An Initial Value Approach to the Inverse Helmholtz Problem at Fixed Frequency

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1 Introduction

The inverse Helmholtz problem calls for the determination of the function f in $\Omega \subseteq \mathbb{R}^n$ from the boundary values $g(\theta, x) = u(x)$, $x \in \partial\Omega$ of the solution u of the Helmholtz equation

$$\begin{aligned} \Delta u(x) + k^2(1 + f(x))u(x) &= 0 \\ u(x) &= e^{ikx\cdot\theta} + w(x) \end{aligned} \tag{1.1}$$

in $\mathbb{R}^n$, where w satisfies the Sommerfeld radiation condition. The number $k > 0$ is the frequency of the incident plane wave with direction $\theta \in S^{n-1}$. We assume $f = 0$ outside Ω. g is measured for a single fixed frequency k and all $\theta \in S^{n-1}$.

The mathematical theory of this inverse problem is fairly well developed [10]. There exists an extensive literature on numerical methods. If the Born or Rytov approximation is valid, the algorithms of diffraction tomography [3] can be used. For the general case, iterative ([1], [2], [4], [8]) and direct [13] methods have been suggested.

The purpose of the present paper is to give a rigourous mathematical justification of the method already described in [12] and to demonstrate its efficacy by computer simulations.

An essential feature in [12] is the use of initial value methods for the Helmholtz equation. Initial value methods, namely the parabolic equation approximation, are also used in [1]. In contrast to [1] we work with initial value problems of second order. The stability analysis is given in §2. In §3 we describe our method for the inverse problem in a continuous setting. The necessary discretizations have been described in [12]. In §4 we report the results of the reconstruction of a breast phantom from simulated data.

2 The Initial Value Problem for the Helmholtz Equation

We consider the $2D$ Helmholtz equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + k^2(1 + f)u = r \tag{2.1}$$

in the upper half plane $x_2 > 0$, subject to the initial conditions

$$u = g \quad , \quad \frac{\partial u}{\partial x_2} = h \tag{2.2}$$

on $x_2 = 0$. This problem is notoriously unstable. However, this instability is a purely high frequency phenomenon. As soon as we restrict everything to sufficiently low frequencies, this instability disappears. More precisely, let u_κ be a low pass filtered version of u with cut off frequency κ, i.e.

$$\hat{u}_\kappa(\xi_1, x_2) = \begin{cases} \hat{u}(\xi_1, x_2) \ , & |\xi_1| < \kappa \ , \\ 0 & , \text{otherwise} \end{cases}$$

where $\hat{u}$ is the Fourier transform

$$\hat{u}(\xi_1, x_2) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{-ix_1 \cdot \xi_1} u(x_1, x_2) dx_1$$

of u with respect to x_1, then u_κ admits a perfectly reasonable estimate in terms of the initial values g, h provided κ is chosen properly. For $f = 0$, $r = 0$ and $\kappa < k$ this follows easily from the explicit solution

$$\hat{u}(\xi_1, x_2) = \hat{g}(\xi_1) \cos(x_2 \sqrt{k^2 - \xi_1^2}) + \frac{\hat{h}(\xi_1)}{\sqrt{k^2 - \xi_1^2}} \sin(x_2 \sqrt{k^2 - \xi_1^2}) \ .$$

In the following we extend this stability result to the general case.

We need the following estimate:

Proposition 1. *Let $v \in C_2([0, \infty), H)$ be a solution to*

$$v'' + A(t)v = r \ , \qquad t > 0 \tag{2.3}$$

where $A(t)$ is a linear bounded operator in the Hilbert space H with the following properties: We have $A = A_1 + A_2$ with $A_1^* = A_1$ and

$$\alpha_1^2(v, v) \le (v, A_1 v) \ , \tag{2.4}$$

$$(v, A_1' v) \le \gamma_1(v, v) \ , \tag{2.5}$$

$$||A_2|| \le \beta_2 \tag{2.6}$$

with $\alpha_1 > 0$. Let $\phi = (v', v') + (v, A_1 v)$. Then,

$$\phi(t) \le \left(\phi(0) + \int_0^t ||r(\tau)||^2 d\tau \right) e^{(1 + 2\beta_2/\alpha_1 + \gamma_1/\alpha_1^2)t} \ . \tag{2.7}$$

