

Dispersive effects in quantum kinetic equations

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Abstract

This paper is concerned with a global-in-time well-posedness analysis for the Wigner-Poisson-Fokker-Planck system, a kinetic evolution equation for an open quantum system with a non-linear Hartree potential. The purely kinetic L^2 -analysis here presented, allows a unified treatment of the elliptic and hypoelliptic cases. The crucial novel tool of the analysis is to exploit in the quantum framework the dispersive effects of the free transport equation. It yields an a-priori estimate on the electric field for all time which allows a new nonlocal-in-time definition of the self-consistent potential and field. Thus, one can circumvent the lacking v -integrability of the Wigner function, which is a central problem in quantum kinetic theory. Due to the (degenerate) parabolic character of this system, the C^∞ -regularity of the Wigner function, its macroscopic density, and the field are established for positive times.

1 Introduction

In this paper we present a new strategy for the well-posedness analysis of quantum kinetic problems that include a Hartree-type nonlinearity. We will focus here on the 3-dimensional Wigner-Poisson-Fokker-Planck (WPFP) system, but we expect this new approach to be suitable for a broad range of quantum kinetic problems.

The Wigner function $w = w(x, v, t)$ is one of several equivalent formalisms to describe the state of a physical quantum system (cf. [Wi]). It is a real-valued quasi-distribution function in the position-velocity (x, v) phase space at time t . In collision-free regimes, the quantum equivalent of the Liouville equation of classical kinetic theory governs the time evolution of w . It reads

$$\partial_t w + v \cdot \nabla_x w - \Theta[V]w = 0, \quad t > 0, \quad (x, v) \in \mathbb{R}^6, \quad (1.1)$$

where the (real-valued) potential $V = V(x)$ enters through the pseudo-differential operator $\Theta[V]$ defined by

$$\begin{aligned} (\Theta[V]w)(x, v, t) &= \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \delta V(x, \eta) \mathcal{F}_{v \rightarrow \eta} w(x, \eta, t) e^{iv \cdot \eta} d\eta \\ &= \frac{i}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta V(x, \eta) w(x, v', t) e^{i(v-v') \cdot \eta} dv' d\eta. \end{aligned} \quad (1.2)$$

Here, $\delta V(x, \eta) := V(x + \frac{\eta}{2}) - V(x - \frac{\eta}{2})$, and $\mathcal{F}_{v \rightarrow \eta} w$ denotes the Fourier transform of w with respect to v . The Cauchy problem for Wigner equations like (1.1) is augmented by the initial condition $w(x, v, t = 0) = w_0(x, v)$, $(x, v) \in \mathbb{R}^6$. Observe that we set, for simplicity, all physical constants equal to 1. In order to describe the non-reversible interaction of a quantum system with its environment, a possible modification of (1.1) consists in introducing a Fokker-Planck type operator on the right hand side (cf. [CL, CEFM] for a derivation):

$$\partial_t w + v \cdot \nabla_x w - \Theta[V]w = \beta \operatorname{div}_v(vw) + \sigma \Delta_v w + 2\gamma \operatorname{div}_v(\nabla_x w) + \alpha \Delta_x w, \quad t > 0. \quad (1.3)$$

Here, $\beta \geq 0$ is the friction parameter and the coefficients $\alpha, \gamma \geq 0, \sigma > 0$ constitute the phase-space diffusion matrix of the system. In the Fokker-Planck equation of classical mechanics (cf. [Ri, CSV]) one would have $\alpha = \gamma = 0$. For the Wigner-Fokker-Planck (WFP) equation (1.3), the so-called Lindblad condition

$$\begin{pmatrix} \alpha & \gamma + \frac{i}{4}\beta \\ \gamma - \frac{i}{4}\beta & \sigma \end{pmatrix} \geq 0$$

has to hold: it guarantees that the evolution of the system is “quantum mechanically correct” (i.e., it corresponds to a positive density matrix, cf. [Di, ALMS]). For the mathematical analysis, however, it suffices that (1.3) is parabolic or degenerate parabolic. Thus, we shall only assume $\alpha\sigma \geq \gamma^2$ henceforth.

Both the Wigner (1.1) and the WFP (1.3) equations constitute valuable models for simulations in semiconductor device theory (cf. [GGKS, MRS, RA] and references therein), for quantum Brownian motion and quantum optics (cf. [CL, Di, De, Va, Val]).

We consider the case when $V = V(x, t)$ models the mean-field interaction in the quantum system: how to define rigorously the Hartree-potential in a quantum kinetic framework is indeed one of the crucial points in this paper. We start with the following version that is the standard definition in classical kinetic theory, but is purely formal in the quantum context. In the next section we will formulate a more appropriate definition.

Definition 1.1 (Standard definition of mean-field quantities) *To a Wigner function $w(t)$ is associated*

- *the position density*

$$n = n(x, t) := \int_{\mathbb{R}^3} w(x, v, t) dv, \quad x \in \mathbb{R}^3, \quad t > 0, \quad (1.4)$$

- *the potential*

$$V = V(x, t) := -\frac{1}{4\pi|x|} * n(x, t), \quad (1.5)$$

which solves the Poisson equation

$$-\Delta V(t) = n(t), \quad x \in \mathbb{R}^3, \quad t > 0, \quad (1.6)$$

- *and the field $E(x, t) := \nabla_x V(x, t)$.*

In classical kinetic theory the phase space density typically satisfies $f(\cdot, \cdot, t) \in L^1(\mathbb{R}^6)$ which yields a position density $n(\cdot, t) = \int f dv \in L^1(\mathbb{R}^3)$. In quantum kinetic theory, however, the natural framework is $w(\cdot, \cdot, t) \in L^2(\mathbb{R}^6)$, which makes (1.4) meaningless and hence also the above definition of the mean-field potential. Solving or circumventing this problem is one of the key points for analyzing self-consistent quantum kinetic models. Accordingly, in order to establish well-posedness of the system (1.1), (1.6) or (1.3), (1.6), two strategies have been used so far. The first possibility is to reformulate the WP or WFP systems either in terms of Schrödinger wave-function sequences (cf. [BM, Ca1]) or in terms of density matrices (cf. [Ar, AS]). In such a framework, all physical quantities are well-defined, in particular $n(t) \in L^1_+(\mathbb{R}^3)$ and the physical conservation laws for mass and energy play a crucial role in the analysis for large time. Alternatively, one can keep to the kinetic formulation and to the use of kinetic tools, with the perspective of later tackling boundary-value problems, which are more reasonable models for real simulations.

The literature related to the latter approach can be split into two groups: in several articles (cf. [AR, ACD, Ma, ADM]), a L^2 -setting is chosen for $w(t)$, such that $w(t)$ satisfies at least the necessary condition to describe a quantum system (cf. [MRS, LiPa]). Then, v -weights are introduced in order to enforce integrability in the v -variable, so to give sense to (1.4). In other articles (cf. [ALMS, CLN]), instead, a L^1 -setting is chosen with the same motivation. In order to prove global-in-time results for nonlinear quantum kinetic models, one might want to exploit the physical conservation laws. However, in neither of the two above approaches they can be exploited directly, since both the mass $\iint w dx dv$ and the kinetic energy $\frac{1}{2} \iint |v|^2 w dx dv$ are *not* positive functionals under the assumptions made at the kinetic level. This is the second crucial point in the quantum kinetic analysis.

A third aspect that differentiates quantum from classical kinetic theory, is the lack of a maximum principle for the Wigner function under time evolution. Indeed, $\|w(t)\|_{L^2(\mathbb{R}^6)}$ is the only conserved norm of (1.1). Due to the described differences, the analytic approach used for classical kinetic models like Vlasov-Poisson (VP) or Vlasov-Poisson-Fokker-Planck (VPFP) can *not* be adapted to quantum kinetic problems and a novel strategy is required. In order to achieve a global-in-time result for the WFP system (1.3)-(1.6), the authors of [ADM] exploit dispersive effects of the free-streaming operator jointly with the parabolic regularization of the Fokker-Planck term, since this yields a-priori estimates for the solution $w(t)$ in a weighted L^2 -space. Such dispersive techniques for kinetic equations were first developed for the VP system (cf. [LiPe, Pe]) and then adapted to the VPFP equation (cf.

[Bo1, Ca2]). In [ADM] these tools were extended to quantum kinetic theory.

In the present article, we will achieve as well a global-in-time well-posedness result for the WPFP system in the space $L^2(\mathbb{R}^6)$, *but without* introducing weights. This is possible thanks to an alternative strategy that relies first of all on an a-priori estimate for the field $\nabla_x V(t)$ in terms of $\|w(t)\|_{L^2(\mathbb{R}^6)}$ only. This estimate was derived in [ADM] using dispersive effects of the free-streaming operator. It allows a *novel definition* of the macroscopic quantities (namely, the self-consistent field, the potential, and the density), which, in contrast to the Definition 1.1, is now non-local in time. This way, no v -integrability of w is needed, and hence no moments in v either. Secondly, we shall use the (degenerate) parabolic regularization of the Fokker-Planck term in order to construct (by a fixed point map) a global-in-time solution. These techniques allow to overcome the described analytical difficulties and they yield –a-posteriori– some L^p -estimates on the density.

In conclusion, our purely kinetic L^2 -analysis solves both main problems of quantum kinetic theory, namely the definition of the density (due to the missing v -integrability of w) and the lack of usable a-priori estimates on w (due to its non-definite sign). Finally, we point out that we expect that this approach could also be a crucial step towards developing a kinetic analysis for the Wigner-Poisson system (1.3),(1.6), which has been an open problem for 15 years.

This paper is organized as follows: In §2 we motivate the new, non-local redefinition of the self-consistent field, and present the main results of this article. In §3 we derive a-priori estimates on the potential and the field which are the crucial ingredients for the global well-posedness analysis of §4. In the two different versions of the WPFP system, namely the elliptic ($\alpha > 0$) and hypoelliptic ($\alpha = 0$) cases, the solution exhibits a different asymptotic behaviour close to the initial time, and hence different analytical strategies will have to be applied. In §5 we establish –a-posteriori– the C^∞ -regularity of the solution, and in §6 decay estimates on the particle density.

