Ultrasonic image reconstruction from backscatter

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1 Introduction

We consider the following inverse scattering problem in $\mathbb{R}^n$, $n = 2, 3$: Let u be the solution of the scattering problem

$$\begin{aligned} &\Delta u + k^2(1 + f)u = 0 \quad \text{in } x_n > 0, \\ &-\frac{\partial u}{\partial x_n} = \delta_{y'} \qquad \text{on } x_n = 0 \,, \\ &\frac{\partial u}{\partial r} - iku \to 0 \,, \quad r = |x| \to \infty \,. \end{aligned} \tag{1.1}$$

Here, f is a complex valued function compactly supported in $x_2 > 0$, and δ'_y is the δ-function on $x_n = 0$ supported at y', and k is the wave number of the time harmonic irradiating wave emanating from the source y. Throughout the paper, we write $x = (x', x_n)$ with an $n-1$–dimensional vector x' for all n–dimensional Vectors x. The inverse scattering problem calls for recovering f from the values of $u(x', 0) = g(x', y')$, x', $y' \in \mathbb{R}^{n-1}$. Possible applications include exploration geophysics (see [1]) and medical ultrasound. Since we make use only of backscatter we can not expect a faithful reconstruction of the object f.

The outline of the paper is as follows. In section 2 we solve the problem within the Born approximation. Based on the Born approximation we investigate the possible resolution in section 3. It will turn out that we basically can obtain a high frequency reconstruction of f, but some low frequency information is available, too. In section 4 we extend the PBP algorithm of ultrasound transmission tomography (see [2]) to the backscatter case. In section 5 we describe the implementation of the algorithm by finite differences, and in section 6 we present numerical examples.

2 The Born approximation

We put $u = u^i + v$ where u^i is the solution of (1.1) for $f = 0$, obtaining

$$\begin{aligned} &\Delta v + k^2(1+f)v = -k^2 f u^i \quad \text{in } x_n > 0 \\ &-\frac{\partial v}{\partial x_n} = 0 \qquad \text{on } x_n = 0\ , \\ &\frac{\partial v}{\partial r} - ikv = 0\ , \quad r \to \infty\ . \end{aligned} \tag{2.1}$$

The Born approximation is obtained by neglecting vf, i.e.

$$\begin{aligned} &\Delta v + k^2 v = -k^2 f u^i \quad \text{in } x_n > 0 \\ &-\frac{\partial v}{\partial x_n} = 0 \qquad \text{on } x_n = 0\ , \\ &\frac{\partial v}{\partial r} - ikv = 0\ , \quad r \to \infty\ . \end{aligned} \tag{2.2}$$

Let G_n be the fundamental solution for the Helmholtz operator in $\mathbb{R}^n$, i.e.

$$\begin{aligned} G_3(x) &= \frac{e^{ik|x|}}{4\pi|x|}\ , \\ G_2(x) &= \frac{i}{4} H_0(k|x|)\ . \end{aligned}$$

Then,

$$u^i(x) = 2G_n(x-y)\ ,$$

and (2.2) is equivalent to

$$v(x) = -k^2 \int_{\mathrm{R}^n} G_n(x-z) \Big(f(z', z_n) u^i(z', z_n) + f(z', -z_n) u^i(z', z_n) \Big)\, dz\ .$$

For $x_n = 0$ this reduces to

$$g(x', y) = -4k^2 \int_0^\infty \int_{\mathrm{R}^{n-1}} G_n(x'-z', z_n) f(z', z_n) G_n(z'-y, z_n) dz' dz_n\ . \tag{2.3}$$

