Scatter Correction in PET Based on Transport Models

F. Natterer, F. Wübbeling
Institut für Numerische und instrumentelle Mathematik
Westf. Wilhelms-Universität Münster
Einsteinstrasse 62, D-48149 Münster, Germany
e-mail: nattere@math.uni-muenster.de

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1 Introduction

Correcting PET data for scatter is a major problem in nuclear medicine. A plethora of approximate methods have been suggested; see e. g. the thesis [1]. All these methods are based on heuristics. We mention only the convolution-subtraction method of Bergström [2] and the Single Scattering Simulation (SSS) of Watson [3].

In the present note we derive a comprehensive mathematical model of PET based on the Boltzmann equation. This model adequately represents attenuation and scatter provided the attenuation coefficient and the scattering kernel are known. If scatter is neglected our model reduces to the usual straight line model of PET.

The derived model is computationally too demanding for practical purposes. In an attempt to find a computationally feasable algorithm we make use of the single scattering approximation to the transport equation. It is hoped that the ensuing simplifications permit the derivation of a reasonably efficient algorithm.

2 The transport model.

The standard model for PET is the fully discrete model developed by Shepp and Vardi [7]; see [8] for a more recent exposition. Below we suggest a much more comprehensive model.

Let $u(x, \theta, v)$ be the density of particles at x travelling in direction θ with velocity v. u satisfies the Boltzmann equation

$$\begin{aligned} v\theta \cdot \nabla_x u(x,\theta,v) + a(x)u(x,\theta,v) &= \int\int k(x,\theta,\theta',v,v',)u(x,\theta',v')d\theta'\,dv' \\ &+ q(x,\theta,v). \end{aligned} \tag{2.1}$$

Here, a is the attenuation coefficient and q the source density. k is the scattering kernel , i. e. $k(x, \theta, \theta', v, v')$ is the probability that a particle arriving at x with direction θ' and speed v' continuous its trip with direction θ and speed v. In order to determine the solution of (2.1) uniquely we have to impose boundary conditions. Assuming that the PET scanner is the infinite cylinder C with boundary

$$\Gamma = \left\{ x : x_1^2 + x_2^2 = R^2, x_3 \in \mathbb{R} \right\}$$

and radius R we put for each $\theta \in S^2$

$$\Gamma_\theta = \{x \in \Gamma : \nu_x \cdot \theta \geq 0\}$$

where ν_x is the interior normal of $x \in \Gamma$. Then we stipulate

$$u(x, \theta, v) = 0 \ \text{ for } \ x \in \Gamma_\theta. \tag{2.2}$$

The boundary value problem (2.1), (2.2) is uniquely solvable subject to reasonable conditions on a, k; see e. g. [4].

The purpose of nuclear medicine imaging is to determine the source term q from measurements of u on Γ. PET is characterized by the following special features:

1. The source ejects particles pairwise in equally distributed opposite directions at a speed c.

2. The detectors work in coincidence, i. e. the detector pair $x, y \in \Gamma$ responds only if particles hit the detectors at the same time.

To see what this means for our transport model we consider the number of counts at detectors x, y as random variables ξ, η, respectively. Assume that a source at z emits a particle in direction ω at speed c. This corresponds to the source term

$$q(x, \theta, v) = \delta(x - z)\delta(\omega - \theta)\delta(v - c). \tag{2.3}$$

Let $G_{z,\omega,c}$ be the solution of (2.1), (2.2) with the source term q from (2.3). Further, let

$$G_{z,\omega}(x) = \int\int G_{z,\omega,c}(x, \theta, v) d\theta \ \ dv.$$

In deterministic language, $G_{z,\omega}(x)$ is the density of particles of any velocity hitting detector x from any direction. (Here we assume that the detector is not direction or energy sensitive. Otherwise weight functions have to be encorporated in this integral). In probabilistic language $G_{z,\omega}(x)$ is the expectation of the random variable

$$\xi^+ = \begin{cases} 1, & \text{a particle hits detector } x, \\ 0, & \text{otherwise} \end{cases}$$