2 Strategy and main results

We shall prove existence and uniqueness of a mild solution $w(t) \in L^2(\mathbb{R}^6) = L^2(\mathbb{R}^6; dx dv)$ to the WPFP problem (1.3)-(1.6) on the time interval $[0, T]$, with $T > 0$ arbitrary, but fixed for the sequel. Accordingly, the solution has to satisfy, for all $t \in [0, T]$, the integral equation

$$\begin{aligned} w(x, v, t) &= \iint G(t, x - x_0, v, v_0) w_0(x_0, v_0) dx_0 dv_0 \\ &\quad + \int_0^t \iint G(s, x - x_0, v, v_0) (\Theta[V]w)(x_0, v_0, t - s) dx_0 dv_0 ds. \end{aligned} \quad (2.1)$$

Here, the Green's function G is the weak solution of the linear equation

$$\begin{aligned} \partial_t w &= Aw, \quad t > 0, \\ Aw &:= -v \cdot \nabla_x w + \beta \operatorname{div}_v(vw) + \sigma \Delta_v w + 2\gamma \operatorname{div}_v(\nabla_x w) + \alpha \Delta_x w \end{aligned} \quad (2.2)$$

with the initial condition

$$\lim_{t \rightarrow 0} G(t, x - x_0, v, v_0) = \delta(x - x_0, v - v_0), \quad \forall (x_0, v_0) \in \mathbb{R}^6.$$

The Green's function reads (cf. [SCDM])

$$G(t, x - x_0, v, v_0) = e^{3\beta t} g(t, X_{-t}(x, v) - x_0, \dot{X}_{-t}(x, v) - v_0), \quad (2.3)$$

with

$$g(t, x, v) = \frac{1}{(2\pi)^3 (4\lambda(t)\nu(t) - \mu^2(t))^{3/2}} \exp \left\{ -\frac{\nu(t)|x|^2 + \lambda(t)|v|^2 + \mu(t)(x \cdot v)}{4\lambda(t)\nu(t) - \mu^2(t)} \right\}. \quad (2.4)$$

The characteristic flow $\Phi_t(x, v) = (X_t(x, v), \dot{X}_t(x, v))$ of the first order part of (2.2), is given for $\beta > 0$ by

$$X_t(x, v) = x + v \left(\frac{1 - e^{-\beta t}}{\beta} \right), \quad \dot{X}_t(x, v) = v e^{-\beta t},$$

and for $\beta = 0$ by

$$X_t(x, v) = x + vt, \quad \dot{X}_t(x, v) = v.$$

The functions $\lambda(t)$, $\nu(t)$, $\mu(t)$ in (2.4) are given for $\beta > 0$ by

$$\begin{aligned} \lambda(t) &= \alpha t + \sigma \left(\frac{e^{2\beta t} - 4e^{\beta t} + 3}{2\beta^3} + \frac{1}{\beta^2} t \right) + \gamma \left(\frac{2}{\beta} t - \frac{2}{\beta^2} (e^{\beta t} - 1) \right), \\ \nu(t) &= \sigma \frac{e^{2\beta t} - 1}{2\beta}, \\ \mu(t) &= \sigma \left(\frac{1 - e^{\beta t}}{\beta} \right)^2 + \gamma \frac{2(1 - e^{\beta t})}{\beta}. \end{aligned}$$

In case $\beta = 0$ they respectively read

$$\lambda(t) = \alpha t + \sigma \frac{t^3}{3} - \gamma t^2, \quad \nu(t) = \sigma t, \quad \mu(t) = \sigma t^2 - 2\gamma t.$$

The main difficulty in analyzing the WFPF system consists in defining the density $n(t)$ and the potential $V(t)$. As mentioned before, the standard, local-in-time definition (1.4) is unfeasible for a Wigner function $w(t) \in L^2(\mathbb{R}^6)$. We will show that it is possible to by-pass the definition of $n(t)$ by defining the potential $V[w]$ corresponding to a Wigner trajectory $w \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$. This *non-local* in time definition of $V[w]$ relies on dispersive effects of kinetic equations and it is inspired by a-priori estimates on the self-consistent field $\nabla_x V$ derived in [ADM].

To motivate our alternative definition of $V[w]$ we first recall from [ADM] how to reformulate the pseudo-differential operator $\Theta[V]$ in terms of $\nabla_x V$. We have

$$\Theta[V]w(x, v) = \mathcal{F}_{\eta \rightarrow v}^{-1} \left(i \delta V(x, \eta) \mathcal{F}_{v \rightarrow \eta} w(x, \eta) \right),$$

where

$$\delta V(x, \eta) = \int_{x-\eta/2}^{x+\eta/2} \nabla_x V(z) \cdot dz = \int_{-1/2}^{1/2} \eta \cdot \nabla_x V(x - r\eta) dr = \eta \cdot W[\nabla_x V](x, \eta), \quad (2.5)$$

with the vector-valued function $W[\cdot]$ defined by

$$W[\nabla_x V](x, \eta) := \int_{-1/2}^{1/2} \nabla_x V(x - r\eta) dr, \quad \forall (x, \eta) \in \mathbb{R}^6. \quad (2.6)$$

By introducing a new vector-valued operator

$$\Gamma[\nabla_x V]u(x, v) := \mathcal{F}_{\eta \rightarrow v}^{-1} \left(W[\nabla_x V](x, \eta) \mathcal{F}_{v \rightarrow \eta} u(x, \eta) \right), \quad (2.7)$$

we formally obtain

$$\Theta[V]u(x, v) = \operatorname{div}_v (\Gamma[\nabla_x V]u)(x, v). \quad (2.8)$$

For the conditions under which this redefinition of $\Theta[V]$ holds rigorously the reader is referred to Section 4.1 in [ADM].

Moreover, we shall exploit that

$$\int_{\mathbb{R}^3} G(t, x - x_0, v, v_0) dv = R(t)^{-3/2} \mathcal{N} \left(\frac{x - x_0 - \vartheta(t)v_0}{\sqrt{R(t)}} \right), \quad (2.9)$$

with

$$\mathcal{N}(x) := (2\pi)^{-3/2} \exp \left(-\frac{|x|^2}{2} \right), \quad (2.10)$$

$$\vartheta(t) := \frac{1 - e^{-\beta t}}{\beta} = \mathcal{O}(t), \quad \text{for } t \rightarrow 0; \quad \vartheta(t) := t, \quad \text{if } \beta = 0, \quad (2.11)$$

$$R(t) := 2\alpha t + \sigma \left(\frac{4e^{-\beta t} - e^{-2\beta t} + 2\beta t - 3}{\beta^3} \right) + 4\gamma \left(\frac{e^{-\beta t} + \beta t - 1}{\beta^2} \right). \quad (2.12)$$

The parameter α (more precisely, $\alpha > 0$ or $\alpha = 0$) determines the asymptotic behaviour at $t = 0$ of the function $R(t)$, and hence the singularity of the convolution kernel (2.9) at $t = 0$. Since this convolution represents the parabolic regularization of the quantum Fokker-Planck operator, we have to distinguish the following two cases for the subsequent analysis:

$$\alpha\sigma \geq \gamma^2 \quad \text{and} \quad \alpha > 0 \quad \Rightarrow \quad R(t) = \mathcal{O}(t), \quad \text{for } t \rightarrow 0, \quad (\text{I})$$

$$\alpha = \gamma = 0, \quad \beta \geq 0 \quad \text{and} \quad \sigma > 0 \quad \Rightarrow \quad R(t) = \mathcal{O}(t^3), \quad \text{for } t \rightarrow 0. \quad (\text{II})$$

We now obtain the following expression for the density n by formally integrating equation (2.1) in v and using the redefinition (2.8):

$$\begin{aligned} n(x, t) &= \int_{\mathbb{R}^3} w(x, v, t) dv \\ &= \frac{1}{R(t)^{3/2}} \iint \mathcal{N} \left(\frac{x - x_0 - \vartheta(t)v_0}{\sqrt{R(t)}} \right) w_0(x_0, v_0) dx_0 dv_0 \\ &\quad + \int_0^t \frac{1}{R(s)^{3/2}} \iint \mathcal{N} \left(\frac{x - x_0 - \vartheta(s)v_0}{\sqrt{R(s)}} \right) \operatorname{div}_{v_0} (\Gamma[\nabla_{x_0} V]w)(x_0, v_0, t - s) dx_0 dv_0 ds \\ &= \frac{1}{R(t)^{3/2}} \iint \mathcal{N} \left(\frac{x - x_0}{\sqrt{R(t)}} \right) w_0(x_0 - \vartheta(t)v_0, v_0) dx_0 dv_0 \\ &\quad + \int_0^t \frac{\vartheta(s)}{R(s)^2} \iint (\nabla_x \mathcal{N}) \left(\frac{x - x_0 - \vartheta(s)v_0}{\sqrt{R(s)}} \right) \cdot (\Gamma[\nabla_{x_0} V]w)(x_0, v_0, t - s) dx_0 dv_0 ds. \end{aligned}$$

Following common practice for the VP (cf. [LiPe]) and VPFP systems (cf. [Bo1]), next we split the density into two terms: $n = n_0 + n_1$, where

$$n_0(x, t) := \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right) *_x n_0^\vartheta(x, t), \quad n_0^\vartheta(x, t) := \int w_0(x - \vartheta(t)v, v) dv, \quad (2.13)$$

and

$$n_1(x, t) := \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x \operatorname{div}_x \int (\Gamma[E]w)(x - \vartheta(s)v, v, t - s) dv ds. \quad (2.14)$$

Analogously, we can split the self-consistent field into $E = E_0 + E_1$, where

$$E_0(x, t) := \lambda \frac{x}{|x|^3} *_x n_0(x, t), \quad (2.15)$$

$$E_1(x, t) := \lambda \frac{x}{|x|^3} *_x n_1(x, t), \quad (2.16)$$

with $\lambda = \frac{1}{4\pi}$. (2.14) now allows to rewrite E_1 as

$$(E_1)_j(x, t) = \lambda \sum_{k=1}^3 \frac{-3x_j x_k + \delta_{jk} |x|^2}{|x|^5} *_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F_k[w](x, t, s) ds, \quad (2.17)$$

$$\text{with } F_k[w](x, t, s) := \int (\Gamma_k[E_0 + E_1]w)(x - \vartheta(s)v, v, t - s) dv. \quad (2.18)$$

This is a linear Volterra integral equation of the second kind for the self-consistent field E_1 . Note that all coefficients in the r.h.s. of (2.17) only depend on w_0 and w (and not on n). The advantage of the reformulation (2.8) of the pseudo-differential operator is precisely to obtain a closed equation for E_1 , if w_0 and w are given. Starting with $w \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$, we shall prove that this integral equation has a unique solution. We remark that (2.14), instead, is not a closed equation for n_1 (for w_0 and w given); its r.h.s. also depends on the self-consistent field E . These motivations lead to our new definition of the Hartree-potential:

Definition 2.1 (New definition of mean-field quantities) *To a Wigner trajectory $w \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$ we associate*

- *the field $E[w] := E_0 + E_1[w]$, with E_0 given by (2.15), and $E_1[w]$ the unique solution of (2.17),*
- *the potential $V[w] := V_0 + V_1[w]$ with*

$$V_0(x, t) := \lambda \sum_{i=1}^3 \frac{x_i}{|x|^3} *_x (E_0)_i(x, t), \quad (2.19)$$

$$V_1[w](x, t) := \lambda \sum_{i=1}^3 \frac{x_i}{|x|^3} *_x (E_1[w])_i(x, t), \quad (2.20)$$

- *and the position density $n[w] := -\operatorname{div} E[w]$ (at least in a distributional sense).*

In contrast to the standard definitions (1.4)-(1.6), these new definitions are *non-local in time*. Also, the map $w \mapsto V[w]$ is now *non-linear*. For a given Wigner-trajectory these two definitions clearly differ in general. However, they coincide if w is the solution of the WPFP

system. These new definitions of the self-consistent field and potential have the advantage that they only require $w \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$ and not $w(x, \cdot, t) \in L^1(\mathbb{R}^3)$. If $w(t = 0)$ only lies in $L^2(\mathbb{R}^6)$, the corresponding field and the potential will consequently only be defined for $t > 0$.

