{R}^6)$ *and* $E \in L^2(\mathbb{R}^3)$ *the following estimate holds*

$$\Big\|\int_{\mathbb{R}^3_v} (\Gamma[E]u)\,(x-sv,v)\,dv\Big\|_{L^2(\mathbb{R}^3_x)} \;\le\; Cs^{-3/2}\|E\|_2\|u\|_2\,, \quad \forall\, s>0. \qquad (4.14)$$

Proof. Since the operator $\Gamma[.]$ was originally defined for $E \in \mathcal{C}_B(\mathbb{R}^3)$, we shall first derive (4.14) for $E \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ and conclude by a density argument.
By the definition (4.8) and by several changes of variables, the following chain of equalities holds:

$$\begin{aligned}(\Gamma[E]u)(x,v) &= (2\pi)^{3/2}\Big[\mathcal{F}^{-1}_{\eta\to v}\,(W(x,\eta)) *_v u\Big](x,v)\\ &= \iint\int_{-1/2}^{1/2} E(x-r\eta)\,e^{i\eta\cdot z}\,dr\,d\eta\,u(x,v-z)\,dz \;=\; \iint\int_{-1/2}^{1/2}\frac{1}{|r|^3}E(x-\tilde\eta)\,e^{i\tilde\eta\cdot\frac{z}{r}}\,dr\,d\tilde\eta\,u(x,v-z)\,dz\\ &= \iint E(x-\tilde\eta)\,e^{i\tilde\eta\cdot\tilde z}\,d\tilde\eta \int_{-1/2}^{1/2} u(x,v-r\tilde z)\,dr\;d\tilde z \;=\; \iint E(-\hat\eta)\,e^{i\hat\eta\cdot\tilde z}\,d\hat\eta\,e^{ix\cdot\tilde z}\int_{-1/2}^{1/2} u(x,v-r\tilde z)\,dr\;d\tilde z\\ &= (2\pi)^{3/2}\int \mathcal{F}_{\eta\to\tilde z}E(\tilde z)\int_{-1/2}^{1/2} u(x,v-r\tilde z)\,dr\,e^{ix\cdot\tilde z}\,d\tilde z.\end{aligned}$$

Hence

$$\int (\Gamma[E]u)\,(x-sv,v)\,dv \;=\; (2\pi)^{\frac{3}{2}} \int \mathcal{F}_{\eta\to\tilde z}E(\tilde z) \left(\iint_{-1/2}^{1/2} u(x-sv,v-r\tilde z)\,dr\, e^{-isv\cdot\tilde z}\,dv \right) e^{ix\cdot\tilde z}\,d\tilde z$$
$$= \frac{1}{(2\pi s)^3} \int \mathcal{F}_{\eta\to\tilde z}E(\tilde z)\,\mathcal{F}_{v\to\tilde z}\left(\int_{-1/2}^{1/2} u(x-v,\frac{v}{s}-r\tilde z)\,dr \right) e^{ix\cdot\tilde z}\,d\tilde z\,.$$

Then,

$$\Big\| \int (\Gamma[E]u)\,(x-sv,v)\,dv\,\Big\|_{L^2(\mathbb{R}^3_x)} \;\le\; \frac{\|E\|_2}{(2\pi s)^3} \Big\| \int_{-1/2}^{1/2} \mathcal{F}_{v\to\tilde z}\Big(u(x-v,\frac{v}{s}-r\tilde z)\Big)\,dr \Big\|_{L^2(\mathbb{R}^3_{\tilde z}\times\mathbb{R}^3_x)}$$
$$\le\; \frac{\|E\|_2}{(2\pi s)^3} \left(\int_{-1/2}^{1/2} \|\mathcal{F}_{v\to\tilde z}\Big(u(x-v,\frac{v}{s}-r\tilde z)\Big)\|^2_{L^2(\mathbb{R}^3_{\tilde z}\times\mathbb{R}^3_x)} dr \right)^{\frac{1}{2}},$$

by applying Hölder's inequality first in the $\tilde z$ integral and then in the r integral. Finally, it remains to prove that

$$\int_{-1/2}^{1/2} \|\mathcal{F}_{v\to z}\Big(u(x-v,\frac{v}{s}-rz)\Big)\|^2_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_z)} dr \;=\; s^3 \|u(x,v)\|^2_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v)}.$$

This is obtained by using repeatedly Plancherel's equality:

$$\int_{-1/2}^{1/2} \|\mathcal{F}_{v\to z}\Big(u(x-v,\frac{v}{s}-rz)\Big)\|^2_{L^2_{x,z}} dr \;=\; \int_{-1/2}^{1/2} \|\mathcal{F}_{x\to\xi}\Big[\mathcal{F}_{v\to z}\Big(e^{-iv\xi}u(x,\frac{v}{s}-rz)\Big)\Big]\|^2_{L^2_{\xi,z}} dr$$
$$= \int_{-1/2}^{1/2} \|\mathcal{F}_{x\to\xi}\Big(s^3 e^{-is(\xi+z)rz}\mathcal{F}_{v\to s(\xi+z)}u(x,v)\Big)\|^2_{L^2_{\xi,z}} dr \;=\; s^6 \int_{-1/2}^{1/2} \|\mathcal{F}_{x\to\xi}\Big(\mathcal{F}_{v\to s(\xi+z)}u(x,v)\Big)\|^2_{L^2_{\xi,z}} dr$$
$$= \; s^3 \|u(x,v)\|^2_{L^2_{x,v}}.$$

□

Remark 4.7 *Observe that the exponent of the variable s recovered in the Lemma is the same as obtained for the VP case (cf. [Pe], e.g.) in the L^2-estimate of $\int_{\mathbb{R}^3_v} Eu(x-sv,v)\,dv$. In the classical case, analogous L^p-estimates hold in addition. In the quantum counterpart, instead, the L^2-framework is the only possible, since the estimate had to be derived in Fourier space. Moreover, to derive a more refined version of this basic estimate (cf. [LP, Bo1]), the non-negativity of the classical distribution is a crucial ingredient. And this non-negativity does not hold for Wigner functions.*

The following lemma is an immediate consequence of Lemma 4.6. We shall need the notation

$$V_{T,\omega} := \{E \in C((0,T];\, L^2_x(\mathbb{R}^3) \;\big|\; \|E\|_{V_{T,\omega}} < \infty\}$$

with

$$\|E\|_{V_{T,\omega}} := \sup_{0<t\le T} t^{\omega} \|E(t)\|_{L^2}.$$

Lemma 4.8 *For any fixed $T>0$, let $w \in \mathcal{C}([0,T];L^2_{x,v})$, and let w_0 satisfy for some $\omega \in [0,1)$:*

$$\Big\| \int w_0(x-tv,v)\,dv \Big\|_{L^{6/5}(\mathbb{R}^3_x)} \;\le\; C_T t^{-\omega}, \quad \forall\, t\in(0,T]. \qquad \textbf{(B)}$$

Then, there exists a unique vector-field $E \in V_{T,\omega-\frac{1}{2}}$ which satisfies the linear equation

$$E_j(x,t) \;=\; \lambda \sum_{k=1}^{3} \frac{-3x_j x_k + \delta_{jk}|x|^2}{|x|^5} *_x \int_0^t s \int \Big(\Gamma_k[E_0+E]w\Big)(x-sv,v,t-s)\,dv\,ds;\; j=1,2,3 \tag{4.16}$$

with E_0 defined by ($\lambda = \frac{1}{4\pi}$):

$$E_0(x,t) \;:=\; \lambda \frac{x}{|x|^3} *_x \int w_0(x-tv,v)\,dv.$$

Proof. (4.16) has the structure of a Volterra integral equation of the second kind. Hence, we define the (affine) map $M : V_{T,\omega-\frac{1}{2}} \to V_{T,\omega-\frac{1}{2}}$ by

$$(ME)_j(x,t) \;:=\; \lambda \sum_{k=1}^{3} \frac{-3x_j x_k + \delta_{jk}|x|^2}{|x|^5} *_x \int_0^t s \int \Big(\Gamma_k[E_0+E]w\Big)(x-sv,v,t-s)\,dv\,ds.$$

Applying the generalized Young inequality to the definition of E_0 yields

$$\|E_0(t)\|_2 \;\le\; C \Big\| \int w_0(x-vt,v)\,dv \Big\|_{L^{6/5}(\mathbb{R}^3_x)}, \quad \forall\, t \in (0,T]. \tag{4.17}$$

Thus, by Lemma 4.6, the second convolution factor in (4.16) is well-defined and

$$\Big\| \int_{\mathbb{R}^3_v} \Big(\Gamma_k[E_0+E]w\Big)(x-sv,v,t-s)\,dv \Big\|_{L^2(\mathbb{R}^3_x)} \le C s^{-3/2} \|(E_0+E)(t-s)\|_2 \|w(t-s)\|_2\,,\ \forall\, s \in (0,t].$$

By classical properties of the convolution with $\frac{1}{|x|}$ (cf. [St]) and the Young inequality, we get

$$\|(ME)_j(t)\|_2 \;\le\; C \int_0^t \frac{1}{\sqrt{s}} (\|E_0(t-s)\|_2 + \|E(t-s)\|_2) \|w(t-s)\|_2\,ds\,, \quad \forall\, t \in (0,T]. \tag{4.18}$$

Hence, the map M is well-defined from $V_{T,\omega-\frac{1}{2}}$ into itself and satisfies

$$\|ME(t)\|_2 \;\le\; C\Big(C_T + \sup_{s\in(0,T]} s^{\omega-\frac{1}{2}} \|E(s)\|_2\Big) \sup_{s\in[0,T]} \|w(s)\|_2 \left(t^{1-\omega} + t^{\frac{1}{2}-\omega}\right), \quad \forall\, t \in (0,T].$$

