Spatial resolution of the PBP algorithm in acoustical inverse scattering

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1 Introduction

The PBP algorithm [1] solves the inverse scattering problem

$$\begin{aligned} \Delta u + k^2(1+f)u &= 0 \quad \text{in } \mathbb{R}^2 \,, \\ u &= e^{ikx\cdot\theta}(1+v) \end{aligned} \tag{1.1}$$

where $e^{ikx\cdot\theta}v$ satisfies the radiation condition, $\theta \in S^1$, and $k > 0$ fixed. The sought - for function f is supported in $|x| \leq \rho$, and the data is $g_+ = v$ on $\Gamma_+(\theta)$, $g_- = v$ on $\Gamma_-(\theta)$ where

$$\Gamma_\pm(\theta) = \{x = \pm\rho\theta + s\theta^\perp : s \in \mathbb{R}^1\} \,.$$

Hence g_+ is the transmitted signal and g_- the backscatter for incidence direction θ. The spatial resolution we expect is $\lambda/2$ with $\lambda = 2\pi/k$ the wavelength of the irradiating wave. This means that functions f of (essential) bandwidth $2k$ should be recovered reliably. It is well known [2] that this is true within the Born approximation, where the frequency components $< \sqrt{2}k$ are determined by the transmitted signal $g_+(\theta, s)$, and those between $\sqrt{2}k$ and $2k$ by the backscatter $g_-(\theta, s)$. We note that the condition for the Born approximation to hold is

$$|Rf| \ll \frac{2\pi}{k}$$

where R is the Radon transform. This is a slight extension of the condition in [2].

In this paper we analyse PBP within the Born approximation. We show that the algorithm as it stands can't possibly achieve resolution $\lambda/2$ or $2k$. Based on our analysis we suggest a modificantion of PBP and demonstrate by numerical examples taken from [9] that the modified PBP algorithm does achieve resolution $2k$ even in the fully nonlinear case, i.e. when the Born approximation is not valid. We do not aim at superresolution [7].

2 The PBP algorithm

The PBP algorithm is essentially an extension of the ART algorithm of CT (see e.g. [3]) to the nonlinear System

$$R_j(f) = g_+(\theta_j, \cdot) \ , \qquad j = 1, \dots, p \tag{2.1}$$

where $R_j(f) = v^j$ on $\Gamma_+(\theta_j)$ with v^j a low-pass filtered version of the solution v of (1.1) with $\theta = \theta_j$ and initial values

$$v = g_-(\theta_j, \cdot) \ , \qquad \frac{\partial v}{\partial \nu} = \frac{\partial g_-}{\partial \nu}(\theta_j, \cdot) \quad \text{on} \ \ \Gamma_-(\theta_j) \ . \tag{2.2}$$

Here, ν denotes the interior normal on $\Gamma_-(\theta_j)$. It has been shown in [4] that v^j can be determined in a stable way from these initial values provided the cut-off frequency of the low-pass filter is put at some value slightly smaller than k. We remark that $\partial g_- / \partial \nu$ can be computed easily from g_- by solving the exterior Helmholtz problem where $f = 0$. We also remark that (2.1) involves implicitly the backscatter g_- via the initial conditions (2.2) even though g_- does not show up explicitly.

PBP solves (2.1) iteratively. We describe the step $f^0 \to f^1$ where f^0 is an initial approximation:

$$f_0 = f^0$$

For $j = 1, \dots, p$:

$$f_j = f_{j-1} - \omega R_j'^*(f_{j-1}) S_j^{-1}(R_j(f_{j-1}) - g_j) \ , \tag{2.3}$$

$$f^1 = f_p \ .$$

Here we have put $g_j = g_+(\theta_j, \cdot)$. S_j is a certain operator on $L_2(\Gamma_+(\theta_j))$ which is supposed to mimic the action of $R_j'(f_{j-1}) R_j'^*(f_{j-1})$. In [5] we have shown that for linear operators R_j the process converges for $0 < \omega < 2$ provided that $S_j \geq R_j R_j^*$. In our implementation of PBP we use either a multiple of the unit operator or the operator $R_j'(0) R_j'^*(0)$. The latter choice leads to convergence in a few steps provided the initial approximation is chosen reasonably.