We shall now describe in detail our strategy to prove well-posedness of the WPFP system

$$w_t = Aw + \Theta[V[w]]w, \quad t \in (0, T]; \quad w(t = 0) = w_0 \in L^2(\mathbb{R}^6). \quad (2.21)$$

(2.21) will be solved by a contractive fixed point map that is based on the linear equation

$$\tilde{u}_t = A\tilde{u} + \Theta[V[u]]\tilde{u}, \quad t \in (0, T]; \quad \tilde{u}(t = 0) = w_0. \quad (2.22)$$

While such an approach is standard for nonlinear PDEs, the key point is here the non-local definition of $V[u]$ via Definition 2.1. We will proceed as follows:

- (i) the iteration will be considered in the set B_R , the ball of radius R in $\mathcal{C}([0, T]; L^2(\mathbb{R}^6))$, centered in the origin. Due to the a-priori estimate (4.4), we shall choose $R := e^{\frac{3}{2}\beta T} \|w_0\|_2$.
- (ii) We will assume $w_0 \in L^2(\mathbb{R}^6)$ and satisfying **(A)** or **(B)** (see below). This will provide L^p -estimates on the field E_0 , defined in (2.15). Accordingly, for $u \in B_R$, Definition 2.1 will yield a unique potential $V[u]$.
- (iii) The estimates on $V[u]$ from (ii) will allow to prove existence and uniqueness of a mild solution for (2.22) that will satisfy $\tilde{u} \in B_R$.
- (iv) We will finally define the non-linear map $M : B_R \rightarrow B_R$ by $Mu := \tilde{u}$. Its unique fixed point will be the mild solution $w \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$ of the WPFP system (1.3)-(1.6), in the sense of (2.1).

We shall now specify the assumptions on the initial data w_0 that are mentioned in point (ii). We shall make two different assumptions on w_0 , which will lead, however, to similar estimates on $E[u]$ and consequently on $V[u]$ (cf. Section 3).

In Section 4 we shall first prove existence and uniqueness of a mild solution of problem (1.3)-(1.6) under the assumption

$$w_0 \in L^2(\mathbb{R}^6) \quad \text{and} \quad \|n_0^\vartheta(t)\|_{L^\theta(\mathbb{R}_x^3)} \leq C_T t^{-\omega_\theta}, \quad \text{for some } \omega_\theta \geq 0, \quad \forall t \in (0, T]. \quad (\mathbf{A})$$

For example, such an estimate for the “shifted” density n_0^ϑ (cf. (2.13)) can be concluded by the Strichartz estimate for the (free) kinetic equation [CP]

$$\|n_0^\vartheta(t)\|_{L^\theta(\mathbb{R}_x^3)} \leq C t^{-\omega_S(\theta)} \|w_0\|_{L_x^1(L_v^\theta)}, \quad \forall t \in (0, T], \quad (2.23)$$

with $\omega_S(\theta) := 3(1 - 1/\theta)$. At least in case **(I)** this typical example is always included in our main result, Theorem 4.1.

Here we introduce the constants determining the decay of n_0^ϑ that will be admitted in the subsequent analysis (cf. Thm. 4.1, e.g.).

$$\theta \in I_\theta := \begin{cases} [1, \frac{6}{5}], & \text{case (I)}, \\ (\frac{9}{8}, \frac{6}{5}], & \text{case (II)}, \end{cases} \quad \kappa(\theta) := \begin{cases} 2 - \frac{3}{2\theta}, & \text{case (I)}, \\ 4 - \frac{9}{2\theta}, & \text{case (II)}. \end{cases}$$

Alternatively to assumption **(A)** we shall also consider initial data that satisfy:

$$w_0 \in L^2(\mathbb{R}^6) \cap L^1(\mathbb{R}_v^3; L^\theta(\mathbb{R}_x^3)), \quad \text{for some } \theta \in \left[1, \frac{6}{5}\right]. \quad (\mathbf{B})$$

Considering estimate (2.23), this second assumption is complementary to the first one in the sense that the x - and v -integrability of w_0 are interchanged. As we shall see, the well-posedness analysis performed under the assumption **(A)** will immediately extend to initial data satisfying **(B)**.

The main result of this paper is

Theorem 4.1 *Let either **(A)** hold for some $\theta \in I_\theta$ and $0 \leq \omega_\theta < \kappa(\theta)$, or let **(B)** hold for some $\theta \in I_\theta$. Then, there exists a unique mild solution $w \in \mathcal{C}([0, \infty); L^2(\mathbb{R}^6))$ of the WPFP problem (1.3)-(1.6). In case **(I)**, we also get $V[w] \in \mathcal{C}((0, \infty); L^\infty(\mathbb{R}^3))$.*

A-posteriori, we shall obtain the following regularity result for the solution:

Theorem 5.1 *Let **(A)** hold for some $\theta \in I_\theta$ and $0 \leq \omega_\theta < \kappa(\theta)$, or let **(B)** hold for some $\theta \in I_\theta$ (in the latter case set $\omega_\theta := 0$). Then, the unique mild solution $w \in \mathcal{C}([0, \infty); L^2(\mathbb{R}^6))$ of the WPFP problem (1.3)-(1.6) satisfies*

$$w \in \mathcal{C}((0, \infty); \mathcal{C}_B^\infty(\mathbb{R}^6)),$$

with the estimate

$$\|D_x^l D_v^m w(t)\|_{L^2(\mathbb{R}^6)} \leq C\left(T, \|w\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{-\frac{L}{2}} t^{-\frac{M}{2}}, \quad \forall t \in (0, T],$$

for all $T > 0$, and all multiindices $l, m \in \mathbb{N}_0^3$, with $|l| = L, |m| = M \in \mathbb{N}_0$.

Moreover, $E[w] \in \mathcal{C}((0, \infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$, satisfying for all $T > 0$:

$$\|D_x^l E[w](t)\|_{L^2(\mathbb{R}^3)} \leq C\left(T, \|w\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{\frac{3}{2}(\frac{1}{2} - \frac{1}{\theta}) - \frac{L-1}{2}} t^{-\omega_\theta}, \quad \forall t \in (0, T],$$

where $D_x^l E[w]$ represents the derivative with multiindex l of each component of the field $E[w]$. Accordingly (cf. Def. 2.1), the density $n[w] = -\operatorname{div} E[w] \in \mathcal{C}((0, \infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$ satisfies in particular

$$\|n[w](t)\|_{L^2(\mathbb{R}^3)} \leq C\left(T, \|w\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{\frac{3}{2}(\frac{1}{2} - \frac{1}{\theta})} t^{-\omega_\theta}, \quad \forall t \in (0, T].$$

The self-consistent potential $V[w] \in \mathcal{C}((0, \infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$ and its Fourier transform $\widehat{V}[w](t)$ satisfy the estimates

$$\begin{aligned} \|V[w](t)\|_{L^\infty(\mathbb{R}^3)} &\leq C\left(T, \|w\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{1 - \frac{3}{2\theta}} t^{-\omega_\theta}, \quad \forall t \in (0, T], \\ \|\widehat{V}[w](t)\|_{L^1(\mathbb{R}^3)} &\leq C\left(T, \|w\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{1 - \frac{3}{2\theta}} t^{-\omega_\theta}, \quad \forall t \in (0, T]. \end{aligned}$$

Analogous regularity results on the classical VPFP equation were obtained in [Bo2] (Hölder regularity of the density and field) and rather recently in [OS] ($w, n, E \in C^\infty$ for positive time). Under the assumption **(B)**, we shall also show for WPFP that $w \in \mathcal{C}([0, \infty); L_v^1(L_x^\theta))$. Hence, the solution $w(t)$ remains in the space of the initial condition w_0 (cf. **(B)**). This allows to define the position density $n[w]$ in the standard sense (cf. Def. 1.1) and to derive an additional decay estimates for the density:

Theorem 6.1 *Let **(B)** hold for some $\theta \in I_\theta$. Then, the solution of the WPFP problem (1.3)-(1.6) satisfies*

$$(i) \quad w \in \mathcal{C}([0, \infty); L_v^1(L_x^\theta)),$$

(ii) *the density $n(t)$ satisfies for all $T > 0$ and $\theta \leq p \leq 2$:*

$$\|n(t)\|_{L^p(\mathbb{R}^3)} \leq C\left(T, \|w\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, \|w_0\|_{L_v^1(L_x^\theta)}\right) R(t)^{\frac{3}{2}(\frac{1}{p} - \frac{1}{\theta})}, \quad \forall t \in (0, T].$$

3 A-priori estimates for the self-consistent potential

In this section we shall derive a-priori estimates for the previously defined fields $E_0, E_1[w]$ and the potentials $V_0, V_1[w]$. Such estimates rely on dispersive effects of the free-streaming operator *and* on the parabolic regularization of the quantum Fokker-Planck operator. Since the regularization is different in the two cases (I) and (II), the corresponding estimates will also differ. The following a-priori estimates generalize the results in Section 4.2 of [ADM] to the hypoelliptic case (II). To make this paper self-contained we shall include a sketch of the proofs.

We start with an estimate on the field E_0 , defined in (2.15) (cf. Lemma 4.13 in [ADM]):

Proposition 3.1 *Let (A) hold for some $1 \leq \theta \leq 6/5$. Then, for all $p \geq 2$, the estimate*

$$\|E_0(t)\|_{L^p(\mathbb{R}_x^3)} \leq C_T R(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})+\frac{1}{2}} t^{-\omega_\theta}, \quad \forall t \in (0, T] \quad (3.1)$$

holds.