Now remember that the source at z ejects the particles pairwise in opposite directions. This gives rise to a second random variable

$$\eta^- = \begin{cases} 1, & \text{a particle hits detector } y, \\ 0, & \text{otherwise} \end{cases}$$

whose expectation is $G_{z,-\omega}(y)$.The random variables ξ^+, η^- are clearly independent since they refer to random walks of the different particles. In coincidence mode we measure

$$E(\xi^+\eta^-) = E(\xi^+)E(\eta^-) = G_{z,\omega}(x)G_{z,-\omega}(y).$$

Since the direction ω in which the sources eject the particles is random, the measurements for a point source located at z are

$$G_z(x, y) = \int G_{z,\omega}(x)G_{z,-\omega}(y) d\omega.$$

For a continuous distribution f of sources we finally get as measurements for the detector pair x, y

$$g(x,y) = \int_C f(z) G_z(x,y) dz. \tag{2.4}$$

This is our model for PET. The problem consists in determining f from (2.4), g being given for all detector pairs x, y.

3 The case without scatter

In this section we show that for $k = 0$ (2.4) degenerates into

$$g(x,y) = \frac{1}{|x-y|^{n-1}} \; e^{-\int_x^y a\, ds} \int_x^y f\, ds.$$

In other words, $G_z(x,y)$ is the distribution

$$\begin{aligned} G_z(x,y) = G_z^0(x,y) &= \frac{1}{|x-y|^{n-1}} \; e^{-\int_x^y a ds} \, l_{x,y}(z), \\ \int l_{x,y}(z) f(z) dz &= \int_x^y f ds. \end{aligned} \tag{3.1}$$

We treat the aforementioned case of dimension $n = 3$ simultaneously with $n = 2$. In the latter case the cylinder C degenerates into the unit disk and Γ into the unit circle in $\mathbb{R}^2$. This is the usual integral equation of PET in which scatter is neglected.

The derivation of (3.1) is quite technical. For $k = 0$ we can ignore the dependence of u on v, and we can normalize v to be 1. Thus (2.1) assumes the form

$$\theta \cdot \nabla_x u(x,\theta) + a(x) u(x,\theta) = q(x,\theta)$$

The solution that satisfies $u(x,\theta) = 0$ on Γ_θ is

$$u(x,\theta) = \int_{-\infty}^{0} q(x + s\theta, \theta) e^{-\int_s^0 a(x+s'\theta) ds'} ds, x \in C$$

where we have put $q(x,\theta)=0$ for $x\notin C$. For $q(x,\theta)=\delta(x-z)\delta(\theta-\omega)$ we obtain

$$G^0_{z,\omega}(x,\theta)=\int\limits_{-\infty}^{0}\delta(x+s\theta-z)e^{-\int\limits_s^0 a(x+s'\theta)ds'}ds\;\delta(\theta-\omega).$$

Here and below the superscript 0 refers to $k=0$.
Hence

$$G^0_{z,\omega}(x)=\int\limits_{-\infty}^{0}\delta(x+s\omega-z)e^{-\int_s^0 a(x+s'\omega)ds'}ds$$

and

$$G^0_z(x,y)=\int G^0_{z,\omega}(x)G^0_{z,-\omega}(y)d\omega$$

$$=\int\limits_{-\infty}^{0}\int\limits_{-\infty}^{0}\int\limits_{S^{n-1}}\delta(x+s\omega-z)\delta(y-t\omega-z)e^{-\int\limits_s^0 a(x+s'\omega)ds'-\int\limits_t^0 a(y+t'\omega)dt'}d\omega\;ds\;dt.$$

Putting $s\omega=w$ we have $d\omega\;ds=dw/|w|^{n-1},\omega=-\frac{w}{|w|}$. Hence

$$G^0_z(x,y)=\int\limits_{\mathbf{R}^n}\delta(x+w-z)\;e^{-\int\limits_{-|w|}^{0}a\left(x+s'\frac{w}{|w|}\right)ds'}$$

$$\int\limits_{-\infty}^{0}\delta\left(y+t\frac{w}{|w|}-z\right)\;e^{-\int\limits_t^0 a\left(y-t'\frac{w}{|w|}\right)dt'}dt\frac{dw}{|w|^{n-1}}$$

$$=\frac{1}{|z-x|^{n-1}}e^{-\int\limits_{-|z-x|}^{0}a\left(x+s'\frac{z-x}{|z-x|}\right)ds'}\int\limits_{-\infty}^{0}\delta\left(y+t\frac{z-x}{|z-x|}-z\right)\;e^{-\int\limits_t^0 a\left(y-t'\frac{z-x}{|z-x|}\right)dt'}dt \qquad (3.2)$$