Now we make use of the plane wave decomposition of G_n, to wit

$$\begin{aligned} G_n(x) &= ic_n \int_{\mathrm{R}^{n-1}} e^{i(|x_n|\kappa(\xi) \pm x'\cdot\xi)} \frac{d\xi}{\kappa(\xi)} \\ c_2 &= \frac{1}{4\pi}\ , \quad c_3 = \frac{1}{8\pi^2}\ , \quad \kappa(\xi) = \sqrt{k^2 - |\xi|^2}\ ; \end{aligned}$$

see e. g. [3], p. 49. Inserting into (2.3) yields

$$\begin{aligned} g(x', y) = & -4c_n^2 k^2 \int_0^\infty \int_{\mathrm{R}^{n-1}} \int_{\mathrm{R}^{n-1}} e^{i(z_n\kappa(\xi) + (x'-z')\cdot\xi)} \frac{d\xi}{\kappa(\xi)} f(z', z_n) \\ & \int_{\mathrm{R}^{n-1}} e^{i(z_n\kappa(\eta) - (z'-y)\cdot\eta)} \frac{d\eta}{\kappa(\eta)} dz' dz_n\ . \end{aligned}$$

Note that the absolute values could be dropped since the integration is only over $z_n \geq 0$. The integration with respect to z', z_n is just a Fourier transform. Hence

$$g(x', y) = -4c_n^2 k^2 (2\pi)^{n/2} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\mathbb{R}^{n-1}} \hat{f}(\xi + \eta, -\kappa(\xi) - \kappa(\eta)) \, e^{i(x' \cdot \xi + y \cdot \eta)} \frac{d\xi d\eta}{\kappa(\xi)\kappa(\eta)} \, .$$

The integration with respect to ξ, η is a $(n-1)$-dimensional inverse Fourier transform each. Hence we obtain for the $2(n-1)$ dimensional Fourier transform $\hat{g}$ of g

$$\hat{g}(\xi, \eta) = -4c_n^2 k^2 (2\pi)^{n/2} (2\pi)^{n-1} \hat{f}(\xi + \eta, -\kappa(\xi) - \kappa(\eta)) \frac{1}{\kappa(\xi)\kappa(\eta)} \, . \tag{2.4}$$

This is the solution of the inverse scattering problem in the Born approximation.

3 Resolution

From (2.4) we see that $\hat{f}$ is determined by the backscatter on the semispheres of radius k centered on the semisphere around 0, see Fig. 1. The semispheres