Since the map is affine, we have (by induction) for all $t \in (0,T]$

$$\begin{aligned}
\|M^n E(t) - M^n \widetilde{E}(t)\|_2 &\le C \sup_{s\in[0,T]} \|w(s)\|_2 \int_0^t \frac{1}{\sqrt{t-s}} \|M^{n-1}E(s) - M^{n-1}\widetilde{E}(s)\|_2\,ds \\
&\le \left(C \sup_{s\in[0,T]} \|w(s)\|_2\right)^n C_{n-1} \int_0^t \frac{s^{\frac{n}{2}-\omega}}{\sqrt{t-s}}\,ds \sup_{s\in(0,T]} \Big(s^{\omega-\frac{1}{2}} \|E(s) - \widetilde{E}(s)\|_2\Big),
\end{aligned}$$

with

$$\int_0^t \frac{s^{\frac{n}{2}-\omega}}{\sqrt{t-s}}\,ds \;=\; t^{\frac{n+1}{2}-\omega} B\left(\frac{1}{2}, \frac{n+2}{2} - \omega\right),$$

$$C_{n-1} \;=\; \prod_{j=1}^{n-1} B\left(\frac{1}{2}, \frac{j}{2} + 1 - \omega\right) \;=\; \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{3}{2} - \omega\right)}{\Gamma\left(\frac{n}{2} + 1 - \omega\right)},$$

where B denotes the Beta function and Γ the Gamma function. Thus, the map M^n is contractive for n large enough and admits a unique fixed point $E \in V_{T,\omega-\frac{1}{2}}$. □

With $E = E_1^{wp}$ the above lemma yields the regularity of the self-consistent field in the WP equation: It satisfies $\nabla_x V[w^{\mathrm{wp}}] = E_1^{wp} + E_0^{\mathrm{wp}} \in V_{T,\omega-\frac{1}{2}}$, under the assumptions that $w^{\mathrm{wp}} \in \mathcal{C}([0,T]; L^2_{x,v})$ and w_0^{wp} satisfies (**B**).

Proposition 4.9 *For any fixed $T > 0$, let $w^{wp} \in \mathcal{C}([0,T]; L^2_{x,v})$ be a mild solution of the WP equation with $\|w^{wp}(t)\|_2 = \|w_0^{wp}\|_2$, and with the initial value w_0^{wp} satisfying condition (**B**). Then, the self-consistent field satisfies the following estimates for all $t \in (0,T]$:*

$$\begin{aligned} \|E_0^{wp}(t)\|_2 &\leq C\Big\|\int w_0^{wp}(x-vt,v)\,dv\Big\|_{L^{6/5}(\mathbb{R}^3_x)} \leq CC_T t^{-\omega}, &(4.19)\\ \|E_1^{wp}(t)\|_2 &\leq C\left(\|w_0^{wp}\|_2, \sup_{s\in(0,T]}\left\{s^\omega\Big\|\int w_0^{wp}(x-sv,v)dv\Big\|_{L^{6/5}}\right\}, T\right) t^{\frac{1}{2}-\omega}. &(4.20)\end{aligned}$$

Here and in the sequel, the T-dependence of the constants C is continuous (on $T \in \mathbb{R}^+$).

Proof. The first estimate is (4.17) in Lemma 4.8. To derive the second one, we exploit eq. (4.18), the conservation of the L^2-norm of the solution and (4.19):

$$\begin{aligned} \|E_1^{\rm wp}(t)\|_2 &\leq C\int_0^t s^{-1/2}(\|E_0^{\rm wp}(t-s)\|_2 + \|E_1^{\rm wp}(t-s)\|_2)\|w^{\rm wp}(t-s)\|_2\,ds \\ &\leq C\|w_0^{\rm wp}\|_2 \sup_{s\in(0,T]}\left\{s^\omega\Big\|\int w_0^{\rm wp}(x-sv,v)dv\Big\|_{L^{6/5}}\right\} t^{\frac{1}{2}-\omega} \\ &\quad + C\|w_0^{\rm wp}\|_2\int_0^t (t-s)^{-1/2}\|E_1^{\rm wp}(s)\|_2\,ds. \end{aligned}$$

The thesis follows by Gronwall's Lemma. □

We shall now state a simple condition on w_0 that implies both conditions (A), (B). For $w_0 \in L^1_x(L^{6/5}_v)$ a Strichartz inequality for the free transport equation (cf. Th. 2 in [CP]) reads:

$$\Big\|\int w_0(x-tv,v)\,dv\Big\|_{L^{6/5}(\mathbb{R}^3_x)} \leq t^{-\frac{1}{2}}\|w_0\|_{L^1_x(L^{6/5}_v)},\ \ t>0, \tag{4.21}$$

and hence (A) and (B) hold.

Remark 4.10 *Let us again compare the a-priori bounds (4.19), (4.20) with their classical counterparts. Using (4.21) we obtain the same $t^{-\frac{1}{2}}$–singularity of $\|E^{wp}(t)\|_2$ for the Wigner-Poisson system, as it was obtained in [CP] for the VP equation. In the latter case, similar L^p-estimates hold for p in a non-trivial interval. One crucial reason for this difference is the conservation of L^p-norm of the solution: while the WP equation only conserves the L^2-norm, all L^p-norms are constant in the VP case. A second reason is that we cannot exploit any pseudo-conformal law for the quantum case, since the Wigner functions are not non-negative (cf. [Pe] for the classical case).*

As a by-product we obtain the following result for the self-consistent potential V, which follows directly from the splitting $V^{\rm wp} = V_0^{\rm wp} + V_1^{\rm wp}$

$$\begin{aligned} V_0^{\rm wp}(x,t) &:= \lambda\sum_{i=1}^3 \frac{x_i}{|x|^3} *_x (E_0^{\rm wp})_i(x,t), &(4.22)\\ V_1^{\rm wp}(x,t) &:= \lambda\sum_{i=1}^3 \frac{x_i}{|x|^3} *_x (E_1^{\rm wp})_i(x,t). &(4.23)\end{aligned}$$

Corollary 4.11 *Under the assumptions of Proposition 4.9, the self-consistent potential $V^{wp} = V_0^{wp} + V_1^{wp}$ satisfies the following estimates for all $t \in (0,T]$:*

$$\begin{aligned} \|V_0^{wp}(t)\|_6 &\leq CC_T t^{-\omega}, &(4.24)\\ \|V_1^{wp}(t)\|_6 &\leq C\left(\|w_0^{wp}\|_2, \sup_{s\in(0,T]}\left\{s^\omega\Big\|\int w_0^{wp}(x-sv,v)dv\Big\|_{L^{6/5}}\right\}, T\right) t^{\frac{1}{2}-\omega}. &(4.25)\end{aligned}$$

4.2 A-priori estimates for the electric field: the WPFP case

According to Corollary 3.5, the mild solution of the WPFP problem satisfies for all $t \in [0, T]$ $(0 < T < t_{\max})$

$$\begin{aligned} w(x,v,t) &= \iint G(t,x,v,x_0,v_0) w_0(x_0,v_0)\, dx_0\, dv_0 \\ &\quad + \int_0^t \iint G(s,x,v,x_0,v_0)(\Theta[V]w)(x_0,v_0,t-s)\, dx_0\, dv_0\, ds \end{aligned}$$

with the Green's function G from (3.4). According to [SCDM] we have

$$\int_{\mathbb{R}^3} G(t,x,v,x_0,v_0)\, dv = R(t)^{-3/2} \mathcal{N}\left(\frac{x - x_0 - \vartheta(t) v_0}{\sqrt{R(t)}} \right),$$

with

$$\mathcal{N}(x) := (2\pi)^{-3/2} \exp\left(-\frac{|x|^2}{2} \right), \tag{4.26}$$

$$\vartheta(t) = \frac{1 - e^{-\beta t}}{\beta} = \mathcal{O}(t), \quad \text{for } t \to 0, \tag{4.27}$$

$$R(t) := 2\alpha t + \sigma \left[\frac{4e^{-\beta t} - e^{-2\beta t} + 2\beta t - 3}{\beta^3} \right] + 4\gamma \left[\frac{e^{-\beta t} + \beta t - 1}{\beta^2} \right] = \mathcal{O}(t), \quad \text{for } t \to 0. \tag{4.28}$$

By exploiting the redefinition (4.11) of the pseudo-differential operator, we obtain the following expression for the density $n[w]$

$$\begin{aligned} n[w](x,t) &= \int_{\mathbb{R}^3} w(x,v,t)\, dv \\ &= \frac{1}{R(t)^{3/2}} \iint \mathcal{N}\left(\frac{x - x_0 - \vartheta(t) v_0}{\sqrt{R(t)}} \right) w_0(x_0,v_0)\, dx_0\, dv_0 \\ &\quad + \int_0^t \frac{1}{R(s)^{3/2}} \iint \mathcal{N}\left(\frac{x - x_0 - \vartheta(s) v_0}{\sqrt{R(s)}} \right) \operatorname{div}_{v_0} \left(\Gamma[\nabla_{x_0} V] w \right)(x_0,v_0,t-s)\, dx_0\, dv_0\, ds \\ &= \frac{1}{R(t)^{3/2}} \iint \mathcal{N}\left(\frac{x - x_0}{\sqrt{R(t)}} \right) w_0(x_0 - \vartheta(t) v_0, v_0)\, dx_0\, dv_0 \\ &\quad + \int_0^t \frac{\vartheta(s)}{R(s)^2} \iint (\nabla_x \mathcal{N}) \left(\frac{x - x_0 - \vartheta(s) v_0}{\sqrt{R(s)}} \right) \cdot \left(\Gamma[\nabla_{x_0} V] w \right)(x_0,v_0,t-s)\, dx_0\, dv_0\, ds \\ &= n_0(x,t) + n_1(x,t), \end{aligned}$$

where

$$\begin{aligned} n_0(x,t) &:= \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}} \right) *_x n_0^\vartheta(x,t), \quad \text{with} \quad n_0^\vartheta(x,t) := \int w_0(x - \vartheta(t)v, v)\, dv, \\ n_1(x,t) &:= \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}} \right) *_x \operatorname{div}_x \int \Big(\Gamma[\nabla_x V] w \Big)(x - \vartheta(s)v, v, t-s)\, dv\, ds. \end{aligned}$$