We next study the distribution $\delta\left(y+t\frac{x-z}{|z-x|}-z\right)$ as a function of z, assuming $t<0$. Putting $w=y+t\frac{z-x}{|z-x|}-z$ we have $w=0$ if and only if

$$z=\frac{|x-z|y-tx}{|x-z|-t}$$

i. e. if z is on the straight line segment between x, y. In that case, $|y-z| = -t$ and $|x-y| = |x-z| + |y-z| = |x-z| - t$, hence

$$z = z_0 = \frac{(|x-y|+t)y - tx}{|x-y|}. \tag{3.3}$$

Note that $z_0 = z_0(x, y, t)$. Furthermore we have

$$\frac{\partial w}{\partial z} = \left(\frac{t}{|z-x|} - 1\right) I - \frac{t}{|z-x|^3}(z-x)(z-x)^T.$$

Making use of the formula

$$\det(I - aa^T) = 1 - |a|^2, a \in \mathbb{R}^n$$

(see e. g. [5], p. 188) we find that for $z = z_0$,

$$\left|\det\left(\frac{\partial w}{\partial z}\right)\right| = \left(\frac{|x-y|}{|x-z_0|}\right)^{n-1}.$$

Putting all together we obtain for any test function g

$$\int g(z)\delta\left(y + t\frac{z-x}{|z-x|} - z\right) dz = g(z_0)\left(\frac{|x-z_0|}{|x-y|}\right)^{n-1}.$$

We use this in (3.2) to obtain

$$\int_{\mathrm{R}^n} f(z)G_z^0(x,y)dz = \int_{-\infty}^{0} \frac{1}{|z-x|^{n-1}} e^{-\int_{-|z-x|}^{0} a\left(x - s'\frac{z-x}{|z-x|}\right)ds' - \int_{t}^{0} a\left(y - t'\frac{z-x}{|z-x|}\right)dt'}$$
$$f(z)\left(\frac{|x-z|}{|x-y|}\right)^{n-1} dt$$

where z_0 from (3.3) has to be substituted for z on the left hand side. For this z, the integrals in the exponent are the line integrals of a over [z,x], $[z, y]$, respectively. They add up to yield the line integral over $[x, y]$. Hence (3.1) is proved.

We remark that with the methods used in [6] approximate versions of (2.4) can be obtained in the presence of scatter.

4 The single scattering approximation

In this section we derive on approximation to (2.4) which makes use of the single scatter approximation in transport theory. Again we ignore energy dependence, i. e. we consider instead of (2.1)

$$\theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) = \int\limits_{S^{n-1}} k(x,\theta,\theta')u(x,\theta')d\theta' + \delta(x-z)\delta(\theta-\omega) \quad (4.1)$$

From section 3 we know that the solution u that vanishes on Γ_θ satisfies

$$\begin{aligned} u(x,\theta) &= \int\limits_{-\infty}^{0} \left\{ \int\limits_{S^{n-1}} k(x+s\theta,\theta,\theta')u(x+s\theta,\theta')d\theta' + \delta(x+s\theta-z)\delta(\theta-\omega) \right\} \\ &\quad e^{-\int\limits_{s}^{0} a(x+s'\theta)ds'} ds \\ &= \int\limits_{-\infty}^{0} \int\limits_{S^{n-1}} k(x+s\theta,\theta,\theta')u(x+s\theta,\theta')d\theta' e^{-\int\limits_{s}^{0} a(x+s'\theta)ds'} ds + G^0_{z,\omega}(x,\theta). \end{aligned}$$