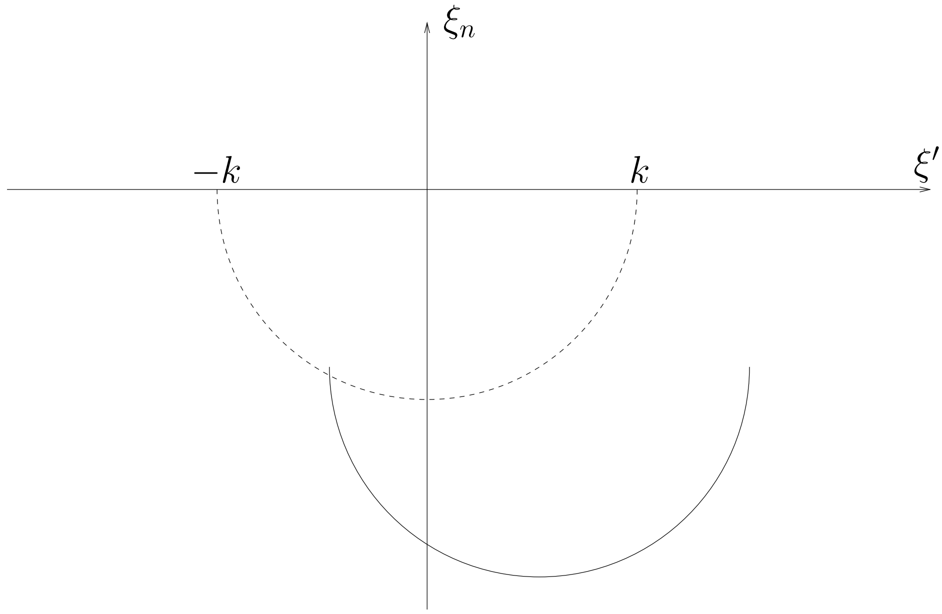

Fig. 1

fill the shaded region in Fig. 2. Thus the information

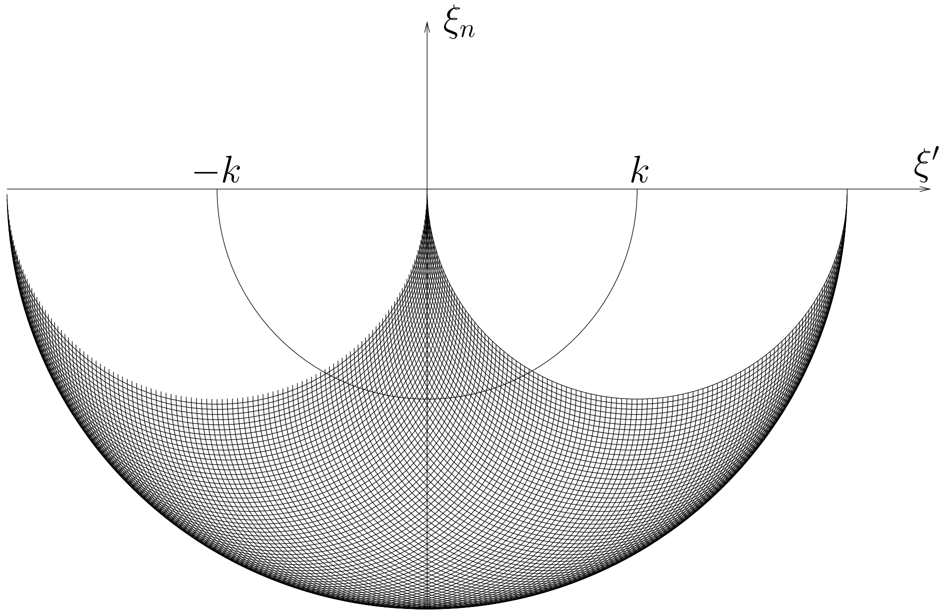

Fig. 2

we get from backscatter is mainly high frequency. However there exist also low frequency contributions. For instance, if f is a real horizontally layered medium it can be recovered with resolution corresponding to the bandwidth $2k$.

4 The PBP algorithm

For the numerical solution we restrict the problem to a finite volume $\Omega = \{x : |x_1| < a, 0 < x_2 < \rho\}$. The boundary $\partial\Omega$ of Ω consists of three parts. The bottom $\Gamma^+ = \{x : x_2 = \rho, |x_1| \leq a\}$, the top on $x_2 = 0$, and the lateral surface where $0 \leq x_2 \leq \rho$ and $|x_1| = a$; see Fig. 3, where, as usual in seismic application, the x_2 axis points downwards.

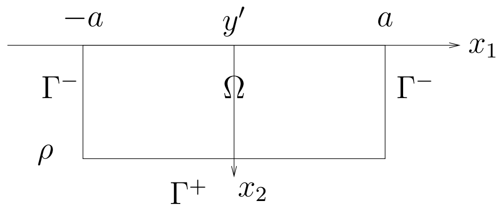

Fig. 3: Geometry of the finitized poblem.

This problem gives rise to a nonlinear operator

$$\begin{aligned} R_{y'} &: L_2(\Omega) \to L_2(\Gamma^+) \ , \\ R_{y'}(f) &= \left.\frac{\partial v}{\partial r} - ikv\right|_{\Gamma^+} \end{aligned}$$

where v is the solution of the initial value problem

$$\begin{aligned} &\Delta v + k^2(1+f)v = -k^2 f u_i \quad \text{in } \ x_2 > 0 \\ &-\frac{\partial v}{\partial x_2} = 0 \quad , \quad v = g(\cdot, y') - u_i \quad \text{on } \ x_2 = 0 \\ &\frac{\partial v}{\partial r} - ikv = 0 \quad \text{on } \ \Gamma \ . \end{aligned} \tag{4.1}$$

where we have put $u = u^i + v$. As in the PBP algorithm, this initial value problem can be stabilized by low-pass filtering u as a function of x_1. The problem we have to solve is now

$$R_y(f) = 0$$

for all y for which measurments are available.