PROOF. The estimate is obtained by applying first the generalized Young inequality with $1/q = 1/p + 1/3$, and then the Young inequality with $1/q = 1/r + 1/\theta - 1$ to (2.15)

$$\begin{aligned} \|E_0(t)\|_p &\leq C \left\| \frac{1}{R(t)^{3/2}} \mathcal{N} \left(\frac{x}{\sqrt{R(t)}} \right) *_x n_0^\vartheta(x, t) \right\|_q \\ &\leq C \left\| \frac{1}{R(t)^{3/2}} \mathcal{N} \left(\frac{x}{\sqrt{R(t)}} \right) \right\|_r \|n_0^\vartheta(x, t)\|_\theta \\ &= C_T R(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})+\frac{1}{2}} \vartheta(t)^{-\omega_\theta}. \end{aligned}$$

The assumed restriction on θ is necessary for the case $p = 2$. □

Let us denote

$$\begin{aligned} N_\theta &\equiv N_\theta(T) := \sup_{s \in (0, T]} \left\{ s^{\omega_\theta} \|n_0^\vartheta(s)\|_{L^\theta} \right\} < \infty, \\ \mu(\theta) &:= \begin{cases} \frac{9}{4} - \frac{3}{2\theta}, & \text{case (I)}, \\ \frac{19}{4} - \frac{9}{2\theta}, & \text{case (II)}. \end{cases} \end{aligned}$$

For a given Wigner-trajectory $u \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$ we now consider the inhomogeneous integral equation for the field $E_1 = E_1[u]$:

$$(E_1)_j(x, t) = \lambda \sum_{k=1}^3 \frac{-3x_j x_k + \delta_{jk} |x|^2}{|x|^5} *_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N} \left(\frac{x}{\sqrt{R(s)}} \right) *_x F_k[u](x, t, s) ds, \quad (3.2)$$

$$\text{with } F_k[u](x, t, s) := \int (\Gamma_k[E_0 + E_1]u)(x - \vartheta(s)v, v, t - s) dv,$$

and the vector valued operator $\Gamma[E_0 + E_1]$ defined in (2.6)-(2.7).

Proposition 3.2 *Let $u \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$ and (A) hold for some $1 \leq \theta \leq 6/5$ and $0 \leq \omega_\theta < \mu(\theta)$. Then, the integral equation (3.2) has a unique solution $E_1 = E_1[u] \in \mathcal{C}((0, T]; L^p(\mathbb{R}_x^3))$, which satisfies*

$$\|E_1[u](t)\|_{L^p(\mathbb{R}_x^3)} \leq C \left(T, \|u\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, N_\theta \right) R(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})+\frac{1}{2}t^{\frac{1}{2}-\omega_\theta}}, \quad \forall t \in (0, T], \quad (3.3)$$

for

$$2 \leq p < p_1 := \begin{cases} 6, & \text{case (I)}, \\ \frac{18}{7}, & \text{case (II)}. \end{cases} \quad (3.4)$$

Hence, $E[u] = E_0 + E_1[u]$ satisfies

$$\|E[u](t)\|_{L^p(\mathbb{R}_x^3)} \leq C \left(T, \|u\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, N_\theta \right) R(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})+\frac{1}{2}t^{-\omega_\theta}}, \quad \forall t \in (0, T], \quad (3.5)$$

for $2 \leq p < p_1$.

PROOF. By Lemma 4.6 in [ADM] it holds for $T \geq t \geq s > 0$:

$$\|F_k[u](\cdot, t, s)\|_{L^2(\mathbb{R}_x^3)} \leq C \vartheta(s)^{-3/2} \|(E_0 + E_1)(t-s)\|_{L^2(\mathbb{R}_x^3)} \|u(t-s)\|_{L^2(\mathbb{R}_x^3)}. \quad (3.6)$$

Proceeding as in the proof of Lemma 4.13 in [ADM] one can show the existence and uniqueness of the solution $E_1 = E_1[u]$ of (3.2) by a Banach fixed point argument in the space

$$\left\{ E \in \mathcal{C}((0, T]; L^2(\mathbb{R}^3)) \left| \sup_{0 < t \leq T} t^{\omega_\theta - \frac{1}{2}} R(t)^{\frac{3}{2\theta} - \frac{5}{4}} \|E(t)\|_{L^2} < \infty \right. \right\}.$$

Moreover, we get as in Proposition 4.15 of [ADM] the following estimate for the solution of (3.2). Using classical properties of the convolution with $\frac{1}{|x|}$ (cf. [St]) yields:

$$\begin{aligned} \|E_1[u](t)\|_{L^p(\mathbb{R}_x^3)} &\leq C \int_0^t \vartheta(s) \left\| \frac{1}{R(s)^{3/2}} \mathcal{N} \left(\frac{x}{\sqrt{R(s)}} \right) *_x F[u](x, t, s) \right\|_{L^p(\mathbb{R}_x^3)} ds \\ &\leq C \int_0^t \vartheta(s) \left\| \frac{1}{R(s)^{3/2}} \mathcal{N} \left(\frac{x}{\sqrt{R(s)}} \right) \right\|_{L^q(\mathbb{R}_x^3)} \|F[u](x, t, s)\|_{L^2(\mathbb{R}_x^3)} ds \\ &\leq C \|u\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))} \int_0^t \frac{R(s)^{\frac{3}{2q} - \frac{3}{2}}}{\sqrt{\vartheta(s)}} \left(\|E_0(t-s)\|_2 + \|E_1[u](t-s)\|_2 \right) ds, \end{aligned} \quad (3.7)$$

with $1/2 + 1/p = 1/q$. For the integrability of the function $\|E_0(t)\|_{L^2(\mathbb{R}_x^3)}$ in $(0, T]$ (cf. Proposition 3.1), we need $\omega_\theta < \mu(\theta)$. The assertion for $p = 2$ follows from Gronwall's Lemma for (3.7). Then, we get (3.3) for $p > 2$ by using (3.7), provided condition (3.4) holds. \square

Via (2.19)-(2.20), the above estimates on the field $E[u] = E_0 + E_1[u]$ immediately yield estimates for the potential $V[u] := V_0 + V_1[u]$:

Corollary 3.3 *Let $u \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$ and (A) hold for some $1 \leq \theta \leq 6/5$ and $0 \leq \omega_\theta < \mu(\theta)$. Then, the potential $V[u] \in \mathcal{C}((0, T]; L^p(\mathbb{R}_x^3))$ satisfies $\forall t \in (0, T]$:*

$$\|V_0(t)\|_p \leq C_T R(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})+1} t^{-\omega_\theta}, \quad (3.8)$$

$$\|V_1[u](t)\|_p \leq C \left(T, \|u\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, N_\theta \right) R(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})+1} t^{\frac{1}{2}-\omega_\theta}, \quad (3.9)$$

$$\|V[u](t)\|_p \leq C \left(T, \|u\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, N_\theta \right) R(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})+1} t^{-\omega_\theta}, \quad (3.10)$$

for $6 \leq p \leq \infty$ in the case (I) or $6 \leq p < 18$ in the case (II).

PROOF. The admissible p -intervals follow immediately from condition (3.4) on E_1 . \square

Remark 3.4 *We note that a-posteriori regularity of the solution $w(t)$ will imply for the self-consistent potential, $V[w](t) \in L^\infty(\mathbb{R}^3)$, $t > 0$ also in case (II) (cf. Theorem 5.1).*

To close this section we shall now derive the analogous a-priori estimates under the assumption (B). For $T > 0$ fixed, we consider the density

$$n_0(x, t) = \iint G(t, x, v, v_0) *_x w_0(x, v_0) dv_0 dv, \quad \forall t \in [0, T]; \quad (3.11)$$

cf. (2.13) for a different representation of n_0 . The following decay estimate for the (classical) Vlasov-FP equation (cf. Lemma 2 in [Car]) carries over to the Green's function G (cf. [SCDM]) for the WFP equation:

$$\| \iint G(t, x, v, v_0) *_x w_0(x, v_0) dv_0 \|_{L_x^p(L_v^1)} \leq CR(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})} \|w_0\|_{L_v^1(L_x^\theta)}, \quad \forall p \geq \theta, \quad \forall t \in (0, T]. \quad (3.12)$$

Hence, if we assume

$$w_0 \in L^2(\mathbb{R}^6) \cap L^1(\mathbb{R}_v^3; L^\theta(\mathbb{R}_x^3)), \quad \text{for some } \theta \in \left[1, \frac{6}{5}\right], \quad (\mathbf{B})$$

then (3.12) implies

$$\|n_0(t)\|_{L_x^p} \leq CR(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})} \|w_0\|_{L_v^1(L_x^\theta)}, \quad \forall p \geq \theta, \quad \forall t \in (0, T]. \quad (3.13)$$

Thus, we can handle the affine term E_0 analogously to Proposition 3.1:

Corollary 3.5 *Let (B) hold for some $1 \leq \theta \leq 6/5$. Then, we have for all $p \geq 2$ and $t \in (0, T]$:*

$$\|E_0(t)\|_{L_x^p} \leq CR(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})+\frac{1}{2}} \|w_0\|_{L_v^1(L_x^\theta)}, \quad (3.14)$$

$$\|V_0(t)\|_{L_x^p} \leq CR(t)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{\theta})+1} \|w_0\|_{L_v^1(L_x^\theta)}. \quad (3.15)$$

PROOF. This follows from (2.15), (2.19) with the generalized Young inequality and (3.13). \square

Remark 3.6 *Note that these decay rates of E_0 and V_0 correspond exactly to case (A) with $\omega_\theta = 0$ (cf. (3.1), (3.8)). Hence, all results of Proposition 3.2 and Corollary 3.3 carry over to case (B) when setting $\omega_\theta = 0$. In the subsequent well-posedness analysis for WFPF, the only relevant information on w_0 is the rate of singularity at $t = 0$ (and hence the integrability on $(0, T]$) of E_0 and V_0 . In this respect, the analysis of the WFPF problem under assumption (B) appears just as a special case of the situation under assumption (A). Therefore, the existence and uniqueness result of Theorem 4.1 for case (A) directly implies an analogous result for case (B).*

4 Existence and uniqueness of a global solution

The goal of this section is to prove the following

Theorem 4.1 *Let either (A) hold for some $\theta \in I_\theta$ and $0 \leq \omega_\theta < \kappa(\theta)$, or let (B) hold for some $\theta \in I_\theta$. Then, there exists a unique mild solution $w \in \mathcal{C}([0, \infty); L^2(\mathbb{R}^6))$ of the WFPF problem (1.3)-(1.6). In case (I), we also get $V \in \mathcal{C}((0, \infty); L^\infty(\mathbb{R}^3))$.*

We will follow the strategy outlined in Section 2. Theorem 4.1 will follow from a sequence of auxiliary results that we derive first. In this section we shall only discuss the analysis according to the assumption **(A)**. Due to Remark 3.6, however, all results of this section apply verbatim (with $\omega_\theta = 0$) to case **(B)**.