Correspondingly, we can split the field (with $\lambda = \frac{1}{4\pi}$):

$$E_0(x,t) := \lambda \frac{x}{|x|^3} *_x n_0(x,t) = \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}} \right) *_x E_0^\vartheta(x,t), \tag{4.29}$$

$$\text{with} \qquad E_0^\vartheta(x,t) \;:=\; \lambda \frac{x}{|x|^3} *_x n_0^\vartheta(x,t),$$

$$E_1(x,t) \;:=\; \lambda \frac{x}{|x|^3} *_x n_1(x,t). \tag{4.30}$$

Remark 4.12 *Note that the splitting of the density (and of the electric field) is the same as in [Bo1, Bo2, Ca2]: in the WPFP case the two components of the decomposition (n_0, n_1, as well as E_0, E_1) are smoothed versions (in fact, convolutions with a Gaussian) of those appearing in the WP case (namely $n_0^{wp}, n_1^{wp}, E_0^{wp}, E_1^{wp}$). Actually, the density $n_0^\vartheta(x,t)$ (which is convoluted with the Gaussian to give n_0) already differs from $n_0^{wp}(x,t)$ in the WP case because the shift contains the function ϑ, which is due to friction (and analogously for $E_0^\vartheta(x,t)$ and $E_0^{wp}(x,t)$).*

From Lemma 4.6 we directly get

$$\Big\| \int_{\mathbb{R}_v^3} (\Gamma[E]u)\,(x-\vartheta(s)v, v, t-s)\,dv \Big\|_{L^2(\mathbb{R}_x^3)} \;\le\; C\vartheta(s)^{-3/2}\|E(t-s)\|_2\|u(t-s)\|_2\,, \quad \forall\, t \ge s > 0. \tag{4.31}$$

To derive an L^2-estimate on the field we shall proceed as in the WP case (Lemma 4.8, Proposition 4.9).

Lemma 4.13 *Let w be the mild solution of the WPFP equation (3.7) and let $w_0 \in X$ satisfy ($\mathbf{A}$) for some $\omega \in [0,1)$. For any fixed $T > 0$ the electric field then satisfies $\nabla_x V \in V_{T,\omega-\frac{1}{2}}$ and the following estimates hold:*

1. *for $2 \le p \le 6$, $\theta = \frac{3(p-2)}{2p}$*

$$\|E_0(t)\|_p \;\le\; C(T)\|w_0\|_X^\theta \|n_0^\vartheta(t)\|_{L^{6/5}}^{1-\theta} = \mathcal{O}(t^{-\omega(1-\theta)}), \quad \forall\, t \in (0,T]; \tag{4.32}$$

2.

$$\|E_1(t)\|_2 \;\le\; C\left(T, \|w_0\|_2\,, \sup_{s\in(0,T]} \left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right) t^{\frac{1}{2}-\omega}, \quad \forall\, t \in (0,T]. \tag{4.33}$$

Proof. The estimate for $\|E_0(t)\|_p$, $p \in [2,6]$ is obtained by applying first the generalized Young inequality and then the Young inequality to the expression (4.29)

$$\begin{aligned}
\|E_0(t)\|_p \;&\le\; C\left\| \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right) *_x n_0^\vartheta(x,t)\right\|_q \\
&\le\; C\left\| \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right)\right\|_1 \|n_0^\vartheta(x,t)\|_q \\
&=\; C\|n_0^\vartheta(t)\|_q, \quad \text{with} \quad q = \frac{3p}{p+3} \in [6/5, 2].
\end{aligned}$$

Next we interpolate n_0^ϑ between L^2 and $L^{6/5}$, use (2.3) and the dissipativity of the operator $-v\cdot\nabla_x - \frac{3}{2}$ in X (cf. Lemma 2.8):

$$\begin{aligned}
\|n_0^\vartheta(t)\|_q \;&\le\; C\|w_0(x-\vartheta(t)v, v)\|_X^\theta \|n_0^\vartheta(t)\|_{6/5}^{1-\theta} \\
&\le\; C\, e^{\frac{3}{2}\theta\vartheta(t)} \|w_0\|_X^\theta \|n_0^\vartheta(t)\|_{6/5}^{1-\theta},
\end{aligned}$$

with $\theta = \frac{5}{2} - \frac{3}{q}$. Hence

$$\|E_0(t)\|_p \;\le\; C(T)\|w_0\|_X^\theta \|n_0^\vartheta(t)\|_{L^{6/5}}^{1-\theta}.$$

We rewrite the function $E_1(x,t)$ as

$$(E_1)_j(x,t) \;=\; \lambda \sum_{k=1}^{3} \frac{-3x_jx_k+\delta_{jk}|x|^2}{|x|^5} *_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F_k(x,t,s)\,ds, \quad (4.34)$$

$$\text{with} \quad F_k(x,t,s) \;:=\; \int (\Gamma_k[E_0+E_1]w)\,(x-\vartheta(s)v,v,t-s)\,dv,$$

For estimating it we exploit classical properties of the convolution with the kernel $\frac{1}{|x|}$ and apply the Young inequality:

$$\begin{aligned}
\|E_1(t)\|_2 &\le C\int_0^t \vartheta(s)\left\|\frac{1}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F(x,t,s)\right\|_2 ds \\
&\le C\int_0^t \vartheta(s)\left\|\frac{1}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right\|_1 \|F(x,t,s)\|_2\,ds \\
&\le C(T)\|w_0\|_2\int_0^t \frac{\|E_0(t-s)\|_2+\|E_1(t-s)\|_2}{\sqrt{\vartheta(s)}}\,ds,
\end{aligned}$$

where the last inequality follows from (4.31) and the L^2–a-priori estimate on the solution w (cf. Lemma 4.2). By applying the estimate (4.32) to $\|E_0(t)\|_2$, we get

$$\begin{aligned}
\|E_1(t)\|_2 \;\le\; C(T)\|w_0\|_2 \Bigg(&\sup_{t\in(0,T]}\left\{\vartheta(t)^\omega \|n_0^\vartheta(t)\|_{L^{6/5}}\right\}\int_0^t \vartheta(s)^{-\frac12}\vartheta(t-s)^{-\omega}\,ds \\
&+\int_0^t \frac{\|E_1(t-s)\|_2}{\sqrt{\vartheta(s)}}\,ds\Bigg), \qquad (4.35)
\end{aligned}$$

where the function $\vartheta(s)=\mathcal{O}(s)$ as $s\to 0$. Thus the integrals are finite.
To establish a solution of (4.34) we introduce the fixed point map

$$\begin{aligned}
(ME)_j(x,t) \;:=\; &\lambda\sum_{k=1}^{3}\frac{-3x_jx_k+\delta_{jk}|x|^2}{|x|^5} *_x \\
&*_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x \int (\Gamma_k[E_0+E]w)\,(x-\vartheta(s)v,v,t-s)\,dv\,ds.
\end{aligned}$$

By using $0<\frac{\vartheta(T)}{T}t\le\vartheta(t)$, $\forall\, t\in(0,T]$ and (4.35), a simple fixed point argument as in the proof of Lemmma 4.8 with the contractivity estimate:

$$\|M^nE(t)-M^n\widetilde{E}(t)\|_2 \le \left(C\sqrt{\frac{T}{\vartheta(T)}}\|w_0\|_2\right)^n t^{\frac{n+1}{2}-\omega}\,\frac{\pi^{\frac n2}\Gamma\left(\frac32-\omega\right)}{\Gamma\left(\frac{n+3}{2}-\omega\right)}\sup_{s\in(0,T]}\left(s^{\omega-\frac12}\|E(s)-\widetilde{E}(s)\|_2\right)$$

shows that the linear equation (4.34) has a unique solution $E_1\in V_{T,\omega-\frac12}$. Hence $\nabla_x V = E_0+E_1\in V_{T,\omega-\frac12}$ and Gronwall's Lemma then yields estimate (4.33). □

Remark 4.14 *For the derivation of the a-priori bound on $\|E\|_2$, we did not use any moments of w (neither in x nor v), nor pseudo-conformal laws (cf. [Bo1, Bo2, Pe, Ca2] for the classical analogue, i.e. VPFP). In fact, the latter are not useful in the quantum case, since the Wigner function typically also takes negative values. Moreover, the convolution with the Gaussian did not play a role there; the estimate (4.33) relies just on the dispersive effect of the free-streaming operator. The parabolic regularization will be exploited in the "post-processing" Proposition 4.15.*

The above lemma was the first crucial step towards proving global existence of the WPFP solution. Next we shall extend this estimates on the field E to a range of L^p-norms:

Proposition 4.15 *Let w be the mild solution of the WPFP equation (3.7) and let $w_0 \in X$ satisfy (**A**) for some $\omega \in [0,1)$. Then, we have for any fixed $T > 0$ and for all $p \in [2,6)$:*

$$\|E_1(t)\|_{L^p} \;\leq\; C\left(T, \|w_0\|_2, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right) t^{\frac{3}{2p}-\frac{1}{4}-\omega}, \quad \forall\, t \in (0,T]. \tag{4.36}$$

Proof. We shall estimate $E_1(t)$ (cf. (4.34)) by using classical properties of the convolution by the kernel $\frac{1}{|x|}$ and the following

$$\left\|\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right\|_q \;=\; C R(s)^{\frac{3}{2q}},\ \forall\, 1 \leq q \leq \infty\,. \tag{4.37}$$

Namely,

$$\begin{aligned}
\|E_1[w](t)\|_{L^p(\mathbb{R}^3_x)} &\leq C\int_0^t \vartheta(s)\left\|\frac{1}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F(x,t,s)\right\|_{L^p(\mathbb{R}^3_x)} ds \\
&\leq C\int_0^t \frac{R(s)^{\frac{3}{2q}-\frac{3}{2}}}{\sqrt{\vartheta(s)}}\Big(\|E_0(t-s)\|_2 + \|E_1[w](t-s)\|_2\Big)\|w(t-s)\|_2\, ds,
\end{aligned}$$

where we used the Young inequality with $1 + 1/p = 1/q + 1/2$ (thus, $p \geq 2$) and, for the L^2-norm, the Lemma 4.6. Then, by applying Lemma 4.13 we get

$$\begin{aligned}
\|E_1[w](t)\|_{L^p(\mathbb{R}^3_x)} \;\leq\; & C\left(T, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\right\}, \|w_0\|_{L^2(\mathbb{R}^6)}\right)\cdot \\
& \cdot\int_0^t \frac{R(s)^{\frac{3}{2q}-\frac{3}{2}}}{\sqrt{\vartheta(s)}}\left(\vartheta(t-s)^{-\omega} + (t-s)^{\frac{1}{2}-\omega}\right) ds
\end{aligned}$$

Since $\vartheta(t) = \mathcal{O}(t)\,, R(t) = \mathcal{O}(t)$ for $t \to 0$ (cf. (4.27), (4.28)), the last integral is finite for all $t > 0$ and for $3/(2q) - 2 > -1 \;\Leftrightarrow\; 3/(2p) - 5/4 > -1 \;\Leftrightarrow\; p < 6$. In fact the integral is $\mathcal{O}(t^{\frac{3}{2p}-\frac{1}{4}-\omega})$.

$\square$

Remark 4.16 *Proposition 4.15 provides a non-trivial interval of L^p-estimates for the electric field in the WPFP case. This is due to the regularizing effect of the FP term. We remark that the corresponding Gaussian is "better behaved" than the classical one, since the quantum FP operator is uniformly elliptic in both x and v variables. On the other hand, exactly as in the WP case, the range of L^p-estimates for the WPFP equation is smaller in comparison to the counterpart VPFP and that depends again on the non-negativity of the classical distribution function.*

As a further result, we obtain an a-posteriori information on the self-consistent potential V, which follows directly from the a-priori estimates on the field. Accordingly, we split the potential as $V = V_0 + V_1$, with

$$V_0(x,t) \;:=\; \lambda\sum_{i=1}^3 \frac{x_i}{|x|^3} *_x (E_0)_i(x,t), \tag{4.38}$$

$$V_1(x,t) \;:=\; \lambda\sum_{i=1}^3 \frac{x_i}{|x|^3} *_x (E_1)_i(x,t). \tag{4.39}$$

Corollary 4.17 *Under the assumptions of Proposition 4.15, the self-consistent potential* $V(t) = V_0(t) + V_1(t)$ *belongs to* $L^p(\mathbb{R}^3_x)$ *with* $6 \leq p \leq \infty$*, and satisfies for all* $t \in (0, T]$:

$$\|V_0(t)\|_p \leq C(T)\|w_0\|_X^\theta \|n_0^\vartheta(t)\|_{L^{6/5}}^{1-\theta} = \mathcal{O}(t^{-\omega(1-\theta)}) \quad \text{with } \theta = \frac{1}{2} - \frac{3}{p}, \tag{4.40}$$

$$\|V_1(t)\|_p \leq C\left(T, \|w_0\|_2, \sup_{s\in(0,T]} \left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right) t^{\frac{1}{2}-\omega}. \tag{4.41}$$

4.3 A-priori estimates for the weighted L^2-norms

A first consequence of the a-priori estimates for the electric field is the following

Lemma 4.18 *For all* $w_0 \in X$ *such that* (**A**) *holds for some* $\omega \in [0, 1)$*, the mild solution of the WPFP equation (3.7) satisfies*

$$\|vw(t)\|_2^2 \leq C\left(T, \|w_0\|_X, \sup_{s\in(0,T]} \left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right), \quad \forall t \in [0, T]. \tag{4.42}$$

Proof. In order to justify the derivation of this a-priori estimate we need again the approximating classical solutions y_n introduced in the proof of Lemma 4.2. Mutiplying both sides of (4.3) by $v_i^2 y_n(t)$ and integrating yields

$$\frac{1}{2}\frac{d}{dt}\|v_i y_n(t)\|_2^2 = \iint v_i^2 y_n(t)\overline{A}y_n(t)\,dx\,dv + \iint v_i^2 y_n(t) f_n(t)\,dx\,dv.$$

By analogous calculations as in the proof of Lemma 2.8 (cf. also (5.3)) we get,

$$\iint |v|^2 y_n(t)\overline{A}y_n(t)\,dx\,dv \leq 3\sigma\|y_n(t)\|_2^2 + \frac{\beta}{2}\|vy_n(t)\|_2^2,$$

and hence

$$\frac{1}{2}\frac{d}{dt}\|vy_n(t)\|_2^2 \leq 3\sigma\|y_n(t)\|_2^2 + \frac{\beta}{2}\|vy_n(t)\|_2^2 + \iint |v|^2 y_n(t) f_n(t)\,dx\,dv, \quad \forall t \in [0, T].$$

By integrating in t, letting $n \to \infty$, and using (4.1), we have

$$\begin{aligned}\|vw(t)\|_2^2 &\leq \|vw_0\|_2^2 + \frac{2\sigma}{\beta}(e^{3\beta t} - 1)\|w_0\|_2^2 + \beta\int_0^t \|vw(s)\|_2^2\,ds \\ &\quad + 2\int_0^t\iint |v|^2 w(s) f(s)\,dx\,dv\,ds, \quad \forall t \in [0, T].\end{aligned}$$

Using again the skew-symmetry of the pseudo-differential operator and the Hölder inequality yields

$$\begin{aligned}\int_0^t\iint v_i w(s) v_i f(s)\,dx\,dv\,ds &= \frac{1}{2}\int_0^t\iint v_i w(s)\Omega[\partial_i V[w(s)]]w(s)\,dx\,dv\,ds \\ &\leq \frac{1}{2}\int_0^t \|v_i w(s)\|_2\,\|\Omega[\partial_i V[w(s)]]w(s)\|_2\,ds,\end{aligned}$$

with the operator Ω defined in (2.10). Estimating as in (2.12) and using the Sobolev inequality we obtain for $t \in [0, T]$:

$$\begin{aligned}\|\Omega[\partial_i V[w(t)]]w(t)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v)} &\leq C\|\partial_i V[w(t)]\hat{w}(t)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\eta)} \\ &\leq C\|\partial_i V[w(t)]\|_3\|\hat{w}(t)\|_{L^2(\mathbb{R}^3_x;L^6(\mathbb{R}^3_\eta))} \\ &\leq C\|\partial_i V[w(t)]\|_3\|\nabla_\eta\hat{w}(t)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\eta)} \\ &\leq C\|\partial_i V[w(t)]\|_3\|vw(t)\|_2,\end{aligned} \tag{4.43}$$

where $\hat{w}(x,\eta,t) := \mathcal{F}_{v\to\eta}(w(x,v,t))$. Finally, using Prop. 4.15 (estimate (4.36) with $p=3$) yields

$$\begin{aligned}\|vw(t)\|_2^2 \quad &\le \quad C(T)\left(\|vw_0\|_2^2 + \|w_0\|_2^2\right) + C\left(T, \|w_0\|_X, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right)\\ &\cdot \int_0^t \left(s^{-\frac{\omega}{2}} + s^{-\omega+\frac{1}{4}} + \beta\right)\|vw(s)\|_2^2\, ds, \quad t\in[0,T], \qquad (4.44)\end{aligned}$$

and the Gronwall Lemma gives the result. □

With this result we can proceed to derive the a-priori estimate for $\||v|^2 w(t)\|_2$.