As in [2], this nonlinear system is solved by the Kaczmarz method. The update for a source y is

$$f \to f - \omega R_y'(f)^* C^{-1} R_y(f) \tag{4.2}$$

where C is a positive definite operator close to $R_y'(f)R_y'(f)^*$ and $0 < \omega < 2$. The operator $R_y'(f) : L_2(\Omega) \to L_2(\Gamma^+)$ is given by

$$R_y'(f)h = \left.\left(\frac{\partial w}{\partial r} - ikw\right)\right|_{\Gamma^+} \tag{4.3}$$

where w is the solution of

$$
\begin{aligned}
&\Delta w + k^2(1+f)w = -k^2 hu \quad \text{in } \Omega \\
&\frac{\partial w}{\partial r} - ikw = 0 \quad \text{on } \Gamma \, , \quad w = 0 \, , \quad \frac{\partial w}{\partial x_2} = 0 \quad \text{on } x_2 = 0 \, .
\end{aligned}
\tag{4.4}
$$

In order to compute the adjoint $R'_y(f)^* : L_2(\Gamma^+) \to L_2(\Omega)$ we start out from

$$
\int\limits_{\Omega} \Big((\Delta w + k^2(1+f)w)z - w(\Delta z + k^2(1+f)z) \Big) \, dx = \int\limits_{\partial\Omega} \left(\frac{\partial w}{\partial \nu} z - w \frac{\partial z}{\partial \nu} \right) ds
\tag{4.5}
$$

which holds for sufficiently regular functions w, z. On $\partial\Omega$ we have

$$
\frac{\partial}{\partial r} = \alpha \frac{\partial}{\partial \nu} + \beta \frac{\partial}{\partial s}
$$

where $r = |x|$ and s is the arc length. We have

$$
\alpha = \frac{x}{|x|} \cdot \nu \, , \qquad \beta = \frac{x}{|x|} \cdot \tau
$$

where $x = x(s)$ is a parametric representation of $\partial\Omega$ and ν, τ are normal and tangent unit vectors, resp. With these notations we get, with L the length of $\partial\Omega$,

$$
\begin{aligned}
\int\limits_{\partial\Omega} \left(\frac{\partial w}{\partial \nu} z - w \frac{\partial z}{\partial \nu} \right) ds &= \int\limits_0^L \left\{ \left(\frac{1}{\alpha} \frac{\partial w}{\partial r} - \frac{\beta}{\alpha} \frac{\partial w}{\partial s} \right) z - w \frac{\partial z}{\partial \nu} \right\} ds \\
&= \int\limits_0^L \left\{ \frac{1}{\alpha} \frac{\partial w}{\partial r} z + w \left(\frac{\partial}{\partial s} \left(\frac{\beta}{\alpha} z \right) - \frac{\partial z}{\partial \nu} \right) \right\} ds \\
&= \int\limits_0^L \left\{ \frac{1}{\alpha} \left(\frac{\partial w}{\partial r} - ikw \right) z + w \left(\frac{\partial}{\partial s} \left(\frac{\beta}{\alpha} z \right) - \frac{\partial z}{\partial \nu} + \frac{ik}{\alpha} z \right) \right\} ds \, .
\end{aligned}
$$