Let $T > 0$ be arbitrary but fixed, $w_0 \in L^2(\mathbb{R}^6)$, and set $R := e^{\frac{3}{2}\beta T} \|w_0\|_2$. Let us denote with B_R the ball of radius R centered in the origin of $\mathcal{C}([0, T]; L^2(\mathbb{R}^6))$ and let u belong to B_R . Then, we consider the linear equation

$$\tilde{u}_t = A\tilde{u} + \Theta[V[u]]\tilde{u}, \quad t \in (0, T], \quad (4.1)$$

with the initial value

$$\tilde{u}(t=0) = w_0.$$

In case **(I)**, we have $V[u](t) \in L^\infty(\mathbb{R}^3)$ for $t > 0$ (cf. (3.10)) and hence $\Theta[V[u](t)]$ is a bounded linear operator on $L^2(\mathbb{R}^6)$, which satisfies

$$\|\Theta[V[u](t)]\|_{\mathcal{B}(L^2(\mathbb{R}^6))} \leq C\left(T, \|u\|_{\mathcal{C}([0, T]; L^2(\mathbb{R}^6))}, N_\theta\right) t^{1-\frac{3}{2\theta}-\omega_\theta}, \quad \forall t \in (0, T]. \quad (4.2)$$

Moreover, if $\omega_\theta < \kappa(\theta)$, then $\Theta[V[u](\cdot)] \in L^1((0, T); \mathcal{B}(L^2(\mathbb{R}^6)))$.

In case **(II)**, however, we lack an a-priori bound for $\|V[u](t)\|_\infty$. Thus, $\Theta[V[u](t)]$ will *not* be a bounded operator on $L^2(\mathbb{R}^6)$. Instead, we shall exploit the a-priori bound for $\|V[u](t)\|_6$, jointly with the regularization of the semigroup e^{tA} . This is the key-idea of the following

Proposition 4.2 *Let **(A)** hold for some $\theta \in I_\theta$ and $0 \leq \omega_\theta < \kappa(\theta)$. Also assume that $u \in B_R$. Then, the equation (4.1) has a unique mild solution $\tilde{u} \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$, which satisfies*

$$\begin{aligned} \tilde{u}(x, v, t) &= \int G(t, x, v, v_0) *_x w_0(x, v_0) dv_0 \\ &+ \int_0^t \int G(s, x, v, v_0) *_x (\Theta[V[u]]\tilde{u})(x, v_0, t-s) dv_0 ds \end{aligned} \quad (4.3)$$

and

$$\|\tilde{u}(t)\|_2 \leq e^{\frac{3}{2}\beta t} \|w_0\|_2, \quad \forall t \in [0, T]. \quad (4.4)$$

PROOF. In case **(I)**, $A - \frac{3}{2}\beta I$ generates a C_0 semigroup of contractions on $L^2(\mathbb{R}^6)$ (see Section 2.2 in [ADM] for the details) and $\Theta[V[u](\cdot)]$ is a bounded perturbation, integrable in time. The assertion then follows from standard semigroup theory (cf. Thm. 6.1.2 in [Pa]).

In case **(II)**, we define the affine map P for all $\tilde{z} \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$:

$$\begin{aligned} P\tilde{z}(x, v, t) &:= \int G(t, x, v, v_0) *_x w_0(x, v_0) dv_0 \\ &+ \int_0^t \int G(s, x, v, v_0) *_x (\Theta[V[u]]\tilde{z})(x, v_0, t-s) dv_0 ds. \end{aligned} \quad (4.5)$$

We will show that it has a unique fixed point $\tilde{u} \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$. The crucial step is to prove that P maps into $\mathcal{C}([0, T]; L^2(\mathbb{R}^6))$.

To this end we first state the following estimate on the x -derivatives (with multiindex l) of the Green's function g (cf. (2.4)) that can be proved directly by calculating the integral. It reflects the regularization of the semigroup e^{At} .

$$\|D_{x,v}^l g(t)\|_{L_{x,v}^1} \leq C_T R(t)^{-\frac{L}{2}}, \quad \forall t \leq T, |l| = L, L \geq 0. \quad (4.6)$$

Thus, by the Sobolev embedding $W^{\frac{1}{2},1}(\mathbb{R}_x^3) \hookrightarrow L^{6/5}(\mathbb{R}_x^3)$ and interpolation in (4.6) between $L = 0$ and $L = 1$, we get

$$\|g(t)\|_{L_v^1(L_x^{6/5})} \leq C_T R(t)^{-\frac{1}{4}}, \quad \forall t \leq T. \quad (4.7)$$

Next we note that G does *not* act in (4.5) as a convolution in the v -variable. However, it *is* a convolution in the characteristic variables $\bar{x} := x + v \left(\frac{1-e^{\beta t}}{\beta} \right)$, $\bar{v} := v e^{\beta t}$ (cf. (2.3)):

$$\iint G(t, x - x_0, v, v_0) \phi(x_0, v_0) dx_0 dv_0 = e^{3\beta t} (g(t) *_{x,v} \phi)(\bar{x}, \bar{v}). \quad (4.8)$$

By using the Jacobian of the transformation, $\left| \frac{d(x,v)}{d(\bar{x},\bar{v})} \right| = e^{-\frac{3}{2}\beta t}$, it follows for all p, q such that $\frac{1}{p} + \frac{1}{q} = \frac{3}{2}$:

$$\begin{aligned} \left\| \int G(t, x, v, v_0) *_{x,v} \phi(x, v_0) dv_0 \right\|_{L_{x,v}^2} &= e^{\frac{3}{2}\beta t} \|g(t) *_{x,v} \phi\|_{L_{x,v}^2} \\ &\leq e^{\frac{3}{2}\beta t} \|g(t)\|_{L_v^1(L_x^p)} \|\phi\|_{L_v^2(L_x^q)}, \quad \forall \phi \in L_v^2(L_x^q). \end{aligned} \quad (4.9)$$

In addition, it can be checked by duality that

$$\|(\Theta[V[u]]\tilde{z})(t-s)\|_{L_v^2(L_x^{3/2})} \leq 2\|V[u](t-s)\|_{L_x^6} \|\tilde{z}(t-s)\|_{L_{x,v}^2}. \quad (4.10)$$

Now we estimate the L^2 -norm of (4.5) by applying the results (4.6)-(4.10) and the a-priori bound (3.10) with $p = 6$. We finally get

$$\begin{aligned} \|P\tilde{z}(t)\|_{L_{x,v}^2} &\leq e^{\frac{3}{2}\beta t} \|w_0\|_{L_{x,v}^2} + 2e^{\frac{3}{2}\beta t} \int_0^t \|g(s)\|_{L_v^1(L_x^{6/5})} \|V[u](t-s)\|_{L_x^6} \|\tilde{z}(t-s)\|_{L_{x,v}^2} ds \\ &\leq e^{\frac{3}{2}\beta t} \|w_0\|_{L_{x,v}^2} + C\left(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \|\tilde{z}\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))} \int_0^t s^{-\frac{3}{4}} (t-s)^{\frac{9}{2}(\frac{1}{6}-\frac{1}{\theta})+3-\omega_\theta} ds. \end{aligned} \quad (4.11)$$

The condition $\omega_\theta < \kappa(\theta)$ guarantees that the last integral is in $\mathcal{C}[0, T]$.

Concerning the contractivity of P , we obtain analogously for all $\tilde{z}_1, \tilde{z}_2 \in \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$ by induction:

$$\begin{aligned} \|(P^n \tilde{z}_1 - P^n \tilde{z}_2)(t)\|_{L_{x,v}^2} &\leq C\left(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \\ &\quad \times \int_0^t s^{-\frac{3}{4}} (t-s)^{\frac{9}{2}(\frac{1}{6}-\frac{1}{\theta})+3-\omega_\theta} \|(P^{n-1} \tilde{z}_1 - P^{n-1} \tilde{z}_2)(t-s)\|_{L_{x,v}^2} ds \\ &\leq C\left(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right)^n C_{n-1} \|\tilde{z}_1 - \tilde{z}_2\|_{\mathcal{C}([0,T];L_{x,v}^2)} \int_0^t \frac{s^{(n-2)(1-a-b)-a}}{(t-s)^b} ds, \end{aligned}$$

for $n \in \mathbb{N}$ and some $a, b \geq 0$ with $a + b < 1$. Further,

$$\int_0^t \frac{s^{(n-2)(1-a-b)-a}}{(t-s)^b} ds = t^{(n-1)(1-a-b)} B\left(1-b, (n-2)(1-a-b) + 1-a\right),$$

and

$$C_{n-1} = \prod_{j=1}^{n-1} B\left(1-b, j(1-a-b) + 1-a\right) = \frac{\Gamma(1-b)^{n-1} \Gamma(1-a)}{\Gamma((n-1)(1-a-b) + 1)},$$

where B denotes the Beta function and Γ the Gamma function. Clearly,

$$C\left(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right)^n C_{n-1} \int_0^t \frac{s^{(n-2)(1-a-b)-a}}{(t-s)^b} ds < 1$$

for n large enough. Thus, the map P^n is contractive and admits a unique fixed point in B_R .

Formally, the L^2 -bound (4.4) follows from the dissipativity of the operator $A - \frac{3}{2}\beta I$ in $L^2(\mathbb{R}^6)$ and the skew-symmetry of $\Theta[V[u]]$. This can be justified as follows:

Applying $|\nabla_x|^{1/2}$ to (4.3) yields by using (4.6):

$$\begin{aligned} \| |\nabla_x|^{1/2} \tilde{u}(t) \|_{L^2_{x,v}} &\leq C(T) \| |\nabla_x|^{1/2} g(t) \|_{L^1_{x,v}} \|w_0\|_{L^2_{x,v}} \\ &\quad + C(T) \int_0^t \| |\nabla_x|^{1/2} g(s) \|_{L^1_{x,v}} \|(\Theta[V[u]]\tilde{u})(t-s)\|_{L^2_{x,v}} ds. \\ &\leq C(T) t^{-\frac{3}{4}} \|w_0\|_{L^2_{x,v}} \\ &\quad + C(T) \int_0^t s^{-\frac{3}{4}} \|V[u](t-s)\|_{L^6_x} \|\tilde{u}(t-s)\|_{L^2_v(\dot{H}^{1/2}_x)} ds, \end{aligned} \quad (4.12)$$

where $\|\cdot\|_{\dot{H}^{1/2}} = \| |\nabla|^{1/2} \cdot \|_{L^2}$ denotes the $H^{1/2}$ -seminorm on \mathbb{R}^3 . Applying the Gronwall Lemma to (4.12) then yields $t^{\frac{3}{4}} \tilde{u} \in \mathcal{C}([0, T]; L^2_v(H^{1/2}_x))$ and hence (using (3.10) with $p = 6$) $f := \Theta[V[u]]\tilde{u} \in \mathcal{C}((0, T]; L^2(\mathbb{R}^6)) \cap L^1((0, T); L^2(\mathbb{R}^6))$. Now (4.1) can be written as

$$\tilde{u}_t = A\tilde{u} + f(t), \quad t \in (0, T], \quad \tilde{u}(t=0) = w_0, \quad (4.13)$$

and the L^2 -estimate on \tilde{u} can finally be obtained by a standard approximation of \tilde{u} by classical solutions of (4.13) (cf. Thm. 4.2.7 in [Pa], Lemma 4.2 of [ADM] for the details). \square

We now consider the non-linear map $M : B_R \longrightarrow B_R$ defined as

$$Mu := \tilde{u}, \quad (4.14)$$

which is well-defined by the previous proposition. The next goal is to prove that the map M admits a unique fixed point in $B_R \subset \mathcal{C}([0, T]; L^2(\mathbb{R}^6))$, which will be the mild solution of our WPFP problem. To this end we need the following result.