Lemma 4.19 *For all $w_0 \in X$ such that* (**A**) *holds for some $\omega\in[0,1)$, the mild solution of the WPFP equation (3.7) satisfies*

$$\||v|^2 w(t)\|_2^2 \quad \le \quad C\left(T, \|w_0\|_X, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right), \quad \forall\, t\in[0,T]. \qquad (4.45)$$

Proof. In order to control the term $\||v|^2 w(t)\|_2$, we shall use the same strategy as in the Lemmata 4.2 and 4.18. Multiplying both sides of (4.3) by $v_i^4 y_n(t)$ and integrating we get by using (5.3) and repeating the same limit procedure as in the previous lemma:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3 \iint v_i^4 w(t)^2\, dx\, dv \quad &\le \quad 9\sigma\|w(t)\|_2^2 + \left(3\sigma - \frac{1}{2}\beta\right)\sum_{i=1}^3 \iint v_i^4 w(t)^2\, dx\, dv\\ &+\sum_{i=1}^3 \iint v_i^4 w(t) f(t)\, dx\, dv, \quad \forall\, t\in[0,T].\end{aligned}$$

By integrating in t, using $C_1|v|^4 \le \sum v_i^4 \le C_2|v|^4$ and (4.1), we have

$$\begin{aligned}\||v|^2 w(t)\|_2^2 \quad &\le \quad C\Bigg(\||v|^2 w_0\|_2^2 + \frac{6\sigma}{\beta}(e^{3\beta t}-1)\|w_0\|_2^2 + (6\sigma-\beta)\int_0^t \||v|^2 w(s)\|_2^2\, ds\\ &+2\sum_{i=1}^3 \int_0^t \iint v_i^4 w(s) f(s)\, dx\, dv\, ds\Bigg), \quad \forall\, t\in[0,T]. \qquad (4.46)\end{aligned}$$

Using again the skew-symmetry of the pseudo-differential operator Θ, the equation (2.9) and the Hölder inequality, we have

$$\begin{aligned}\int_0^t \iint v_i^2 w(s) v_i^2 f(s) dx\, dv\, ds \quad &\le \quad \frac{1}{4}\int_0^t \|v_i^2 w(s)\|_2\, \|\Theta[\partial_i^2 V[w(s)]]w(s)\|_2\, ds\\ &+\int_0^t \|v_i^2 w(s)\|_2\, \|\Omega[\partial_i V[w(s)]]v_i w(s)\|_2\, ds. \qquad (4.47)\end{aligned}$$

Since $\hat{w}(x,.,t)\in H^2(\mathbb{R}^3_\eta)$, the Gagliardo-Nirenberg inequality yields for $t\in[0,T]$

$$\begin{aligned}\|\hat{w}(x,.,t)\|_{L^\infty(\mathbb{R}^3_\eta)} \quad &\le \quad C\|\hat{w}(x,.,t)\|_{L^6(\mathbb{R}^3_\eta)}^{1/2}\|\widehat{|v|^2 w}(x,.,t)\|_{L^2(\mathbb{R}^3_\eta)}^{1/2}\\ &\le \quad C\|\widehat{vw}(x,.,t)\|_{L^2(\mathbb{R}^3_\eta)}^{1/2}\|\widehat{|v|^2 w}(x,.,t)\|_{L^2(\mathbb{R}^3_\eta)}^{1/2}. \qquad (4.48)\end{aligned}$$

Using

$$\|\Delta V[w(t)]\|_2 \;=\; \|n[w(t)]\|_2 \;=\; C\|\hat{w}(.,\eta=0,t)\|_{L^2(\mathbb{R}^3_x)} \;\le\; C\left(\int \|\hat{w}(x,.,t)\|_{L^\infty(\mathbb{R}^3_\eta)}^2\, dx\right)^{1/2},$$

(4.48), the Hölder inequality, and (4.42) we can estimate:

$$\begin{aligned}\|\Theta[\partial_i^2 V[w(t)]]w(t)\|_2 &\le C\|\Delta V[w(t)]\|_2\Big(\int\|\hat{w}(x,.\,,t)\|^2_{L^\infty(\mathrm{I\!R}^3_\eta)}\,dx\Big)^{1/2}\\ &\le C\int\|\hat{w}(x,.\,,t)\|^2_\infty\,dx\\ &\le C\int\|\widehat{vw}(x,.\,,t)\|_{L^2(\mathrm{I\!R}^3_\eta)}\,\|\widehat{|v|^2w}(x,.\,,t)\|_{L^2(\mathrm{I\!R}^3_\eta)}\,dx\\ &\le C\Big(T,\|w_0\|_X,\sup_{s\in(0,T]}\Big\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\Big\}\Big)\,\||v|^2w(t)\|_2. \quad (4.49)\end{aligned}$$

For the second term of the r.h.s. of (4.47) we proceed as in (4.43) and use the estimate (4.36):

$$\|\Omega[\partial_i V[w(t)]]v_i w(t)\|_2 \le C\|\partial_i V[w(t)]\|_3\|\widehat{v_i w}(t)\|_{L^2(\mathrm{I\!R}^3_x;L^6(\mathrm{I\!R}^3_\eta))}$$

$$\le C\Big(T,\|w_0\|_X,\sup_{s\in(0,T]}\Big\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\Big\}\Big)\Big(t^{-\frac{\omega}{2}}+t^{-\omega+\frac14}\Big)\,\||v|^2w(t)\|_2, \quad \forall\, t\in[0,T]. \quad (4.50)$$

Analogously to (4.44), combining the estimates (4.47), (4.49) and (4.50) the Gronwall Lemma gives the assertion. □

Proof of Theorem 4.1.
The Lemmata 4.2 and 4.19 show that

$$\|w(t)\|_X \le C\Big(T,\|w_0\|_X,\sup_{s\in(0,T]}\Big\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\Big\}\Big), \quad \forall\, t\in[0,T], \quad \forall\, 0<T<t_{\max},$$

with C being continuous in $T\in[0,t_{\max}]$. Then, Corollary 3.5 shows that the mild solution w exists on $[0,\infty)$. □

4.4 Regularity

The following result concerns the smoothness of the solution of WPFP, the macroscopic density and the force field, for positive times.

Corollary 4.20 *Under the assumptions of Theorem 4.1, the mild solution of the WPFP equation (3.7) satisfies*

$$w\in\mathcal{C}((0,\infty);\mathcal{C}^\infty_{\mathcal{B}}(\mathrm{I\!R}^6)),$$

$$n(t),E(t),V(t)\in\mathcal{C}((0,\infty);\mathcal{C}^\infty_{\mathcal{B}}(\mathrm{I\!R}^3)).$$

Proof. Obviously, $w(t)\in\mathcal{C}(\mathrm{I\!R}^6)$ $\forall t>0$, because of the Green's function representation in (3.22), (3.3). If we differentiate equation (3.7) with respect to x_i and, resp., v_i, we obtain the following linear, inhomogeneous problems for any fixed $t_1>0$.

$$\begin{aligned} z_t(t) &= \overline{A}z(t)+\Theta[V[z(t)]]w(t)+\Theta[V[w(t)]]z(t), \quad \forall\, t>t_1, \quad z(t_1)=\partial_{x_i}w(t_1)\in X,\\ y_t(t) &= \overline{A}y(t)+\beta y(t)-\partial_{x_i}w(t)+\Theta[V[w(t)]]y(t), \quad \forall\, t>t_1, \quad y(t_1)=\partial_{v_i}w(t_1)\in X.\end{aligned}$$

By arguments analogous to Lemma 3.3, there exists a unique mild solution

$$z = \partial_{x_i} w \in \mathcal{C}([t_1, \infty); H^1(\mathrm{I\!R}^6; (1+|v|^2)^2\, dx\, dv)). \tag{4.51}$$

By an induction procedure, the derivatives $\nabla_x^\alpha \nabla_v^\beta w$, for $\alpha, \beta \in \mathrm{I\!N}^3$, $|\alpha| + |\beta| = m > 1$ are also mild solutions of similar problems with additional well-defined inhomogeneities and with initial times $0 < t_1 < t_2 < ... < t_m$. This yields $\nabla_x^\alpha \nabla_v^\beta w \in \mathcal{C}([t_m, \infty); H^1(\mathrm{I\!R}^6; (1+|v|^2)^2\, dx\, dv))$, and thus $\nabla_x^\alpha \nabla_v^\beta w \in \mathcal{C}((0,\infty); X)$. Hence, the statement about smoothness of the density and the electric field is straightforward from Propositions 2.1 and 2.3 and Sobolev embeddings.

□

5 Appendix

Proof of Lemma 2.8

For $u \in D(A)$ we have

$$< Au, u >_{\tilde{X}} \;=\; < Au, u >_{L^2(\mathrm{I\!R}^6)} + \sum_{i=1}^{3} \iint v_i^4 u A u, \tag{5.1}$$

where $\iint f$ denotes the integral $\int_{\mathrm{I\!R}^3}\int_{\mathrm{I\!R}^3} f(x,v)\, dv\, dx$, and the norm $\|.\|_{\tilde{X}}$ is defined by (2.7). Using integrations by parts we shall calculate the three terms on the right hand side separately.

$$\begin{aligned}
< Au, u >_{L^2(\mathrm{I\!R}^6)} \;&=\; \sum_{i=1}^{3} \Big(-\iint v_i u_{x_i} u + \beta \iint (v_i u)_{v_i} u \\
&\quad + \sigma \iint u_{v_i v_i} u + 2\gamma \iint u_{x_i v_i} u + \alpha \iint u_{x_i x_i} u \Big) \\
&\le\; \sum_{i=1}^{3} \Big[3\beta \iint u^2 + \beta \iint v_i u_{v_i} u - \sigma \iint u_{v_i}^2 \\
&\quad + \gamma \left(\epsilon \iint u_{x_i}^2 + \frac{1}{\epsilon} \iint u_{v_i}^2 \right) - \alpha \iint u_{x_i}^2 \Big] \\
&=\; \frac{3}{2} \beta \|u\|_2^2 + \left(\frac{\gamma}{\epsilon} - \sigma \right) \|\nabla_v u\|_2^2 + (\epsilon\gamma - \alpha) \|\nabla_x u\|_2^2 .
\end{aligned}$$

With $\epsilon = \frac{\gamma}{\sigma}$ we obtain

$$< Au, u >_{L^2(\mathrm{I\!R}^6)} \;\le\; \frac{3}{2} \beta \|u\|_2^2 . \tag{5.2}$$