Now we take w from (4.4) and above z as solution of the initial value problem

$$
\begin{aligned}
\Delta z + k^2(1+f)z &= 0 \quad \text{in } \Omega \\
\frac{\partial}{\partial s} \left(\frac{\beta}{\alpha} z \right) - \frac{\partial z}{\partial \nu} + \frac{ik}{\alpha} z &= 0 \quad \text{on } \Gamma \cup \Gamma^+ \\
z &= \alpha \overline{g} \quad \text{on } \Gamma^+ \, ,
\end{aligned}
\tag{4.6}
$$

obtaining from (4.5), (4.6)

$$
-k^2 \int\limits_{\Omega} huz dx = \int\limits_0^L (R'_y(f)h) \overline{g} ds
$$

or

$$-k^2(h, \overline{uz})_{L_2(\Omega)} = (R_y'(f)h, g)_{L_2(\partial\Omega)} \ .$$

Hence,

$$R_y'(f)^* g = -k^2 \overline{uz}$$

with z the solution of (4.6).

5 Implementation

We first describe the propagation step, i.e. the evaluation of $R_y(f)$ for y and f given. We have to solve the initial value problem (4.1). Introducing the grid $x_1 = hm$, $x_2 = h\ell$, $m = -M, \ldots, M$, $\ell = 0, \ldots, L$ we can replace the differential equation by

$$\begin{aligned} &v_{\ell+1,m} + v_{\ell-1,m} + v_{\ell,m+1} + v_{\ell,m-1} - 4v_{\ell,m} \\ &\qquad + \varepsilon^2(1 + f_{\ell,m})v_{\ell,m} = -k^2 f_{\ell,m} u^i_{\ell,m} \ , \\ &m = -M+1, \ldots, M-1 \ , \quad \ell = 0, \ldots, L-1 \ , \quad \varepsilon = hk \ . \end{aligned} \tag{5.1}$$

For $\ell = 0$, $v_{\ell,m}$ is given by the initial condition. The level $\ell = -1$ is eliminated by the initial condition $\partial v / \partial x_2 = 0$, yielding $v_{-1,m} = v_{1,m}$, hence

$$v_{1,m} = \frac{1}{2}\left((4 - \varepsilon^2)v_{0,m} - v_{0,m+1} - v_{0,m-1}\right) \ . \tag{5.2}$$

For $m = \pm M$ we stipulate $v_{\ell,m} = 0$. This requires the grid to be sufficiently extended horizontally.

The backpropagation step, i.e. the evaluation of $R'_y(f)^* g$, requires the solution of the initial value problem (4.6). With Γ^+ the line $x_2 = \rho$ it reads

$$\Delta z + k^2(1 + f)z = 0 \ , \quad 0 < x_2 < \rho \ , \tag{5.3}$$

$$z = \frac{x_2}{r}\overline{g} \ , \quad \frac{\partial}{\partial x_1}\left(\frac{x_1}{x_2} z\right) - \frac{\partial z}{\partial x_2} + \frac{ik}{x_2/r} z = 0 \ , \quad r = |x| \quad \text{on} \quad x_2 = \rho \tag{5.4}$$

The initial conditions on $x_2 = \rho$ can be written as

$$z = \frac{\rho}{r}\overline{g} \ , \quad \frac{\partial z}{\partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{x_1}{r}\overline{g}\right) + ik\overline{g} \ . \tag{5.5}$$

The initial value problem (5.3), (5.5) can be solved by finite differences on the same grid, introducing an artificial level $\ell = -L-1$ and eliminating $z_{\ell,m}$ for $\ell = -L-1$ with the help of the second equation (5.5). Again we put $z_{\ell,m} = 0$ for $m = \pm M$.

In order to preserve stability, low-pass filtering with bandwidth k has to carried out after each ℓ step.

References

[1] Bleistein, N., J.K. Cohen, J.W. Stockwell, Jr.: Mathematics of Multidimensional Seismic Imaging, Migration, and Inversion. Springer 2001.

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[0] LaTeX: /u/berg/f/berger/paperfne1/April2002.tex