Lemma 4.3 *Let (A) hold for some $1 \leq \theta \leq 6/5$ and $0 \leq \omega_\theta < \mu(\theta)$. Then, for $u_1, u_2 \in B_R$,*

$$\|V_1[u_1](t) - V_1[u_2](t)\|_{L^6(\mathbb{R}^3_x)} \leq C(T, R, N_\theta) R(t)^{\frac{5}{4} - \frac{3}{2\theta} t^{\frac{1}{2} - \omega_\theta}} \|u_1 - u_2\|_{\mathcal{C}([0,t]; L^2(\mathbb{R}^6))} \quad (4.15)$$

holds for all $t \in (0, T]$. In the case (I), it also holds

$$\|V_1[u_1](t) - V_1[u_2](t)\|_{L^\infty(\mathbb{R}^3_x)} \leq C(T, R, N_\theta) t^{\frac{3}{2} - \frac{3}{2\theta} - \omega_\theta} \|u_1 - u_2\|_{\mathcal{C}([0,t]; L^2(\mathbb{R}^6))}, \quad (4.16)$$

for all $t \in (0, T]$.

PROOF.

$$\begin{aligned} (E_1[u_1] - E_1[u_2])_j(x, t) &= \\ &= \lambda \sum_{k=1}^3 \frac{-3x_j x_k + \delta_{jk} |x|^2}{|x|^5} *_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x (F_k[u_1, u_1] - F_k[u_2, u_2])(x, t, s) ds, \end{aligned}$$

with

$$F_k[u, \tilde{u}](x, t, s) := \int (\Gamma_k[E_0 + E_1[u]]\tilde{u})(x - \vartheta(s)v, v, t - s) dv.$$

By classical properties of the convolution with $\frac{1}{|x|}$ (cf. [St]) and the Young inequality, we get

$$\|E_1[u_1](t) - E_1[u_2](t)\|_2 \leq C \int_0^t \vartheta(s) \| (F_k[u_1, u_1] - F_k[u_2, u_2])(t, s) \|_2 ds.$$

We write $F_k[u_1, u_1] - F_k[u_2, u_2] = F_k[u_1, u_1] - F_k[u_1, u_2] + F_k[u_1, u_2] - F_k[u_2, u_2]$.

By (3.6) (cf. Lemma 4.6 in [ADM]) we have for $t \geq s > 0$:

$$\|F_k[u_1, u_2](t, s)\|_{L^2(\mathbb{R}_x^3)} \leq C \vartheta(s)^{-3/2} \|(E_0 + E_1[u_1])(t - s)\|_{L^2(\mathbb{R}_x^3)} \|u_2(t - s)\|_{L^2(\mathbb{R}^6)}.$$

Then, we get

$$\begin{aligned} \|E_1[u_1](t) - E_1[u_2](t)\|_2 &\leq C \int_0^t \frac{1}{\sqrt{\vartheta(s)}} \left(\|(E_0 + E_1[u_1])(t - s)\|_2 \|(u_1 - u_2)(t - s)\|_2 \right. \\ &\quad \left. + \|(E_1[u_1] - E_1[u_2])(t - s)\|_2 \|u_2(t - s)\|_2 \right) ds. \end{aligned} \quad (4.17)$$

By (3.5) we obtain

$$\begin{aligned} \|E_1[u_1](t) - E_1[u_2](t)\|_2 &\leq \\ &\leq C(T, R, N_\theta) \int_0^t \frac{1}{\sqrt{\vartheta(s)}} R(t - s)^{\frac{5}{4} - \frac{3}{2\theta}} (t - s)^{-\omega_\theta} \|(u_1 - u_2)(t - s)\|_2 ds \\ &\quad + CR \int_0^t \frac{1}{\sqrt{\vartheta(s)}} \|(E_1[u_1] - E_1[u_2])(t - s)\|_2 ds. \end{aligned} \quad (4.18)$$

By Gronwall's Lemma we get for $t \in (0, T]$:

$$\begin{aligned} \|E_1[u_1](t) - E_1[u_2](t)\|_2 &\leq \\ &\leq C(T, R, N_\theta) \|u_1 - u_2\|_{C([0, t]; L^2(\mathbb{R}^6))} \left(\int_0^t \frac{1}{\sqrt{\vartheta(s)}} R(t - s)^{\frac{5}{4} - \frac{3}{2\theta}} (t - s)^{-\omega_\theta} ds \right. \\ &\quad \left. + e^{C_T R t^{1/2}} \int_0^t \int_0^s \frac{1}{\sqrt{\vartheta(s)\vartheta(\tau)}} R(s - \tau)^{\frac{5}{4} - \frac{3}{2\theta}} (s - \tau)^{-\omega_\theta} d\tau ds \right) \\ &\leq C(T, R, N_\theta) \|u_1 - u_2\|_{C([0, t]; L^2(\mathbb{R}^6))} R(t)^{\frac{5}{4} - \frac{3}{2\theta}} t^{\frac{1}{2} - \omega_\theta}. \end{aligned} \quad (4.19)$$

With

$$\|V_1[u_1](t) - V_1[u_2](t)\|_6 \leq C \|E_1[u_1](t) - E_1[u_2](t)\|_2$$

the assertion (4.15) follows.

To prove (4.16) in case **(I)** we proceed analogously and obtain by using Young's inequality

$$\begin{aligned} \|E_1[u_1](t) - E_1[u_2](t)\|_4 &\leq \\ &\leq C \int_0^t \vartheta(s) \left\| \frac{1}{R(s)^{3/2}} \mathcal{N} \left(\frac{x}{\sqrt{R(s)}} \right) \right\|_{4/3} \| (F_k[u_1, u_1] - F_k[u_2, u_2])(t, s) \|_2 ds \\ &\leq C \int_0^t \frac{R(s)^{-3/8}}{\sqrt{\vartheta(s)}} \left(\|(E_0 + E_1[u_1])(t - s)\|_2 \|(u_1 - u_2)(t - s)\|_2 \right. \\ &\quad \left. + \|(E_1[u_1] - E_1[u_2])(t - s)\|_2 \|u_2(t - s)\|_2 \right) ds. \end{aligned} \quad (4.20)$$

By applying the estimates (3.5) and (4.19), we then get

$$\|E_1[u_1](t) - E_1[u_2](t)\|_{L^4(\mathbb{R}_x^3)} \leq C(T, N_\theta, R) t^{\frac{11}{8} - \frac{3}{2\theta} - \omega_\theta} \|u_1 - u_2\|_{C([0,t]; L^2(\mathbb{R}^6))}. \quad (4.21)$$

The second assertion (4.16) then follows from the Gagliardo–Nirenberg inequality using estimates (4.15) and (4.21). \square

Proposition 4.4 *Let (A) hold for some $\theta \in I_\theta$ and $0 \leq \omega_\theta < \kappa(\theta)$. Then, the map M , defined by (4.14), has a unique fixed point in B_R .*

PROOF. We give the proof only for the case (II); case (I) is easier due to the boundedness of $\Theta[V[u]]$ (cf. (4.2) and (4.16)).

For $u_1, u_2 \in B_R$ we start from equation (4.3). Estimating like in the proof of Proposition 4.2 and with (4.15), we obtain

$$\begin{aligned} \|Mu_1(t) - Mu_2(t)\|_{L_{x,v}^2} &\leq \\ &\leq \int_0^t \left\| \int G(s, x, v, v_0) *_x (\Theta[V[u_1]] - \Theta[V[u_2]]) Mu_2(x, v_0, t-s) dv_0 \right\|_{L_{x,v}^2} ds \\ &\quad + \int_0^t \left\| \int G(s, x, v, v_0) *_x (\Theta[V[u_1]])(Mu_1 - Mu_2)(x, v_0, t-s) dv_0 \right\|_{L_{x,v}^2} ds \\ &\leq C(T, R, N_\theta) \int_0^t s^{-\frac{3}{4}}(t-s)^{\frac{17}{4} - \frac{9}{2\theta} - \omega_\theta} \|u_1 - u_2\|_{C([0,t-s]; L^2(\mathbb{R}^6))} ds \\ &\quad + C(T, R, N_\theta) \int_0^t s^{-\frac{3}{4}}(t-s)^{\frac{15}{4} - \frac{9}{2\theta} - \omega_\theta} \|(Mu_1 - Mu_2)(t-s)\|_{L_{x,v}^2} ds. \end{aligned}$$

By applying Gronwall's Lemma we get

$$\begin{aligned} \|Mu_1(t) - Mu_2(t)\|_{L_{x,v}^2} &\leq C(T, R, N_\theta) \int_0^t s^{-\frac{3}{4}}(t-s)^{\frac{17}{4} - \frac{9}{2\theta} - \omega_\theta} \|u_1 - u_2\|_{C([0,t-s]; L^2(\mathbb{R}^6))} ds \\ &\quad + C(T, R, N_\theta) \int_0^t s^{(\frac{9}{2} - \frac{9}{2\theta} - \omega_\theta) - \frac{3}{4}}(t-s)^{\frac{15}{4} - \frac{9}{2\theta} - \omega_\theta} \|u_1 - u_2\|_{C([0,s]; L^2(\mathbb{R}^6))} ds \end{aligned}$$

Then, the result follows by a contraction argument like in Proposition 4.2: The map M^n is contractive for n large enough. Thus, M admits a unique fixed point in B_R . \square

The above auxiliary results directly yield the

PROOF OF THEOREM 4.1.

The fixed point of the map M satisfies the equation (2.1) for all $T > 0$. Thus, it is the unique *global-in-time* mild solution of problem (1.3)–(1.6), in the sense of (2.1). \square

5 Regularity of the solution

In this section we shall establish the C^∞ -regularity of the unique, global, mild solution $(w, n, V, E = E_0 + E_1)$ of the WFPF-system (1.3)–(1.6). Based on a bootstrapping argument, our main result is

Theorem 5.1 *Let (A) hold for some $\theta \in I_\theta$ and $0 \leq \omega_\theta < \kappa(\theta)$, or let (B) hold for some $\theta \in I_\theta$ (in the latter case set $\omega_\theta := 0$). Then, the unique mild solution $w \in \mathcal{C}([0, \infty); L^2(\mathbb{R}^6))$ of the WFPF problem (1.3)–(1.6) satisfies*

$$w \in \mathcal{C}((0, \infty); \mathcal{C}_B^\infty(\mathbb{R}^6)),$$

with the estimate

$$\|D_x^l D_v^m w(t)\|_{L^2(\mathbb{R}^6)} \leq C\left(T, \|w\|_{C([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{-\frac{L}{2}} t^{-\frac{M}{2}}, \quad \forall t \in (0, T], \quad (5.1)$$

for all $T > 0$, and all multiindices $l, m \in \mathbb{N}_0^3$, with $|l| = L, |m| = M \in \mathbb{N}_0$.

