Acoustic tomography in a slab

F. Natterer

Institut für Numerische und Angewandte Mathematik

Westf. Wilhelms-Universität Münster

Einsteinstrasse 62, D-48149 Münster, Germany

e-mail: nattere@math.uni-muenster.de

December 2004

1 Introduction

We consider a slab whose material properties are described by a complex valued function $f(x)$. Transducers that can operate as sources and/or receivers are sitting on the two opposite surfaces of the slab. We want to determine f from the receiver responses generated by activating one of the sources.

A mathematical model is as follows. The slab $S \subseteq \mathbb{R}^n, n = 2, 3$ is defined by $0 \leq x_n \leq D$. The field u is a solution of the boundary value problem

$$\Delta u + k^2(1 + f)u = 0 \ , \ 0 < x_n < D, \tag{1.1}$$

$$\frac{\partial u}{\partial x_n}(x', 0) = q(x' - y') \ , \ \frac{\partial u}{\partial x_n}(x', D) = 0 \ , \ x' \in \mathbb{R}^{n-1}. \tag{1.2}$$

Here, q is the source activity, typically a δ-like function, and $(0, y'), y' \in \mathbb{R}^{n-1}$ is the source position. The data is

$$g(x', y', x_n) = u(x', x_n) \ , \ x', y' \in \mathbb{R}^{n-1} \tag{1.3}$$

where either $x_n = 0$ (reflection mode) or $x_n = D$ (transmission mode).

The difference to the case treated in [1] is that the reflections at the transducers are taken into account.

2 The Born approximation

Let u_0 be the solution of (1.1), (1.2) for $f = 0$ and $y' = 0$. Then $v(x) = u(x) - u_0(x' - y', x_n)$ satisfies

$$-\Delta v - k^2 v = k^2 f u \ , \ 0 < x_n < D,$$
$$\frac{\partial v}{\partial x_n} = 0 \text{ on } x_n = 0, D.$$

The Born approximation is obtained by replacing u by u_0 on the right hand side of the differential equation, obtaining

$$-\Delta v - k^2 v = k^2 f u_0(x' - y', x_n) \ , \ 0 < x_n < D,$$
$$\frac{\partial v}{\partial x_n} = 0 \text{ on } x_n = 0, D.$$

Let $G(x', x_n, z_n)$ be the Green's function for this boundary value problem, i. e.

$$-\Delta G(x', x_n, z_n) - k^2 G(x', x_n, z_n) = \delta(x')\delta(x_n - z_n)$$
$$\frac{\partial}{\partial x_n} G(x', x_n, z_n) = 0 \ , \ x_n = 0, D.$$

Then

$$g(x', y', x_n) \ = \ k^2 \int\limits_{\mathrm{R}^{n-1}} \int\limits_0^D G(x' - z', x_n, z_n) f(z', z_n) u_0(z' - y', z_n) dz' dz_n. \quad (2.1)$$

Doing $(n-1)$-dimensional Fourier transforms with respect to x', y' we obtain

$$\hat{g}(\xi', \eta', x_n) = (2\pi)^{-(n-1)} k^2 \int\limits_{\mathrm{R}^{n-1}} \int\limits_0^D \hat{G}(\xi', x_n, z_n) e^{-i\xi' \cdot z'} f(z', z_n) \hat{u}_0(-\eta', z_n) e^{-iz' \cdot \eta'} dz' dz_n.$$

The integral with respect to z' is an $(n-1)$-dimensional Fourier transform. Carrying it out leads to

$$\hat{g}(\xi', \eta', x_n) = (2\pi)^{-(n-1)/2} k^2 \int\limits_0^D \hat{G}(\xi', x_n, z_n) \hat{f}(\eta' + \xi', z_n) \hat{u}_0(-\eta', z_n) dz_n \quad (2.2)$$

We need explicit expressions for $\hat{u}_0$ and $\hat{G}$. u_0 satisfies

$$-\Delta u_0 - k^2 u_0 = 0\ ,\quad 0 < x_n < D$$
$$\frac{\partial}{\partial x_n} u_0(x',0) = q(x'),\quad \frac{\partial}{\partial x_n} u_0(x',D) = 0\ ,\ x' \in \mathbb{R}^{n-1}.$$

By an $(n-1)$-dimensional Fourier transform we obtain

$$-\frac{d^2}{dx_n^2}\hat{u}_0(\xi',x_n) - (k^2 - |\xi'|^2)\hat{u}_0(\xi',x_n) = 0,$$
$$\frac{\partial}{\partial x_n}\hat{u}_0(\xi',0) = \hat{q}(\xi'), \frac{\partial}{\partial x_n}\hat{u}_0(\xi',D) = 0.$$

The solution is

$$\hat{u}_0(\xi',x_n) = \frac{-\hat{q}(\xi')}{a(\xi')\sin\left(a(\xi')D\right)}\cos\left(a(\xi')(D-x_n)\right), \qquad (2.3)$$
$$a(\xi') = \sqrt{k^2 - |\xi'|^2}.$$

This makes sense for $a(\xi')D$ not a multiple of π.