Proof: We use the energy method, see e.g. [5]. Multiplying (2.3) with v' we obtain

$$(v', v'') + (v', Av) = (v', r) \; . \qquad (2.8)$$

Using

$$\frac{1}{2}\,\frac{d}{dt}(v', v') = Re(v', v'') \; ,$$
$$\frac{1}{2}\,\frac{d}{dt}(v, A_1 v) = Re(v', A_1 v) + \frac{1}{2}(v, A_1' v)$$

we can rewrite the real part of (2.8) as

$$\frac{1}{2}\,\frac{d}{dt}(v', v') + \frac{1}{2}\frac{d}{dt}(v, A_1 v)$$
$$= Re(v', r) + \frac{1}{2}(v, A_1' v) - Re(v', A_2 v) \; .$$

Integrating we obtain

$$\phi(t) = \phi(0) + 2\int\limits_0^t \left\{ Re(v', r) + \frac{1}{2}(v, A_1' v) - Re(v', A_2 v) \right\} dt \; .$$

Cauchy-Schwarz and (2.5), (2.6) yield

$$\phi(t) \le \phi(0) + \int\limits_0^t \{\; 2\|v'\|\|r\| + \gamma_1 \|v\|^2 + 2\beta_2 \|v'\|\|v\| \} \; dt \; .$$

By an appropriate use of the inequality

$$ab \le \frac{1}{2}(\delta^2 a^2 + \delta^{-2} b^2) \; , \quad \delta > 0$$

we obtain

$$\phi(t) \le \phi(0) + \int\limits_0^t \left\{ \|v'\|^2 + \|r\|^2 + \gamma_1 \|v\|^2 + \frac{\beta_2}{\alpha_1}\|v'\|^2 + \beta_2 \alpha_1 \|v\|^2 \right\} dt$$
$$= \phi(0) + \int\limits_0^t \|r\|^2 dt + \int\limits_0^t \left\{ \left(1 + \frac{\beta_2}{\alpha_1}\right) \|v'\|^2 + (\gamma_1 + \beta_2 \alpha_1) \|v\|^2 \right\} dt \; .$$

Using

$$\phi \ge \|v'\|^2 + \alpha_1^2 \|v\|^2$$

we obtain

$$\phi(t) \le c_1(t) + c_2 \int\limits_0^t \phi(t) dt \; ,$$

$$c_1(t) = \phi(0) + \int\limits_0^t \|r\|^2 dt \; , \quad c_2 = 1 + 2\frac{\beta_2}{\alpha_1} + \frac{\gamma_1}{\alpha_1^2} \; .$$

Now (2.7) follows from Gronwall's inequality (see e.g. [6], p. 24).

□

Now we can prove our stability estimate for u_κ.

Theorem 2. *Let $f \in C^1(\mathbb{R}^2)$ and $f = f_1 + \frac{i}{k} f_2$ with f_1, f_2 real, and let m_ν, M_ν be constants such that*

$$-1 < m_1 \le f_1 \le M_1 \ , \qquad \left| \frac{\partial f_1}{\partial x_2} \right| \le M_1 \ , \quad |f_2| \le M_2 \ .$$

Then, for $\kappa < k\sqrt{1+m_1}$, we have

$$\|u'_\kappa(\cdot, x_2)\|^2 \le e^{\alpha x_2} \left(\|h\|^2 + k^2(1+M_1)\|g\|^2 + 2\|r\|^2 + 2k^4 \|f(u-u_\kappa)\|^2 \right) \quad (2.9)$$

where

$$\alpha = 1 + \frac{2M_2}{\vartheta} + \frac{M_1}{\vartheta^2} \ , \quad \vartheta = \sqrt{1 + m_1 - \left(\frac{\kappa}{k}\right)^2} \ .$$

Proof: Taking in (2.1) the Fourier transform with respect to x_1 we obtain

$$(k^2 - \xi_1^2)\hat{u}(\xi_1, x_2) + \frac{\partial^2}{\partial x_2^2} \hat{u}(\xi_1, x_2) + (2\pi)^{-1/2} k^2 \int_{-\infty}^{+\infty} \hat{f}(\xi_1 - \eta_1, x_2)\hat{u}(\eta_1, x_2) d\eta_1$$
$$= \hat{r}(\xi_1, x_2) \ .$$

We consider this equation only for $|\xi_1| \le \kappa$ and decompose the integral accordingly, i.e.

$$(k^2 - \xi_1^2)\hat{u}_\kappa(\xi_1, x_2) + \frac{\partial^2}{\partial x_2^2} \hat{u}_\kappa(\xi_1, x_2) + (2\pi)^{-1/2} k^2 \int_{-\infty}^{+\infty} \hat{f}(\xi_1 - \eta_1, x_2)\hat{u}_\kappa(\eta_1, x_2) d\eta_1$$
$$= \hat{r}(\xi_1, x_2) + \hat{\varepsilon}(\xi_1, x_2) \ , \qquad \hat{\varepsilon}(\xi_1, x_2) = -(2\pi)^{-1/2} k^2 \int_{|\eta_1| \ge \kappa} \hat{f}(\xi_1 - \eta_1, x_2)\hat{u}(\eta_1, x_2) d\eta_1 \ .$$