Moreover, $E, V \in \mathcal{C}((0, \infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$, satisfying for all $T > 0$:

$$\|D_x^l E(t)\|_{L^2(\mathbb{R}^3)} \leq C\left(T, \|w\|_{C([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\theta})-\frac{L-1}{2}} t^{-\omega_\theta}, \quad \forall t \in (0, T], \quad (5.2)$$

where $D_x^l E$ represents the derivative with multiindex l of each component of the field E . Accordingly, the density $n = -\operatorname{div} E \in \mathcal{C}((0, \infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$ satisfies

$$\|n(t)\|_{L^2(\mathbb{R}^3)} \leq C\left(T, \|w\|_{C([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\theta})} t^{-\omega_\theta}, \quad \forall t \in (0, T]. \quad (5.3)$$

The self-consistent potential $V(t)$ and its Fourier transform $\widehat{V}(t)$ satisfy the estimates

$$\|V(t)\|_{L^\infty(\mathbb{R}^3)} \leq C\left(T, \|w\|_{C([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{1-\frac{3}{2\theta}} t^{-\omega_\theta}, \quad \forall t \in (0, T], \quad (5.4)$$

$$\|\widehat{V}(t)\|_{L^1(\mathbb{R}^3)} \leq C\left(T, \|w\|_{C([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{1-\frac{3}{2\theta}} t^{-\omega_\theta}, \quad \forall t \in (0, T]. \quad (5.5)$$

PROOF. A calculation as in Proposition 3.1 gives

$$\|D_x^l E_0(t)\|_{L^2(\mathbb{R}^3)} \leq C_T R(t)^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\theta})-\frac{L-1}{2}} t^{-\omega_\theta}, \quad \forall t \in (0, T]. \quad (5.6)$$

In the sequel we shall use the estimate (4.6) for the x -derivatives of the Green's function g and

$$\|D_v^m g(t)\|_{L_{x,v}^1} \leq C_T t^{-\frac{M}{2}}, \quad \forall t \leq T, |m| = M, M \geq 0, \quad (5.7)$$

which again can be proved directly by calculating the integrals.

Step 1:

By induction on $L = |l|$ we shall first prove the estimates for the x -derivatives, i.e. (5.1) for $m = 0$ and (5.2). We fix some multiindex $l = (l_1, l_2, l_3)$ with $L = |l| \geq 1$ and suppose that the inequalities (5.1), (5.2) hold for all multiindices \tilde{l} with $0 \leq |\tilde{l}| < L$.

We apply D_x^l to (2.1) and use (4.9) with $p = 1$ for the first summand and $p = \frac{6}{5}$ for the second one:

$$\begin{aligned} \|D_x^l w(t)\|_2 &\leq e^{\frac{3}{2}\beta t} \|D_x^l g(t)\|_{L_{x,v}^1} \|w_0\|_2 + \int_{t/2}^t e^{\frac{3}{2}\beta s} \|D_x^l g(s)\|_{L_v^1(L_x^{6/5})} \|(\Theta[V]w)(t-s)\|_{L_v^2(L_x^{3/2})} ds \\ &+ \sum_{\substack{0 \leq l_j^k \leq l_j \\ j=1,2,3}} \binom{l_1}{l_1^k} \binom{l_2}{l_2^k} \binom{l_3}{l_3^k} \int_0^{t/2} e^{\frac{3}{2}\beta s} \|g(s)\|_{L_v^1(L_x^{6/5})} \|(\Theta[D_x^{l-l^k} V] D_x^{l^k} w)(t-s)\|_{L_v^2(L_x^{3/2})} ds, \end{aligned} \quad (5.8)$$

where $l^k = (l_1^k, l_2^k, l_3^k)$. Here we had to split the time integral and apply D_x^l , respectively, to the first and second convolution factor. Without this procedure the resulting integrals would diverge either at $s = 0$ or at $s = t$. Using the estimates (4.6), (4.7), (4.10), and (3.10) we

obtain:

$$\begin{aligned}
\|D_x^l w(t)\|_2 &\leq C_T R(t)^{-\frac{L}{2}} \|w_0\|_2 + C_T \int_{t/2}^t R(s)^{-\frac{2L+1}{4}} \|V(t-s)\|_6 \|w(t-s)\|_2 ds \\
&\quad + C_T \sum_{\substack{0 \leq l_j^k \leq l_j \\ j=1,2,3}} \int_0^{t/2} R(s)^{-\frac{1}{4}} \|D_x^{l-l^k} V(t-s)\|_6 \|D_x^{l^k} w(t-s)\|_2 ds \\
&\leq C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \left[R(t)^{-\frac{L}{2}} + R(t)^{-\frac{L}{2} + \frac{3}{2}(\frac{1}{2} - \frac{1}{\theta}) + \frac{1}{4}t^{1-\omega_\theta}} \right] \\
&\quad + C_T \sum_{\substack{0 \leq l_j^k \leq l_j, j=1,2,3 \\ l^k \neq 0, l^k \neq l}} \int_0^{t/2} R(s)^{-\frac{1}{4}} \|D_x^{l-l^k} E(t-s)\|_2 \|D_x^{l^k} w(t-s)\|_2 ds \\
&\quad + C_T \int_0^{t/2} R(s)^{-\frac{1}{4}} \|D_x^l E(t-s)\|_2 \|w(t-s)\|_2 ds \\
&\quad + C_T \int_0^{t/2} R(s)^{-\frac{1}{4}} \|E(t-s)\|_{L_x^2} \|D_x^l w(t-s)\|_2 ds.
\end{aligned}$$

By considering the range of ω_θ , using (3.5), (5.6), (5.1), and (5.2) for $|l| < L$ we obtain

$$\begin{aligned}
\|D_x^l w(t)\|_2 &\leq C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \\
&\quad \times \left[R(t)^{-\frac{L}{2}} + \sum_{k=1}^{L-1} R(t)^{-\frac{1}{4} + \frac{3}{2}(\frac{1}{2} - \frac{1}{\theta}) - \frac{L-k-1}{2}} R(t)^{-\frac{k}{2}} t^{1-\omega_\theta} + R(t)^{\frac{3}{2}(\frac{1}{2} - \frac{1}{\theta}) - \frac{2L-1}{4}} t^{1-\omega_\theta} \right] \\
&\quad + C_T \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))} \int_0^{t/2} R(s)^{-\frac{1}{4}} \|D_x^l E_1(t-s)\|_2 ds \\
&\quad + C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \int_0^{t/2} R(s)^{-\frac{1}{4}} R(t-s)^{\frac{3}{2}(\frac{1}{2} - \frac{1}{\theta}) + \frac{1}{2}} (t-s)^{-\omega_\theta} \|D_x^l w(t-s)\|_2 ds \\
&\leq C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \left[R(t)^{-\frac{L}{2}} + \int_0^{t/2} R(s)^{-\frac{1}{4}} \|D_x^l E_1(t-s)\|_2 ds \right. \\
&\quad \left. + \int_0^{t/2} R(s)^{-\frac{1}{4}} R(t-s)^{\frac{3}{2}(\frac{1}{2} - \frac{1}{\theta}) + \frac{1}{2}} (t-s)^{-\omega_\theta} \|D_x^l w(t-s)\|_2 ds \right]. \tag{5.9}
\end{aligned}$$

Since this inequality for $\|D_x^l w(t)\|_2$ is not closed, we have to consider in parallel the derivatives of the field. As before, we apply D_x^l to (3.2) and use (3.5), (3.6), (3.7) together with (5.6), (5.1) and (5.2) for $|l| < L$:

$$\begin{aligned}
\|D_x^l E_1(t)\|_2 &\leq C \int_{t/2}^t \vartheta(s) R(s)^{-\frac{L}{2}} \|F[w](t,s)\|_2 ds + C \int_0^{t/2} \vartheta(s) \|D_x^l F[w](t,s)\|_2 ds \\
&\leq C \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))} \int_{t/2}^t \vartheta(s)^{-\frac{1}{2}} R(s)^{-\frac{L}{2}} \|E(t-s)\|_2 ds \\
&\quad + C \sum_{\substack{0 \leq l_j^k \leq l_j \\ j=1,2,3}} \int_0^{t/2} \vartheta(s)^{-\frac{1}{2}} \|D_x^{l-l^k} E(t-s)\|_2 \|D_x^{l^k} w(t-s)\|_2 ds \\
&\leq C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \left(R(t)^{\frac{3}{2}(\frac{1}{2} - \frac{1}{\theta}) - \frac{L-1}{2}} t^{\frac{1}{2}-\omega_\theta} \right. \\
&\quad \left. + \sum_{k=1}^{L-1} R(t)^{\frac{3}{2}(\frac{1}{2} - \frac{1}{\theta}) - \frac{L-k-1}{2}} R(t)^{-\frac{k}{2}} t^{\frac{1}{2}-\omega_\theta} + \int_0^{t/2} \vartheta(s)^{-\frac{1}{2}} \|D_x^l E_1(t-s)\|_2 ds \right)
\end{aligned}$$

$$\begin{aligned}
& + \int_0^{t/2} \vartheta(s)^{-\frac{1}{2}} R(t-s)^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\theta})+\frac{1}{2}} (t-s)^{-\omega_\theta} \|D_x^l w(t-s)\|_2 ds \Big) \\
& \leq C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \left(R(t)^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\theta})-\frac{L-1}{2}} t^{\frac{1}{2}-\omega_\theta} + \int_0^{t/2} \vartheta(s)^{-\frac{1}{2}} \|D_x^l E_1(t-s)\|_2 ds \right. \\
& \quad \left. + \int_0^{t/2} \vartheta(s)^{-\frac{1}{2}} R(t-s)^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\theta})+\frac{1}{2}} (t-s)^{-\omega_\theta} \|D_x^l w(t-s)\|_2 ds \right). \tag{5.10}
\end{aligned}$$

By applying Gronwall's Lemma to the coupled system (5.9)-(5.10), the estimates (5.1) for $m = 0$ and (5.2) follow.