Next we estimate the second term of (5.1):

$$\begin{aligned}
\sum_{i=1}^{3} \iint v_i^4 u A u \;&=\; \sum_{i,j=1}^{3} \Big(-\iint v_i^4 v_j u_{x_j} u + \beta \iint v_i^4 (v_j u)_{v_j} u \\
&\quad + \sigma \iint v_i^4 u_{v_j v_j} u + 2\gamma \iint v_i^4 u_{x_j v_j} u + \alpha \iint v_i^4 u_{x_j x_j} u \Big) \\
&\le\; \sum_{i,j=1}^{3} \Big[\beta \iint v_i^4 u^2 + \beta \iint v_i^4 v_j u_{v_j} u - \sigma \iint v_i^4 u_{v_j}^2 \\
&\quad - \frac{4}{3} \sigma \iint v_i^3 u_{v_i} u + \gamma \left(\epsilon \iint v_i^4 u_{x_j}^2 + \frac{1}{\epsilon} \iint v_i^4 u_{v_j}^2 \right) - \alpha \iint v_i^4 u_{x_j}^2 \Big]
\end{aligned}$$

$$\leq \sum_{i=1}^{3}\left(-\frac{1}{2}\beta\iint v_i^4 u^2 + 6\sigma\iint v_i^2 u^2\right)$$

$$\leq 9\sigma\|u\|_2^2 + \left(-\frac{1}{2}\beta + 3\sigma\right)\sum_{i=1}^{3}\iint v_i^4 u^2, \tag{5.3}$$

by choosing $\epsilon = \frac{\gamma}{\sigma}$ and by an interpolation.
Collecting the two estimates yields

$$< Au, u >_{\tilde{X}} \;\leq\; \left(\frac{3}{2}\beta + 9\sigma\right)\|u\|_2^2 + 3\sigma\sum_{i=1}^{3}\iint v_i^4 u^2 \;\leq\; \left(\frac{3}{2}\beta + 9\sigma\right)\|u\|_{\tilde{X}}^2.$$

Thus, the operator $A - \kappa I$ is dissipative. □

Proof of Lemma 2.10

To prove the assertion we shall construct for each $f \in D(\overline{P}) \subset L^2(\mathbb{R}^6)$ a sequence $\{f_n\} \subset D(P)$ such that $f_n \to f$ in the graph norm
$\|f\|_P = \|f\|_{L^2} + \||x|^2 f\|_{L^2} + \||v|^2 f\|_{L^2} + \|Pf\|_{L^2} + \||x|^2 Pf\|_{L^2} + \||v|^2 Pf\|_{L^2}$.
To shorten the proof we shall consider here only the case

$$P = \theta + \nu v\cdot\nabla_x + \mu x\cdot\nabla_v + \beta v\cdot\nabla_v + \alpha\Delta_x + \sigma\Delta_v + \gamma\,\mathrm{div}_v\nabla_x$$

(cf. the definition of the operator A in (2.13)), but exactly the same strategy extends to the case, where P is a general quadratic polynomial.
First we define the mollifying delta sequence

$$\phi_n(x, v) := n^6\phi(nx, nv), \qquad n \in \mathbb{N},\ x, v \in \mathbb{R}^3,$$

where

$$\phi \in C_0^\infty(\mathbb{R}^6), \quad \phi(x, v) \geq 0,$$
$$\iint \phi(x, v)\,dx dv = 1, \quad \text{and} \quad \mathrm{supp}\,\phi \subset \{|x|^2 + |v|^2 \leq 1\}.$$

By definition we have the following properties:

(I) $\phi_n \to \delta$ in $\mathcal{D}'(\mathbb{R}^6)$,

(II) $\frac{1}{n}\partial_{x_i}\phi_n$, $\frac{1}{n}\partial_{v_i}\phi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^6)$, $i = 1, 2, 3$,

(III) $(x, v)^\alpha\partial^\beta_{(x,v)}\left[(x, v)^\gamma\phi_n(x, v)\right] \to 0$ in $\mathcal{D}'(\mathbb{R}^6)$, with $\alpha, \beta, \gamma \in \mathbb{N}_0^6$ multi-indexes and $|\gamma| > 0$, since $(x, v)^\gamma\phi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^6)$.

The cutoff sequence is

$$\psi_n(x, v) := \psi\left(\frac{|(x, v)|}{n}\right), \qquad n \in \mathbb{N},\ x, v \in \mathbb{R}^3,$$

where ψ satisfies

$$\psi \in C_0^\infty(\mathbb{R}), \quad 0 \leq \psi(z) \leq 1, \quad \mathrm{supp}\psi \subset [-1, 1], \quad \psi|_{[-\frac{1}{2},\frac{1}{2}]} \equiv 1,$$

and

$$|\psi^{(j)}(z)| \leq C_j, \quad \forall z \in \mathbb{R},\ j = 1, 2.$$

The sequence ψ_n has the following properties:

(IV) $\psi_n \to 1$ pointwise,

(V) $(x,v)^\alpha \partial^\beta_{(x,v)} \psi_n(x,v) = \frac{1}{n} \frac{(x,v)^\alpha (x,v)^\beta}{|(x,v)|} \psi' \left(\frac{|(x,v)|}{n} \right)$, with $\alpha, \beta \in \mathbb{N}_0^6$, $|\alpha| = |\beta| = 1$, are supported in the annulus

$$\mathrm{supp}\left(\psi'\left(\frac{|(x,v)|}{n}\right)\right) \;=\; \{(x,v) \mid n/2 \le |(x,v)| \le n\} \;=:\; V_n,$$

and they are in $L^\infty(\mathbb{R}^6)$, uniformly in $n \in \mathbb{N}$.

(VI) $n\partial^\alpha_{(x,v)} \psi_n(x,v) = \frac{(x,v)^\alpha}{|(x,v)|} \psi' \left(\frac{|(x,v)|}{n} \right)$, with $\alpha \in \mathbb{N}_0^6$, $|\alpha| = 1$, are uniformly bounded in $L^\infty(\mathbb{R}^6)$.

(VII) $\partial^\alpha_{(x,v)} \psi_n(x,v) = \frac{(x,v)^\alpha}{n^2 |(x,v)|^2} \psi'' \left(\frac{|(x,v)|}{n} \right) + \left(\frac{1}{n^2 |(x,v)|} - \frac{(x,v)^\alpha}{n^3 |(x,v)|^3} \right) \psi' \left(\frac{|(x,v)|}{n} \right)$, with $|\alpha| = 2$ have support on V_n and converge uniformly to 0 in $L^\infty(\mathbb{R}^6)$.

We now define the approximating sequence

$$f_n(x,v) := (f * \phi_n)(x,v) \cdot \psi_n(x,v), \quad n \in \mathbb{N},$$

where '$*$' denotes the convolution in x and v.
By construction we have $f_n \in C_0^\infty(\mathbb{R}^6) = D(P)$.
Since we can split our operator as

$$\begin{aligned} P \;&=\; \sum_{i=1}^{3} \left[\frac{\theta}{3} + \nu v_i \partial_{x_i} + \mu x_i \partial_{v_i} + \beta v_i \partial_{v_i} + \alpha \partial^2_{x_i} + \sigma \partial^2_{v_i} + \gamma \partial_{v_i} \partial_{x_i} \right] \\ &=\; \sum_{i=1}^{3} \tilde{p}(x_i, v_i, \partial_{x_i}, \partial_{v_i}), \end{aligned}$$

we shall in the sequel only consider

$$\tilde{P} \;=\; \tilde{p}(y, z, \partial_y, \partial_z), \quad y, z \in \mathbb{R}$$

acting in one spatial direction $y = x_j$ and one velocity direction $z = v_j$.
We have to prove that $f_n(x,v) \to f(x,v)$ in the graph norm

$$\|f\|_{\tilde{P}} = \|f\|_{L^2} + \||x|^2 f\|_{L^2} + \||v|^2 f\|_{L^2} + \|\tilde{P} f\|_{L^2} + \||x|^2 \tilde{P} f\|_{L^2} + \||v|^2 \tilde{P} f\|_{L^2}.$$

According to the 6 terms of the graph norm we split the proof into 6 steps:
Step 1: By applying (P1) and (P4), we have

$$f_n \to f \quad \text{in} \quad L^2(\mathbb{R}^6).$$

Step 2: For the second term of the graph norm we write

$$x_i^2 f_n = (x_i^2 f * \phi_n)\psi_n + 2(x_i f * x_i \phi_n)\psi_n + (f * x_i^2 \phi_n)\psi_n.$$

The first summand converges to $x_i^2 f$ in $L^2(\mathbb{R}^6)$ and both the second and the third terms converge to 0 by (III), since also $x_i f$ belongs to $L^2(\mathbb{R}^6)$ by interpolation.