In the single scatter approximation we replace u in the integral by $G^0_{z,\omega}$, obtaining

$$\begin{aligned} u(x,\theta) &= \int\limits_{-\infty}^{0} \int\limits_{S^{n-1}} k(x+s\theta,\theta,\theta')G^0_{z,\omega}(x+s\theta,\theta')d\theta' e^{-\int\limits_{s}^{0} a(x+s'\theta)ds'} ds + G^0_{z,\omega}(x,\theta) \\ &= \int\limits_{-\infty}^{0} \int\limits_{S^{n-1}} k(x+s\theta,\theta,\theta') \int\limits_{-\infty}^{0} \delta(x+s\theta+t\theta'-z)e^{\int\limits_{t}^{0} a(x+s\theta+s'\theta')ds'} dt\, d\theta' \\ &\quad \delta(\theta'-\omega)e^{-\int_s^0 a(x+s'\theta)ds'} ds + G^0_{z,\omega}(x,\theta) \\ &= \int\limits_{-\infty}^{0} \int\limits_{-\infty}^{0} k(x+s\theta,\theta,\omega)\delta(x+s\theta+t\omega-z)e^{-\int\limits_{t}^{0} a(x+s\theta+s'\omega)ds' - \int\limits_{s}^{0} a(x+s'\theta)ds'} \\ &\quad ds\, dt + G^0_{z,\omega}(x,\theta). \end{aligned}$$

Integrating over θ and substituting $w = s\theta$, $|w| = -s$ we obtain

$$G^1_{z,w}(x) = \int\limits_{S^{n-1}} u(x,\theta)d\theta$$

$$= \int_{-\infty}^{0} \int_{\mathbb{R}^n} k(x+w, -\frac{w}{|w|}, \omega)\delta(x+w+t\omega-z) e^{-\int_t^0 a(x+w+s'\omega)ds' - \int_{-|w|}^0 a(x-s'\frac{w}{|w|})ds'}$$
$$\frac{dw}{|w|^{n-1}} dt + G^0_{z,\omega}(x);$$

the superscript 1 refers to the single scatter approximation. Carrying out the w integral yields

$$G^1_{z,\omega}(x) = \frac{1}{|w|^{n-1}} \int_{-\infty}^{0} k(x+w, -\frac{w}{|w|}, \omega) e^{-\int_t^0 a(x+w+s'\omega)ds' - \int_{-|w|}^0 a(x-s'\frac{w}{|w|})ds'} dt + G^0_{z,\omega}(x)$$

where $w = z - x - t\omega$. Hence $G^1_{z,\omega}(x) = H_{z,\omega}(x) + G^0_{z,\omega}(x)$ where

$$H_{z,\omega}(x) = \int_{-\infty}^{0} k\left(z - t\omega, \frac{x-z+t\omega}{|x-z+t\omega|}, \omega\right) e^{-\int_\ell a\,ds} \frac{dt}{|x-z+t\omega|^{n-1}} \qquad (4.2)$$

where ℓ is the union of the line segments $[z, z-t\omega]$ and $[z-t\omega, x]$. Thus $H_{z,\omega}(x)$ is the density of the particles ejected at z in direction ω, being scattered at some point $z - t\omega, t < 0$, and hitting the detector at x; see Fig. 1.

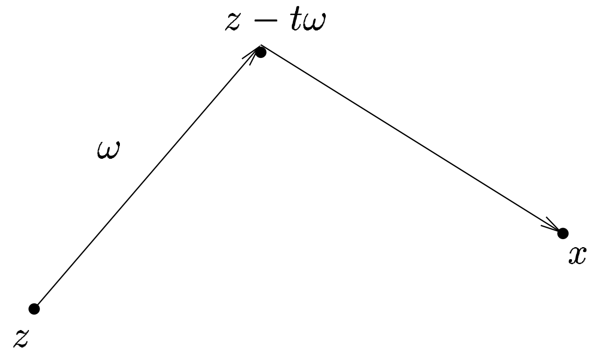

Fig. 1: Single scatter. The path ℓ.