Step 2:

Next we consider the v -derivatives of the solution $w(t)$ for some fixed multiindex $m \in \mathbb{N}_0^3$ with $|m| = M \geq 1$. As in Step 1, we assume that (5.1) holds for all multiindices \tilde{m} with $0 \leq |\tilde{m}| \leq M-1$ and $l = 0$. By interpolation with the result of Step 1, the estimate (5.1) then also holds for all mixed derivatives $D_x^l D_v^{\tilde{m}} w(t)$ with $0 \leq |\tilde{m}| \leq M-2$ and $l \in \mathbb{N}_0^3$.

We apply D_v^m to (2.1), introduce the characteristic coordinates $\bar{x}_t := x + v \left(\frac{1-e^{\beta t}}{\beta} \right)$, $\bar{v}_t := v e^{\beta t}$ (and, analogously, \bar{x}_s, \bar{v}_s), and use (4.8). Here and in the sequel, $\frac{1-e^{\beta t}}{\beta}$ has to be replaced by its limit $-t$ if $\beta = 0$. This yields

$$\begin{aligned}
D_v^m w(x, v, t) &= \int [D_v^m G(t, x, v, v_0)] *_x w_0(x, v_0) dv_0 \\
&\quad + \int_0^t \int [D_v^m G(s, x, v, v_0)] *_x (\Theta[V]w)(x, v_0, t-s) dv_0 ds \\
&= e^{3\beta t} \left[e^{\beta t} D_v + \frac{1-e^{\beta t}}{\beta} D_x \right]^m (g(t) *_x w_0)(\bar{x}_t, \bar{v}_t) \\
&\quad + \int_{t/2}^t e^{3\beta s} \left(\left[e^{\beta s} D_v + \frac{1-e^{\beta s}}{\beta} D_x \right]^m g(s) *_x (\Theta[V]w)(t-s) \right) (\bar{x}_s, \bar{v}_s) ds \\
&\quad + \int_0^{t/2} e^{3\beta s} \left(g(s) *_x \left[e^{\beta s} D_v + \frac{1-e^{\beta s}}{\beta} D_x \right]^m (\Theta[V]w)(t-s) \right) (\bar{x}_s, \bar{v}_s) ds.
\end{aligned}$$

Since $\|\cdot\|_{L_{\bar{x}, \bar{v}}^2} = e^{\frac{3}{2}\beta t} \|\cdot\|_{L_{x,v}^2}$, we then obtain by estimating like in (4.10)-(4.11), using $D_v^{\tilde{m}} \Theta[V]w = \Theta[V]D_v^{\tilde{m}} w$ and (5.1):

$$\begin{aligned}
\|D_v^m w(t)\|_2 &\leq e^{\frac{3}{2}\beta t} \left\| \left[e^{\beta t} D_v + \frac{1-e^{\beta t}}{\beta} D_x \right]^m g(t) \right\|_{L_{x,v}^1} \|w_0\|_2 \\
&\quad + \int_{t/2}^t e^{\frac{3}{2}\beta s} \left\| \left[e^{\beta s} D_v + \frac{1-e^{\beta s}}{\beta} D_x \right]^m g(s) \right\|_{L_v^1(L_x^{6/5})} \|(\Theta[V]w)(t-s)\|_{L_v^2(L_x^{3/2})} ds \\
&\quad + C_T \sum_{\substack{|\tilde{m}| \leq 1 \\ \tilde{m}, m-\tilde{m} \in \mathbb{N}_0^3}} \int_0^{t/2} \left(\frac{e^{\beta s}-1}{\beta} \right)^{|\tilde{m}|} \|g(s)\|_{L_v^1(L_x^{6/5})} \|(\Theta[V]D_x^{\tilde{m}} D_v^{m-\tilde{m}} w)(t-s)\|_{L_v^2(L_x^{3/2})} ds \\
&\quad + C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \sum_{k=2}^M \left(\frac{e^{\beta t}-1}{\beta} \right)^k R(t)^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\theta})-\frac{k-1}{2}} t^{-\omega_\theta+1} t^{-\frac{M-k}{2}}.
\end{aligned} \tag{5.11}$$

In the last integral we only kept the v -derivatives of the order M and $M-1$, as the estimates of mixed lower order v -derivatives of w are already known. For the second factor of the last

integral we use (4.10) and the following interpolation (if $|\tilde{m}| = 1$):

$$\begin{aligned} & \left(\frac{e^{\beta s} - 1}{\beta} \right)^{|\tilde{m}|} \|(\Theta[V] D_x^{\tilde{m}} D_v^{m-\tilde{m}} w)(t-s)\|_{L_v^2(L_x^{3/2})} \\ & \leq C \|V(t-s)\|_{L_x^6} \left(\left(\frac{e^{\beta s} - 1}{\beta} \right)^{M|\tilde{m}|} \|D_x^{M\tilde{m}} w(t-s)\|_{L_{x,v}^2} + \sum_{j=1}^3 \|D_v^{Me_j} w(t-s)\|_{L_{x,v}^2} \right), \end{aligned} \quad (5.12)$$

where e_j denote the unit vectors in \mathbb{N}^3 . Since (5.11), (5.12) would not yield a closed inequality for $\|D_v^m w(t)\|_2$, we sum (5.11) over all multiindices with $|m| = M$. Using the estimates (5.7), (4.6), (4.7), and (3.10) we get

$$\begin{aligned} \sum_{|m|=M} \|D_v^m w(t)\|_2 & \leq C_T t^{-\frac{M}{2}} \|w_0\|_2 + C_T \int_{t/2}^t R(s)^{-\frac{1}{4}} s^{-\frac{M}{2}} \|V(t-s)\|_{L_x^6} \|w(t-s)\|_2 ds \\ & + C_T \sum_{|m|=M} \int_0^{t/2} R(s)^{-\frac{1}{4}} \|V(t-s)\|_6 \left(\left(\frac{e^{\beta s} - 1}{\beta} \right)^M \|D_x^m w(t-s)\|_2 + \|D_v^m w(t-s)\|_2 \right) ds \\ & + C \left(T, \|w\|_{C([0,T];L^2(\mathbb{R}^6))}, N_\theta \right) \sum_{k=2}^M \left(\frac{e^{\beta t} - 1}{\beta} \right)^k R(t)^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\theta})-\frac{k-\frac{1}{2}}{2}} t^{-\omega_\theta+1} t^{-\frac{M-k}{2}} \\ & \leq C \left(T, \|w\|_{C([0,T];L^2(\mathbb{R}^6))}, N_\theta \right) \left[t^{-\frac{M}{2}} + R(t)^{\frac{3}{2}(\frac{1}{6}-\frac{1}{\theta})+\frac{3}{4}} t^{-\frac{M-2}{2}-\omega_\theta} \right. \\ & + \sum_{k=2}^M \left(\frac{e^{\beta t} - 1}{\beta} \right)^k R(t)^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\theta})-\frac{k-\frac{1}{2}}{2}} t^{-\omega_\theta+1} t^{-\frac{M-k}{2}} \\ & \left. + \sum_{|m|=M} \int_0^{t/2} R(s)^{-\frac{1}{4}} R(t-s)^{\frac{3}{2}(\frac{1}{6}-\frac{1}{\theta})+1} (t-s)^{-\omega_\theta} \|D_v^m w(t-s)\|_2 ds \right]. \end{aligned}$$

By applying Gronwall's Lemma and considering the range of ω_θ we finally obtain the estimate (5.1) for $l = 0$.

Further, (5.4) follows by using the Gagliardo–Nirenberg inequality

$$\|V(t)\|_{L^\infty(\mathbb{R}^3)} \leq C \|V(t)\|_{L^6(\mathbb{R}^3)}^{1/2} \|D_x^l E(t)\|_{L^2(\mathbb{R}^3)}^{1/2}, \quad \text{with } |l| = 1,$$

and the estimates (3.10), (5.2).

For (5.5) we use the Fourier transformed version of (2.19), (2.20), i.e.

$$\widehat{V}(t, \xi) = -i \frac{\xi \cdot \widehat{E}(\xi)}{|\xi|^2},$$

the estimate (5.2) with $L = 0$, $L = 1$, and the Hölder inequality. \square

6 A-posteriori estimates on the particle density

In this section we present some additional decay results for the particle density that hold only under assumption **(B)**. They are complementary to §5, since we recover estimates for $\|n(t)\|_{L^p(\mathbb{R}^3)}$ with $p \leq 2$. Since we have $\widehat{V}(t) \in L^1(\mathbb{R}^3)$ for $t > 0$ (cf. (5.5)), the following (rigorous) reformulation of the pseudo-differential operator $\Theta[V]$ holds (cf. [ALMS]):

$$\Theta[V(t)]u(x, v) = -\frac{16}{(2\pi)^{3/2}} \operatorname{Re}(ie^{2iv \cdot x} \widehat{V}(t, 2v)) *_v u(x, v). \quad (6.1)$$

Hence, by Young's inequality and (5.5)

$$\begin{aligned} \|\Theta[V(t)](t)\|_{\mathcal{B}(L_v^1(L_x^\theta))} &\leq \frac{16}{(2\pi)^{3/2}} \|\widehat{V}(t)\|_{L^1(\mathbb{R}^3)} \\ &\leq C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, \|w_0\|_{L_v^1(L_x^\theta)}\right) R(t)^{1-\frac{3}{2\theta}}, \quad \forall t \in (0, T]. \end{aligned} \quad (6.2)$$

The main result is

Theorem 6.1 *Let (B) hold for some $\theta \in I_\theta$. Then, the solution of the WPFP problem (1.3)-(1.6) satisfies:*

(i) $w \in C([0, \infty); L_v^1(L_x^\theta))$.

(ii) The density $n(t)$ satisfies for all $T > 0$ and $\theta \leq p \leq 2$:

$$\|n(t)\|_{L^p(\mathbb{R}^3)} \leq C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, \|w_0\|_{L_v^1(L_x^\theta)}\right) R(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)}, \quad \forall t \in (0, T]. \quad (6.3)$$

PROOF.

(i) We estimate (2.1) by using (4.8) and (6.2):

$$\begin{aligned} \|w(t)\|_{L_v^1(L_x^\theta)} &\leq C_T \|g(t)\|_{L_{x,v}^1} \|w_0\|_{L_v^1(L_x^\theta)} + C_T \int_0^t \|g(s)\|_{L_{x,v}^1} \|(\Theta[V]w)(t-s)\|_{L_v^1(L_x^\theta)} ds \\ &\leq C_T \|w_0\|_{L_v^1(L_x^\theta)} + C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, \|w_0\|_{L_v^1(L_x^\theta)}\right) \\ &\quad \times \int_0^t \|w(t-s)\|_{L_v^1(L_x^\theta)} R(t-s)^{1-\frac{3}{2\theta}} ds. \end{aligned}$$

Then, Gronwall's Lemma yields the assertion.

(ii) The estimate (6.3) follows by interpolation between (5.3) and the estimate

$$\|n(t)\|_{L^\theta(\mathbb{R}^3)} \leq \|w(t)\|_{L_v^1(L_x^\theta)}.$$

□

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