Step 3: For the third term of the graph norm the same argument as in previous step can be used. Hence we have

$$f_n \to f \qquad \text{in} \quad Z.$$

Step 4: To prove that $\tilde{P}f_n \to \tilde{P}f$ in $L^2(\mathbb{R}^6)$ we write:

$$\begin{aligned}\tilde{P}f_n &= \frac{\theta}{3}(f*\phi_n)\psi_n + \nu(zf_y*\phi_n)\psi_n + \mu(yf_z*\phi_n)\psi_n + \beta(zf_z*\phi_n)\psi_n \\ &\quad +\alpha(f_{yy}*\phi_n)\psi_n + \sigma(f_{zz}*\phi_n)\psi_n + \gamma(f_{yz}*\phi_n)\psi_n + r_n^1(y,z) \\ &= (\tilde{P}f*\phi_n)\psi_n + r_n^1(y,z).\end{aligned}$$

As we shall show, all thirteen terms of the remainder

$$\begin{aligned}r_n^1 &= \nu(f*\partial_y(z\phi_n))\psi_n + \nu(f*\phi_n)z\partial_y\psi_n + \mu(f*y\partial_z\phi_n)\psi_n \\ &\quad +\mu(f*\phi_n)y\partial_z\psi_n + \beta(f*\partial_z(z\phi_n))\psi_n + \beta(f*\phi_n))z\partial_z\psi_n \\ &\quad +2\alpha(f*(\frac{1}{n}\partial_y\phi_n))(n\partial_y\psi_n) + \alpha(f*\phi_n))(\partial_y^2\psi_n) + 2\sigma(f*\frac{1}{n}\partial_z\phi_n)n\partial_z\psi_n \\ &\quad +\sigma(f*\phi_n)\partial_z^2\psi_n + \gamma(f*(\frac{1}{n}\partial_z\phi_n))(n\partial_y\psi_n) + \gamma(f*(\frac{1}{n}\partial_y\phi_n))(n\partial_z\psi_n) \\ &\quad +\gamma(f*\phi_n)\partial_y\partial_z\psi_n\end{aligned}$$

converge to 0 in $L^2(\mathbb{R}^6)$.
The first, the third and the fifth terms converge to 0 in $L^2(\mathbb{R}^6)$ by (III).
In the second, fourth and the sixth terms, exploiting (V) we have

$$\|(f*\phi_n)(z\partial_y\psi_n)\|_{L^2(R^6)} \le C\|f*\phi_n - f\|_{L^2(V_n)} + \|f\|_{L^2(V_n)} \to 0, \tag{5.4}$$

because $\|f\|_{L^2(R^6)} = \|f\|_{L^2(B_{1/2}(0))} + \sum_{k=0}^{\infty}\|f\|_{L^2(V_{2^k})}$.
For what the seventh, ninth, eleventh and twelfth terms are concerned, we can exploit (VI) and then (II).
The remaining terms can be handled thanks to (VII).

Step 5: To prove that $|x|^2\tilde{P}f_n \to |x|^2Pf$ in $L^2(\mathbb{R}^6)$ we write:

$$\begin{aligned}x_i^2\tilde{P}f_n &= \frac{\theta}{3}(x_i^2f*\phi_n)\psi_n + \nu(x_i^2zf_y*\phi_n)\psi_n + \mu(x_i^2yf_z*\phi_n)\psi_n + \beta(x_i^2zf_z*\phi_n)\psi_n \\ &\quad +\alpha(x_i^2f_{yy}*\phi_n)\psi_n + \sigma(x_i^2f_{zz}*\phi_n)\psi_n + \gamma(x_i^2f_{yz}*\phi_n)\psi_n + r_n^2(y,z) \\ &= (x_i^2\tilde{P}f*\phi_n)\psi_n + r_n^2(y,z).\end{aligned}$$

The remainder r_n^2 can be split in the following way ($y = x_j$, $z = v_j$):

$$\begin{aligned}r_{n,\theta}^2 &= \frac{2}{3}\theta(x_if*y\phi_n)\psi_n + \frac{\theta}{3}(f*x_i^2\phi_n)\psi_n \\ r_{n,\nu}^2 &= 2\nu(zx_if*\partial_y(x_i\phi_n))\psi_n - 2\nu\delta_{ij}(zf*x_i\phi_n)\psi_n + \nu(zf*\partial_y(x_i^2\phi_n))\psi_n \\ &\quad +\nu(x_i^2f*z\partial_y\phi_n)\psi_n + 2\nu(x_if*x_iz\partial_y\phi_n)\psi_n + \nu(f*x_i^2z\partial_y\phi_n)\psi_n \\ &\quad +\nu(x_i^2f*\phi_n)z\partial_y\psi_n + 2\nu(x_if*x_i\phi_n)z\partial_y\psi_n + \nu(f*x_i^2\phi_n)z\partial_y\psi_n \\ r_{n,\mu}^2 &= 2\mu(x_iyf*\partial_z(x_i\phi_n))\psi_n + \mu(yf*\partial_z(x_i^2\phi_n))\psi_n + \mu(x_i^2f*y\partial_z\phi_n)\psi_n \\ &\quad +2\mu(x_if*x_iy\partial_z\phi_n)\psi_n + \mu(f*x_i^2y\partial_z\phi_n)\psi_n + \mu(x_i^2f*\phi_n)y\partial_z\psi_n \\ &\quad +2\mu(x_if*x_i\phi_n)y\partial_z\psi_n + \mu(f*x_i^2\phi_n)y\partial_z\psi_n \\ r_{n,\beta}^2 &= 2\beta(x_izf*x_i\partial_z\phi_n)\psi_n - 2\beta(x_if*x_i\phi_n)\psi_n + \beta(zf*x_i^2\partial_z\phi_n)\psi_n - \beta(f*x_i^2\phi_n)\psi_n \\ &\quad +\beta(x_i^2f*\partial_z(z\phi_n))\psi_n + 2\beta(x_if*x_i\partial_z(z\phi_n))\psi_n + \beta(f*x_i^2\partial_z(z\phi_n))\psi_n \\ &\quad +\beta(x_i^2f*\phi_n)z\partial_z\psi_n + 2\beta(x_if*x_i\phi_n)z\partial_z\psi_n + \beta(f*x_i^2\phi_n)z\partial_z\psi_n\end{aligned}$$

$$\begin{aligned}
r^2_{n,\alpha} &= 2\alpha(x_i f * \partial_y^2(x_i\phi_n))\psi_n - 4\alpha\delta_{ij}(f * \partial_y(x_i\phi_n))\psi_n + \alpha(f * \partial_y^2(x_i^2\phi_n))\psi_n \\
&\quad +2\alpha(x_i^2 f * \frac{1}{n}\partial_y\phi_n)n\partial_y\psi_n + 4\alpha(x_i f * \frac{x_i}{n}\partial_y\phi_n)n\partial_y\psi_n + 2\alpha(f * \frac{x_i^2}{n}\partial_y\phi_n)n\partial_y\psi_n \\
&\quad +\alpha(x_i^2 f * \phi_n)\partial_y^2\psi_n + 2\alpha(x_i f * x_i\phi_n)\partial_y^2\psi_n + \alpha(f * x_i^2\phi_n)\partial_y^2\psi_n \\
r^2_{n,\sigma} &= 2\sigma(x_i f * x_i\partial_z^2\phi_n)\psi_n + \sigma(f * x_i^2\partial_z^2\phi_n)\psi_n + 2\sigma(x_i^2 f * \frac{1}{n}\partial_z\phi_n)n\partial_z\psi_n \\
&\quad +4\sigma(x_i f * \frac{x_i}{n}\partial_z\phi_n)n\partial_z\psi_n + 2\sigma(f * \frac{x_i^2}{n}\partial_z\phi_n)n\partial_z\psi_n + \sigma(x_i^2 f * \phi_n)\partial_z^2\psi_n \\
&\quad +2\sigma(x_i f * x_i\phi_n)\partial_z^2\psi_n + \sigma(f * x_i^2\phi_n)\partial_z^2\psi_n \\
r^2_{n,\gamma} &= 2\gamma(x_i f * \partial_y(x_i\partial_z\phi_n))\psi_n - 2\gamma\delta_{ij}(f * x_i\partial_z\phi_n)\psi_n + \gamma(f * \partial_y\partial_z(x_i^2\phi_n))\psi_n \\
&\quad +\gamma(x_i^2 f * \frac{1}{n}\partial_z\phi_n)n\partial_y\psi_n + 2\gamma(x_i f * \frac{x_i}{n}\partial_z\phi_n)n\partial_y\psi_n + \gamma(f * \frac{x_i^2}{n}\partial_z\phi_n)n\partial_y\psi_n \\
&\quad +\gamma(x_i^2 f * \frac{1}{n}\partial_y\phi_n)n\partial_z\psi_n + 2\gamma(x_i f * \frac{x_i}{n}\partial_y\phi_n)n\partial_z\psi_n + \gamma(f * \frac{x_i^2}{n}\partial_y\phi_n)n\partial_z\psi_n \\
&\quad +\gamma(x_i^2 f * \phi_n)\partial_y\partial_z\psi_n + 2\gamma(x_i f * x_i\phi_n)\partial_y\partial_z\psi_n + \gamma(f * x_i^2\phi_n)\partial_y\partial_z\psi_n.
\end{aligned}$$

By the properties (I)-(VII) and estimate like (5.4), it can be easily seen that each term converges to 0 in $L^2(\mathbb{R}^6)$.