Now we compute

$$G^1_z(x,y) = \int_{S^{n-1}} G^1_{z,\omega}(x) G^1_{z,-\omega}(y) d\omega$$
$$= \int_{S^{n-1}} (G^0_{z,\omega}(x)G^0_{z,-\omega}(y) + G^0_{z,\omega}(x)H_{z,-\omega} + H_{z,\omega}(x)G^0_{z,-\omega}(y)$$
$$+ H_{z,\omega}(x)H_{z,-\omega}(y))d\omega.$$

We have

$$\begin{aligned}
\int\limits_{S^{n-1}} G^0_{z,\omega}(x)G^0_{z,-\omega}(y)d\omega &= G^0_z(x,y),\\
\int\limits_{S^{n-1}} G^0_{z,\omega}(x)H_{z,-\omega}(y)d\omega &= \int\limits_{S^{n-1}}\int\limits_{-\infty}^{0} \delta(x+s\omega-z)e^{-\int\limits_s^0 a(x+s'\omega)ds'} ds H_{z,-\omega}(y)d\omega\\
&= \int\limits_{\mathrm{R}^n} \delta(x+w-z)e^{-\int\limits_{-|w|} a(x-s'w/|w|)ds'} H_{z,w/|w|}(y)\frac{dw}{|w|^{n-1}}\\
&= \frac{1}{|x-z|^{n-1}}e^{-\int\limits_x^z a\, ds} H_{z,(x-z)/|x-z|}(y)\\
&= H^1_z(x,y). \qquad (4.3)
\end{aligned}$$

Hence,

$$\begin{aligned}
G^1_z(x,y) = G^0_z(x,y) + H^1_z(x,y) + H^1_z(y,x) + H^2_z(x,y),\\
H^2_z(x,y) = \int\limits_{S^{n-1}} H_{z,\omega}(x)H_{z,-\omega}(y)d\omega. \qquad (4.4)
\end{aligned}$$

5 The issue of implementation.

The equation we have to solve is (2.4), i. e.

$$g(x,y) = \int\limits_C f(z)G_z(x,y)dz. \qquad (5.1)$$

If G_z is known this can be done by methods such as OSEM or list mode EM. The problem is the computation of $G_z(x,y)$ for each point z in the reconstruction region and for each detector pair x, y. Assuming to do the reconstruction on an $N \times N \times N$ grid we have N^2 detectors, necessitating the computation of N^7 numbers $G_z(x,y)$, each being a complicated multiple integral. So it is clear that we can't solve (in fact we can't even set up) (5.1) as it stands.

A way out is the approximation of (5.1) by terms that can be evaluated more quickly. For instance, if we use the single scattering approximation $G_z^1(x,y)$ for $G_z(x,y)$, then (5.1) reduces to

$$\begin{aligned} g(x,y) &= \frac{1}{|x-z|^2} e^{-\int_x^y a\,ds} \int_x^y f\,ds \\ &+ \int_C f(z)\left(H_z^1(x,y)+H_z^1(y,x)\right)dz + \int_C f(z)H_z^2(x,y)dz. \end{aligned} \quad (5.2)$$

The first term is just the usual line integral model of PET. The remaining terms account for single scattering. (5.2) is reminiscent of SSS; see Watson [3], Werling [1], except that it is much more detailed and in any case based on sound principles rather than heuristics.

Setting up (5.2) is still a challenge. The computation of $H_z^1(x,y), H_z^2(x,y)$ amounts to the computation of the integrals

$$H_{z,\omega}(x) = \int_{-\infty}^{0} k\left(z-t\omega, \frac{x-z+t\omega}{|x-z+t\omega|}, \omega\right) e^{-\int_\ell a\,ds} \frac{dt}{|x-z+t\omega|^2} \quad (5.3)$$

for each $\omega \in S^2$; see Fig. 1 for the path ℓ. Finding fast methods for evaluating (5.3) is the crux of the matter. Of course (5.3) can be evaluated in parallel. One also has to find ways to exploit the structure of (5.3). For instance, for isotropic scatterers we have

$$k(x,\theta,\theta') = k(x,\theta\cdot\theta').$$

The challenge is to combine this (and possibly other features) with methods of parallelization.

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