Obviously,

$$\varepsilon = -k^2 f(u - u_\kappa) \ .$$

We rewrite the last equation for $\hat{u}_\kappa$ as

$$\hat{u}''_\kappa(x_2) + A(x_2)\hat{u}_\kappa(x_2) = \hat{r}(x_2) + \hat{\varepsilon}(x_2) \quad (2.10)$$

In order to apply Proposition 1 to (2.10) we only have to find the constants in (2.4 - 2.6) for the operator A from (2.10). We have

$$(A\hat{u}, \hat{v}) = ((k^2 - \xi_1^2)\hat{u}, \hat{v}) + (2\pi)^{-1/2} k^2 \int_{-\infty}^{+\infty} (\hat{f} \star \hat{u}) \overline{\hat{v}} d\xi_1$$
$$= ((k^2 - \xi_1^2)\hat{u}, \hat{v}) + k^2 \int_{-\infty}^{+\infty} \widehat{fu} \tilde{\overline{v}} d\xi_1$$
$$= ((k^2 - \xi_1^2)\hat{u}, \hat{v}) + k^2 \int_{-\infty}^{+\infty} f u \overline{v} dx_1$$

where we have used Parseval's relation. Writing f as $f_1 + \frac{i}{k} f_2$ it readily follows that

$$
\begin{aligned}
(A_1 \hat{u}, \hat{u}) &= ((k^2 - \xi_1^2)\hat{u}, \hat{u}) + k^2 \int_{-\infty}^{+\infty} f_1 |u|^2 dx_1 \ , \\
(A_2 \hat{u})^{\sim} &= ik f_2 u \ .
\end{aligned}
$$

From these relations we easily read

$$\alpha_1^2 = k^2 - \kappa^2 + k^2 m_1 = \vartheta^2 k^2 \ , \quad \gamma_1 = k^2 M_1 \ , \beta_2 = k M_2 \ .$$

Inserting this into (2.7) and using

$$\|\hat{u}'_\kappa(\cdot, x_2)\|^2 \leq \phi(x_2) \leq \|\hat{u}'_\kappa(\cdot, x_2)\|^2 + k^2(1 + M_1)\|\hat{u}_\kappa(\cdot, x_2)\|^2$$

yields (2.9).

□

We remark that we can replace the left hand side of (2.9) by

$$k^2 \vartheta^2 \|u_\kappa(\cdot, x_2)\|^2 \ .$$

Note that $\kappa = k\sqrt{1 + m_1 - \vartheta^2}$.

The estimate of Theorem 2 is useful only if $u - u_\kappa$ is small in some sense. In the next theorem we show that this is in fact the case for the scattered wave w from (1.1) with $\theta = \binom{0}{1}$. For w we have the Lippmann-Schwinger equation

$$w(x) = \frac{-i}{4} k^2 \int_{\mathbf{R}^2} H_0(k|x - x'|) f(x')(e^{ikx'_2} + w(x'))dx'$$

with the Hankel function of the first kind H_0. Taking the Fourier transform with respect to x_1 we obtain

$$
\begin{aligned}
& \hat{w}(\xi_1, x_2) \\
&= -\frac{i}{4\sqrt{2\pi}} k^2 \int\int\int H_0(k\sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2}) e^{-i\xi_1 x_1} dx_1 \\
& \quad (e^{ikx'_2} + w(x'_1, x'_2)) f(x'_1, x'_2) dx'_1 dx'_2 \ .
\end{aligned}
$$

The x_1 integral can be evaluated as follows. With $a(\xi) = \sqrt{k^2 - \xi^2}$ we have

$$H_0(k\sqrt{u^2 + v^2}) = \frac{-i}{4\pi} \int e^{i(|u|a(\xi) + v\xi)} \frac{d\xi}{a(\xi)} \ ,$$

see e.g. [9], p. 123. Hence

$$\begin{aligned}
&\int H_0\left(k\sqrt{(x_1-x_1')^2+(x_2-x_2')^2}\right)e^{-i\xi_1 x_1}dx_1\\
&=\frac{-i}{4\pi}\int\int e^{i(|x_2-x_2'|a(\xi)+(x_1-x_1')\xi)-i\xi_1 x_1}\frac{dx_1 d\xi}{a(\xi)}\\
&=\frac{-i}{2}\int e^{i(|x_2-x_2'|a(\xi)-x_1'\xi)}\delta(\xi-\xi_1)\frac{d\xi}{a(\xi)}\\
&=-\frac{i}{2}e^{i(|x_2-x_2'|a(\xi_1)-x_1'\xi_1)}\frac{1}{a(\xi_1)}\ .
\end{aligned}$$