Step 6: In analogy to $|x|^2\tilde{P}f_n$, the sequence $|v|^2\tilde{P}f$ can be split as

$$v_i^2\tilde{P}f_n = (v_i^2\tilde{P}f * \phi_n)\psi_n + r_n^3(y,z).$$

Due to the symmetry of the operator $\tilde{P}$ in x and v, the terms of the remainder r_n^3 can be obtained from r_n^2 by interchanging y and z (and changing the coefficients), except for the following term

$$v_i^2[\beta z\partial_z\left((f * \phi_n)\psi_n\right)] = \beta(v_i^2 z f_z * \phi_n)\psi_n + r^3_{n,\beta},$$

where

$$\begin{aligned}
r^3_{n,\beta} &= 2\beta(v_i z f * \partial_z(v_i\phi_n))\psi_n - 2\beta(1+\delta_{ij})(v_i f * v_i\phi_n)\psi_n + \beta(z f * \partial_z(v_i^2\phi_n))\psi_n \\
&\quad -\beta(f * v_i^2\phi_n)\psi_n + \beta(v_i^2 f * \partial_z(z\phi_n))\psi_n + 2\beta(v_i f * v_i\partial_z(z\phi_n))\psi_n \\
&\quad +\beta(f * v_i^2\partial_z(z\phi_n))\psi_n + \beta(v_i^2 f * \phi_n)z\partial_z\psi_n + 2\beta(v_i f * v_i\phi_n)z\partial_z\psi_n \\
&\quad +\beta(f * v_i^2\phi_n)z\partial_z\psi_n
\end{aligned}$$

converges to 0 in $L^2(\mathbb{R}^6)$, since (I)-(VII) and (5.4) can be used. $\square$

Proof of Proposition 3.1

First, we shall prove the following estimates on the derivatives of the Green's function (3.4):

$$|\nabla_v G(t,x,v,x_0,v_0)| \le b\frac{G(t,\frac{x}{2},\frac{v}{2},\frac{x_0}{2},\frac{v_0}{2})}{\sqrt{t}}, \quad \forall\, t \le t_0, \tag{5.5}$$

$$|\nabla_x G(t,x,v,x_0,v_0)| \le b'\frac{G(t,\frac{x}{2},\frac{v}{2},\frac{x_0}{2},\frac{v_0}{2})}{\sqrt{t}}, \quad \forall\, t \le t_1. \tag{5.6}$$

with $b = b(\alpha,\gamma,\sigma), t_0 = t_0(\alpha,\beta,\sigma,\gamma), b' = b'(\alpha,\gamma,\sigma)$ and $t_1 = t_1(\alpha,\beta,\sigma,\gamma)$. The v-derivative of G is given by

$$\nabla_v G(t,x,v,x_0,v_0) = G(t,x,v,x_0,v_0)\Bigg[-\frac{\left(\mu(t)e^{\beta t} - 2\nu(t)\frac{e^{\beta t}-1}{\beta}\right)(x - \frac{e^{\beta t}-1}{\beta}v - x_0)}{f(t)}$$

$$
- \frac{\left(2\lambda(t)e^{\beta t} - \mu(t)\frac{e^{\beta t}-1}{\beta}\right)(e^{\beta t}v - v_0)}{f(t)} \Bigg]. \tag{5.7}
$$

For all real $a, b, c > 0$ such that $c/\sqrt{a} \leq b\sqrt{2e}$, one easily verifies that

$$
c|x| \;\leq\; be^{a|x|^2}, \quad \forall\, x \in \mathbb{R}^3. \tag{5.8}
$$

Since $\alpha, \sigma > 0$, we have for $t > 0$ small enough

$$
\nu(t) - \frac{1}{2}\mu(t) \;>\; 0, \qquad \lambda(t) - \frac{1}{2}\mu(t) \;>\; 0.
$$

In order to apply the estimate (5.8) to the two terms inside the squared bracket in (5.7) we shall use for t small:

$$
\frac{c_1}{\sqrt{a_1}} \;:=\; \frac{\frac{\sqrt{t}}{f(t)}\left|\mu(t)e^{\beta t} - 2\nu(t)\frac{e^{\beta t}-1}{\beta}\right|}{\sqrt{\frac{3}{4}\frac{\nu(t)-\frac{1}{2}\mu(t)}{f(t)}}} \;\sim\; \frac{2\gamma}{\sqrt{3(\alpha\sigma - \gamma^2)(\sigma+\gamma)}} \;\leq\; b_1\sqrt{2e},
$$

with $b_1 = \gamma/\sqrt{3(\alpha\sigma - \gamma^2)(\sigma+\gamma)}$. Similarly,

$$
\frac{c_2}{\sqrt{a_2}} \;:=\; \frac{\frac{\sqrt{t}}{f(t)}\left|2\lambda(t)e^{\beta t} - \mu(t)\frac{e^{\beta t}-1}{\beta}\right|}{\sqrt{\frac{3}{4}\frac{\lambda(t)-\frac{1}{2}\mu(t)}{f(t)}}} \;\sim\; \frac{2\alpha}{\sqrt{3(\alpha\sigma - \gamma^2)(\alpha+\gamma)}} \;\leq\; b_2\sqrt{2e},
$$

with $b_2 = \alpha/\sqrt{3(\alpha\sigma - \gamma^2)(\alpha+\gamma)}$. Then, there exists some $t_0 > 0$ such that, for all $t \leq t_0$, the two inequalities can be combined with $b = \max\{b_1, b_2\}$ to give

$$
\begin{aligned}
&\left|\frac{\left(\mu(t)e^{\beta t} - 2\nu(t)\frac{e^{\beta t}-1}{\beta}\right)\left(x - \frac{e^{\beta t}-1}{\beta}v - x_0\right) + \left(2\lambda(t)e^{\beta t} - \mu(t)\frac{e^{\beta t}-1}{\beta}\right)(e^{\beta t}v - v_0)}{f(t)}\right|\sqrt{t} \;\leq \\
&\leq \frac{\sqrt{t}}{f(t)}\left\{\left|\mu(t)e^{\beta t} - 2\nu(t)\frac{e^{\beta t}-1}{\beta}\right|\left|x - \frac{e^{\beta t}-1}{\beta}v - x_0\right| + \left|2\lambda(t)e^{\beta t} - \mu(t)\frac{e^{\beta t}-1}{\beta}\right||e^{\beta t}v - v_0|\right\} \\
&\leq b\exp\left\{\frac{\left(\nu(t) - \frac{1}{2}\mu(t)\right)\left|x - \frac{e^{\beta t}-1}{\beta}v - x_0\right|^2 + \left(\lambda(t) - \frac{1}{2}\mu(t)\right)\left|e^{\beta t}v - v_0\right|^2}{\frac{4}{3}f(t)}\right\} \\
&\leq b\exp\left\{\frac{\nu(t)\left|x - \frac{e^{\beta t}-1}{\beta}v - x_0\right|^2 + \lambda(t)\left|e^{\beta t}v - v_0\right|^2 + \mu(t)\left(x - \frac{e^{\beta t}-1}{\beta}v - x_0\right)\cdot\left(e^{\beta t}v - v_0\right)}{\frac{4}{3}f(t)}\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|\nabla_v G(t,x,v,x_0,v_0)| \;\leq\; b\frac{G(t,x,v,x_0,v_0)}{\sqrt{t}} \\
&\times\exp\left\{\frac{3}{4}\frac{\nu(t)\left|x - \frac{e^{\beta t}-1}{\beta}v - x_0\right|^2 + \lambda(t)\left|e^{\beta t}v - v_0\right|^2 + \mu(t)\left(x - \frac{e^{\beta t}-1}{\beta}v - x_0\right)\cdot\left(e^{\beta t}v - v_0\right)}{f(t)}\right\}
\end{aligned}
$$

and the decay (5.5) follows by comparison with (3.4).

Next we consider the x-derivative of the Green's function,

$$\nabla_x G(t,x,v,x_0,v_0) \;=\; G(t,x,v,x_0,v_0)\left[-\frac{2\nu(t)(x-\frac{(e^{\beta t}-1)}{\beta}v-x_0)+\mu(t)(e^{\beta t}v-v_0)}{f(t)}\right].$$

Analogously, the decay (5.6) follows by exploiting that for t small enough

$$\frac{\frac{\sqrt{t}}{f(t)}2\nu(t)}{\sqrt{\frac{3}{4}\frac{\nu(t)-\frac{1}{2}\mu(t)}{f(t)}}} \sim \frac{2\sigma}{\sqrt{3(\alpha\sigma-\gamma^2)(\sigma+\gamma)}} \leq b_1'\sqrt{2e},$$

$$\frac{\frac{\sqrt{t}}{f(t)}|\mu(t)|}{\sqrt{\frac{3}{4}\frac{\lambda(t)-\frac{1}{2}\mu(t)}{f(t)}}} \sim \frac{2\gamma}{\sqrt{3(\alpha\sigma-\gamma^2)(\alpha+\gamma)}} \leq b_2'\sqrt{2e},$$

with appropriate $b_1'(\alpha,\gamma,\sigma), b_2'(\alpha,\gamma,\sigma)$.

Since

$$e^{t\overline{A}}w_0(x,v) \;=\; \iint G(t,x,v,x_0,v_0)w_0(x_0,v_0)\,dx_0\,dv_0,$$

we have

$$\begin{aligned}
|\nabla_v e^{t\overline{A}}w_0(x,v)| &\leq \iint |\nabla_v G(t,x,v,x_0,v_0)|\,|w_0(x_0,v_0)|\,dx_0\,dv_0 \\
&\leq bt^{-1/2}\iint G\left(t,\frac{x}{2},\frac{v}{2},\frac{x_0}{2},\frac{v_0}{2}\right)|w_0(x_0,v_0)|\,dx_0\,dv_0 \\
&= 64b\,t^{-1/2}\iint G(t,\tilde{x},\tilde{v},\tilde{x_0},\tilde{v_0})|w_0(2\tilde{x_0},2\tilde{v_0})|\,d\tilde{x_0}\,d\tilde{v_0} \\
&= 64b\,t^{-1/2}e^{t\overline{A}}\tilde{w}_0(\tilde{x},\tilde{v}), \quad \forall\, t\leq t_0. \qquad (5.9)
\end{aligned}$$

Here we used the decay (5.5), and we put $\tilde{x}=\frac{x}{2}$, $\tilde{v}=\frac{v}{2}$ and $\tilde{w}_0(\tilde{x},\tilde{v})=|w_0(2\tilde{x},2\tilde{v})|$. The assertion (3.5) follows directly by applying the estimate (3.2) to (5.9) and choosing $T_0=\min\{t_0,t_1\}$.
The estimate (3.6) can be obtained analogously. □

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