We now give an analysis of PBP in the Born appximation. The algorithm is then essentially identical to the one given in [6], except that $R_j(f_{j-1})$ is evaluated in a different way. The differential equation for v is

$$\Delta v + 2ik\theta \cdot \nabla v + k^2 v f = -k^2 f \ . \tag{2.4}$$

The Born approximation is obtained by dropping $k^2 vf$, yielding

$$\Delta v + 2ik\theta \cdot \nabla v = -k^2 f \ . \tag{2.5}$$

The initial value problem for this equation with initial values on $\Gamma_-(\theta)$ is easily solved by taking the Fourier transform of $v(t\theta + s\theta^\perp)$ with respect to s. Putting $\kappa = \sqrt{k^2 - \sigma^2}$ we obtain for $|\sigma| < k$

$$\begin{aligned} \hat{v}(t\theta + \sigma\theta^\perp) &= -\frac{k^2}{\kappa} \int_{-\rho}^{t} e^{-ik(t-t')} \sin(\kappa(t-t')) \hat{f}(t'\theta + \sigma\theta^\perp) dt' \\ &+ e^{-ik(t+\rho)} \left(\hat{g}_-(\theta,\sigma) \cos(\kappa(t+\rho)) + \frac{1}{\kappa}(\frac{\partial \hat{g}_-}{\partial \nu} + ik\hat{g}_-)(\theta,\sigma) \sin(\kappa(t+\rho)) \right) \ . \end{aligned} \tag{2.6}$$

For $t = \rho$ and $\theta = \theta_j$ the left hand side is just $\hat{v}^j$ on $\Gamma_+(\theta_j)$. Hence

$$\begin{aligned} (R_j f)^\wedge(\sigma) &= -\frac{k^2}{\kappa} \int_{-\rho}^{\rho} e^{-ik(\rho - t)} \sin(\kappa(\rho - t)) \hat{f}(t\theta + \sigma\theta^\perp) dt \\ &+ \hat{G}_-(\theta_j, \sigma) \end{aligned}$$

where G_- depends only on g_-, $\partial g_- / \partial \nu$ on $\Gamma_-(\theta_j)$ but not on f. Expressing $\hat{f}$ by the $2D$ Fourier transform of f which we denote by $\hat{f}$, too, yields

$$\begin{aligned} (R_j f)^\wedge(\sigma) &= -\frac{k^2}{(2\pi)^{1/2}\kappa} \int_{\mathbb{R}^1} \int_{\mathbb{R}^1} e^{-ik(\rho-t)} \sin(\kappa(\rho - t)) \hat{f}(\tau\theta_j + \sigma\theta_j^\perp) e^{it\tau} d\tau dt \\ &+ \hat{G}_-(\theta_j, \sigma) \ . \end{aligned}$$

The integral over $[-\rho, \rho]$ has been extended to all of $\mathbb{R}^1$ because $f(x)$ vanishes outside $|x| < \rho$. The t integral can be evaluated, yielding

$$\begin{aligned} (R_j f)^\wedge(\sigma) &= -\frac{e^{-ik\rho}(2\pi)^{1/2}}{2i\kappa} \int_{\mathbb{R}^1} (e^{i\kappa\rho}\delta(k - \kappa + \tau) - e^{-i\kappa\rho}\delta(k + \kappa + \tau)) \hat{f}(\tau\theta_j + \sigma\theta_j^\perp) d\tau \\ &\quad + \hat{G}_-(\theta_j, \sigma) \\ &= -\frac{e^{-ik\rho}(2\pi)^{1/2}}{2i\kappa} \left(e^{i\kappa\rho} \hat{f}((\kappa - k)\theta_j + \sigma\theta_j^\perp) - e^{-i\kappa\rho} \hat{f}((-\kappa - k)\theta_j + \sigma\theta_j^\perp) \right) \\ &\quad + \hat{G}_-(\theta_j, \sigma) \ . \end{aligned} \tag{2.7}$$

The values of $\hat{f}$ which appear on the right hand side are just the values on the Ewald Spheres (or better Ewald circles in $2D$), see Fig. 1.