It follows that

$$\begin{aligned}
&\hat{w}(\xi_1,x_2)\\
&=\frac{k^2}{8a(\xi_1)}\int\int e^{i(|x_2-x_2'|a(\xi_1)-x_1'\xi_1)}\left(e^{ikx_2'}+w(x_1',x_2')\right)f(x_1',x_2')dx_1'dx_2'\ .
\end{aligned}$$

Now assume that the straight line $x_2'=x_2$ misses $\mathrm{supp}(f)$ by the quantity $\varepsilon>0$ and $\xi_1>k$. Then,

$$|\hat{w}(\xi_1,x_2)|\le\frac{k^2}{8\sqrt{\xi_1^2-k^2}}\,e^{-\varepsilon\sqrt{\xi_1^2-k^2}}(1+\|w\|_\infty)\|f\|_1\ .$$

Hence we have for arbitrary θ:

Theorem 3. *Let w be the scattered wave for incident direction θ, and let L be a straight line perpendicular to θ which misses the support of f. Then, the Fourier transform of w along L decays exponentially beyond k.*

We shall apply Theorem 2 with m_1 negative but close to zero (typically $m_1=-0.01$). Then, Theorem 2 requires κ to be slightly smaller than k, while, according to Theorem 3, $u-u_\kappa$ in Theorem 2 is small if κ is slightly bigger than k. In practice this conflict did not cause any problems. With $\kappa=0.99k$ we observed both stability and accuracy.

3 The Inversion Method

We consider the case $n=2$. Assume that Ω is the ball of radius ρ centered at the origin. Let the scattered field w_j be measured on $\partial\Omega$ for p directions θ_j. We first extend w_j to the exterior of Ω. Since w_j satisfies there the homogeneous Helmholtz equation $\Delta w_j+k^2w_j=0$, this is easily done by expanding $w_j(x)$ for $|x|>\rho$ as

$$w_j(r\cos\varphi,r\sin\varphi)=\sum_\ell c_\ell e^{i\ell\varphi}H_\ell(kr)$$

with H_ℓ the first kind Hankel function of order ℓ. The coefficients c_ℓ are readily obtained from the data by

$$c_\ell = \frac{1}{2\pi H_\ell(k\rho)} \int_0^{2\pi} w_j(\rho\cos\varphi, \rho\sin\varphi) e^{-i\ell\varphi} d\varphi \ .$$

Now let Q_j be the square circumscribed to Ω, two of its edges, named Γ_j, parallel to θ_j. The other edges are denoted by Γ_j^-, Γ_j^+ with θ_j pointing from Γ_j^- to Γ_j^+. We define an operator $R_j : L_2(\Omega) \to L_2(\Gamma_j^+)$ by solving the initial value problem

$$\begin{aligned} \Delta w + k^2(1+f)w &= -k^2 f e^{ikx\cdot\theta} \quad \text{in } Q_j \ , \\ w = w_j \ , \ \frac{\partial w}{\partial \nu} &= \frac{\partial w_j}{\partial \nu} \quad \text{on } \Gamma_j^- \ , \ w = w_j \ \text{ on } \Gamma_j \end{aligned}$$

and putting $R_j(f) = w$ on Γ_j^+. We also put $g_j = w_j|_{\Gamma_j^+}$.

Now we have to solve the nonlinear system

$$R_j(f) = g_j \ , \quad j = 1, \ldots, p$$

for f. We do this by an ART type algorithm.

With f^0 an initial approximation to f we compute a new approximation f^1 by

$$\begin{aligned} f_0 &= f^0 \quad , \\ f_j &= f_{j-1} + \gamma R_j'^*(f_{j-1})(g_j - R_j(f_{j-1})) \ , \ j = 1, \ldots, p \\ f^1 &= f_p \ . \end{aligned}$$

γ is a relaxation parameter.

$R_j'^*$ can be evaluated by solving an initial value problem with initial values on Γ_j^+. This can done by the usual five point finite difference star, combined with filtering along the grid lines perpendicular to θ_j. For details see [12].