Similarly,

$$-\frac{d^2}{dx_n^2}\hat{G}(\xi',x_n,z_n) - (k^2 - |\xi'|^2)\hat{G}(\xi',x_n,z_n) = (2\pi)^{-(n-1)/2}\delta(x_n - z_n),$$
$$\frac{\partial}{\partial x_n}\hat{G}(\xi',x_n,z_n) = 0\ , x_n = 0, D.$$

The Green's function of this one-dimensional boundary value problem can be represented as

$$\hat{G}(\xi',x_n,z_n) = \frac{1}{(2\pi)^{(n-1)/2}W(\xi')}\begin{cases} U(x_n) & V(z_n)\ , x_n \le z_n \\ V(x_n) & U(z_n)\ , x_n \ge z_n \end{cases} \qquad (2.4)$$

where

$$\begin{aligned} U(x_n) &= \cos(a(\xi')x_n) \\ V(x_n) &= \cos(a(\xi')(D-x_n)) \end{aligned}$$

and W is the Wronskian

$$W(\xi') = V'U - U'V = a(\xi')\sin(a(\xi')D).$$

Inserting (2.3), (2.4) into (2.2) we obtain for $x_n = 0$

$$\hat{g}(\xi', \eta', 0) = A(\xi', \eta') \int_0^D \cos(a(\xi')(D - z_n)) \cos(a(\eta')(D - z_n)) \hat{f}(\eta' + \xi', z_n) dz_n,$$

$$A(\xi', \eta') = \frac{-k^2 \hat{q}(-\eta')}{(2\pi)^{n-1} a(\xi') a(\eta') \sin(a(\xi')D) \sin(a(\eta')D)}.$$

Correspondingly, for $x_n = D$ we obtain

$$\hat{g}(\xi', \eta', D) = A(\xi', \eta') \int_0^D \cos(a(\xi') z_n) \cos(a(\eta')(D - z_n) \hat{f}(\eta' + \xi', z_n) dz_n$$

Both integrals with respect to z_n can be expressed by Fourier transforms, leading to the final result

$$\hat{g}(\xi', \eta', 0) = (2\pi)^{1/2} A(\xi', \eta') \frac{1}{4} \left\{ e^{i(a(\xi') + a(\eta'))D} \hat{f}(\eta' + \xi', a(\xi') + a(\eta')) \right. \quad (2.5)$$

$$\begin{aligned} + \quad & e^{-i(a(\xi') + a(\eta'))D} \hat{f}(\eta' + \xi', -a(\xi') - a(\eta')) \\ - \quad & e^{i(-a(\xi') + a(\eta'))D} \hat{f}(\eta' + \xi', -a(\xi') + a(\eta')) \\ - \quad & e^{i(a(\xi') - a(\eta'))D} \hat{f}(\eta' + \xi', a(\xi') - a(\eta')) \left. \right\}, \end{aligned}$$

$$\hat{g}(\xi', \eta', D) = (2\pi)^{1/2} A(\xi', \eta') \frac{1}{4} \left\{ e^{ia(\eta')D} \hat{f}(\eta' + \xi', a(\eta') - a(\xi')) \right. \quad (2.6)$$

$$\begin{aligned} + \quad & e^{-ia(\eta')D} \hat{f}(\eta' + \xi', a(\xi') - a(\eta')) - e^{-ia(\eta')D} \hat{f}(\eta' + \xi', -a(\xi') - a(\eta')) \\ - \quad & e^{ia(\eta')D} \hat{f}(\eta' + \xi', a(\xi') + a(\eta')) \left. \right\} \end{aligned}$$

for reflection and transmission, respectively.

3 Resolution

For each pair ξ', η', the equations (2.5), (2.6) constitute two linear equations for the 4 unknowns

$$\hat{f}_{\ell j} = \hat{f}(\xi' + \eta', (-1)^\ell a(\xi') + (-1)^j a(\eta')), \ell, j = 0, 1.$$

Putting

$$X = e^{ia(\xi')D}, Y = e^{ia(\eta')D}$$
$$B(\xi', \eta', x_n) = 4(2\pi)^{-1/2}\hat{g}(\xi', \eta', x_n)/A(\xi', \eta')$$

these equations read

$$XY\hat{f}_{00} + \frac{1}{XY}\hat{f}_{11} - \frac{Y}{X}\hat{f}_{10} - \frac{X}{Y}\hat{f}_{01} = B(\xi', \eta', 0), \tag{3.1}$$

$$-Y\hat{f}_{00} - \frac{1}{Y}\hat{f}_{11} + Y\hat{f}_{10} + \frac{1}{Y}\hat{f}_{01} = B(\xi', \eta', D). \tag{3.2}$$

Putting the sources on top leads to another set of equations. This can be derived by setting up (3.1, 3.2) for the function

$$f_r(x', x_n) = f(x', D - x_n)$$

and observing that

$$\hat{f}_r(\xi', \xi_n) = e^{-iD\xi_n}\hat{f}(\xi', -\xi_n).$$

With $\hat{f}_{r\ell j}$ defined as $\hat{f}_{\ell j}$ we obtain

$$\hat{f}_{r00} = \frac{1}{XY}\hat{f}_{11},\ \hat{f}_{r11} = XY\hat{f}_{00},\ \hat{f}_{r10} = \frac{X}{Y}\hat{f}_{01},\ \hat{f}_{r01} = \frac{Y}{X}\hat{f}_{10}.$$

Thus we obtain the additional equations

$$\hat{f}_{00} + \hat{f}_{11} + \hat{f}_{10} + \hat{f}_{01} = B_r(\xi', \eta', 0) \tag{3.3}$$

$$X\hat{f}_{00} + \frac{1}{X}\hat{f}_{11} + \frac{1}{X}\hat{f}_{10} + X\hat{f}_{01} = B_r(\xi', \eta', D) \tag{3.4}$$

where B_r is obtained by replacing f in B by f_r. Generically the 4 equations suffice to determine $\hat{f}_{\ell j}$ uniquely. This amounts to determining $\hat{f}$ in the disk or ball of radius $2k$, see [1].

References

[1] Natterer, F.: Ultrasound tomography with fixed linear arrays of transducers. Preprint, Institut für Numerische und Angewandte Mathematik, Universität Münster.