4 Numerical Experiment

We did reconstructions from data obtained from a computer generated breast phantom which has been patterned after a phantom created by Borup et al. [1]. With c the sound speed in the breast tissue, $c_0 = 1500\frac{m}{sec}$ the sound speed in the surrounding water, and α the attenuation coefficient in the breast tissue we have

$$f = \frac{c_0^2}{c^2} - 1 - i\frac{2\alpha c_0}{kc} \quad , \quad \omega = c_0 k \ . \tag{4.1}$$

The phantom is made up of four different kinds of tissue: fat, glandular tissue, tumor, and cyst. The values of c and α (at 1MHZ) are

tissue	$c[\frac{m}{sec}]$	$\alpha[\frac{db}{m}]$	Ref	Imf
fat	1458	41	0.058	$-9.4/k$
glandular tissue	1519	80	-0.025	$-18.4/k$
tumor	1564	118	-0.080	$-27.2/k$
cyst	1568	10	-0.084	$-2.3/k$

The value of α in column 2 have to be converted to those in formula (4.1) by dividing with the factor $20\log_{10}(e)$, yielding α in units $[\frac{1}{m}]$. Therefore, k in column 4 is in units of $[\frac{1}{m}]$, too.

We generated data by solving the forward problem with the initial value method, assuming the backscatter to be zero. This may be considered as "inverse crime". However, in our case, proceeding in this way can be justified. First, we checked our forward solver very carefully for potentials f which admit an exact solution and found satisfactory agreement. Second, then simply doesn't exist a forward solver for the Helmholtz boundary value problem for k as large as in our case.

We worked on a 128×128 grid, using $p = 128$ equally spaced directions in $[0, 2\pi)$. The frequency of the irradiating plane waves was 1MHZ, i.e. $\omega = 2\pi \cdot 10^6$ sec^{-1}, hence $k = 4189 m^{-1}$.

The condition for the Born approximation to hold is

$$R|f| \ll \frac{2\pi}{k}$$

where R is the Radon transform. This is a slight generalisation of the condition given by Kak and Slaney [7]. It is obviously far from being satisfied. As has been discussed in [12], we can't use 0 as initial approximation in this case. We used the reconstruction for the smaller value $k = 1000\ m^{-1}$ instead which can be computed starting out from 0. A smoothed version of the initial approximation thus obtained is shown in Fig. 2.

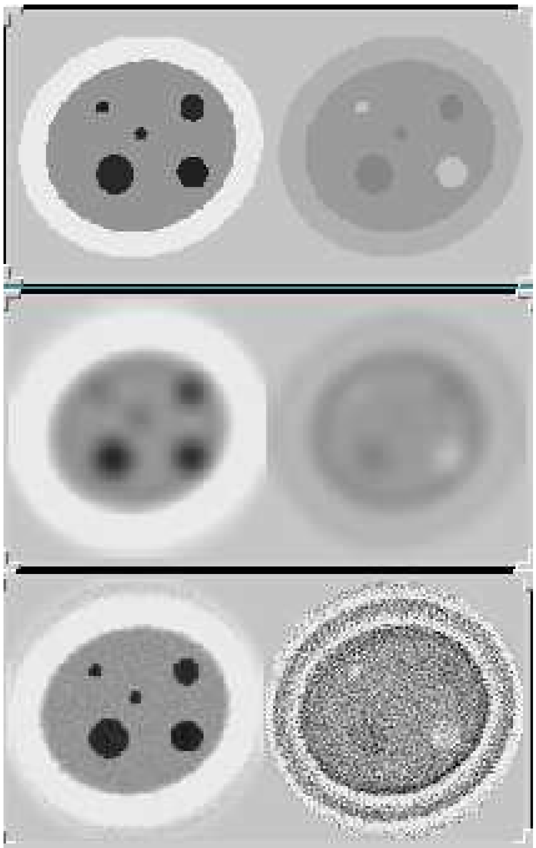

Figure captions

Fig. 1 (top): Breast phantom. Real part left, imaginary part right. Since the imaginary part is much smaller than the real part it has been scaled. The location of the 3 tumors and 2 cysts is clear from the real part, while they can be distinguished by looking at the imaginary part.
Fig. 2 (middle): Initial approximation which has been computed from data corresponding to $k = 1000\ m^{-1}$ using 0 as initial approximation.
Fig. 3 (bottom): Reconstruction after 6 sweeps with $\gamma' = 0.5$.

The reconstruction after 6 sweeps is shown in Fig. 3. The total computation time on a SPARC 20 was 5 minutes. The value of γ has been chosen in the following way. First we determined in [11] the asymptotic value $1/(\rho k^2)$ of the operator $(R_j'(0)R_j'^*(0))^{-1}$ for large k. Then we put $\gamma = \gamma'/(\rho k^2)$ where, in agreement with the convergence properties of the Kaczmarz procedure, $0 < \gamma' < 2$. We used $\gamma' = 0.5$.

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