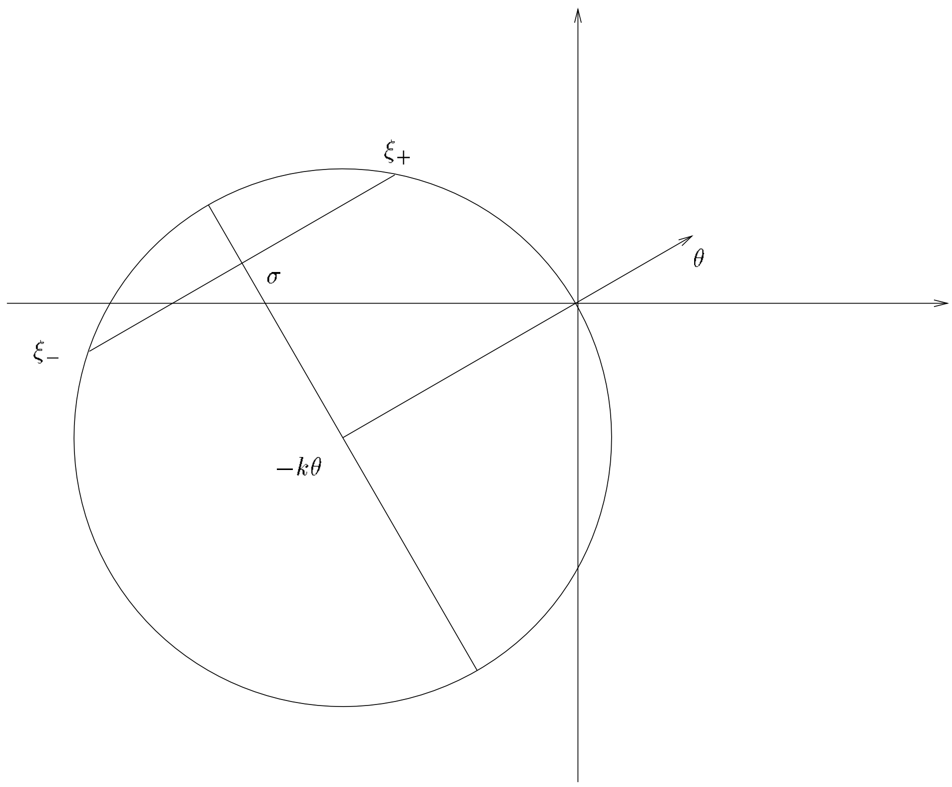

Fig. 1: The points $\xi_\pm(\theta, \sigma) = (\pm\kappa - k)\theta + \sigma\theta^\perp$ in Fourier space where $\hat{f}$ is taken in (2.7).

From (2.7) we draw an important conclusion: Our operator R_j provides information only about a linear combination of $\hat{f}((\pm\kappa - k)\theta_j + \sigma\theta_j^\perp)$. This is in contrast to diffraction tomography where each of these two numbers is determined separately by the transmitted and backscattered fields, resp.. It is clear that $\hat{f}$ can't be determined uniquely from (2.7). For instance, a function $\hat{f}$ with

$$e^{i\kappa\rho}\hat{f}(\xi_+(\theta_j, \sigma)) = e^{-i\kappa\rho}\hat{f}(\xi_-(\theta_j, \sigma))$$

does not contribute to $R_j(f)$. Such a function is easily constructed as a radially symmetric function.

The conclusion is that the operator R_j is not a good one for the solution of the inverse problem. The only reason for this choice is that $R_j(f)$ can be evaluated very efficiently by initial value techniques. In the next section we show how to extend R_j to an operator which still can be evaluated by initial value techniques but which, at least in the Born approximation, is stably invertible.

3 The extended PBP algorithm

Our aim is to separate the two contributions of $\hat{f}$ in (2.7). An obvious way to do this is to include in $R_j(f)$ the derivative of (2.7) with respect to ρ. It will turn out that

$$R_j(f) = \left(\begin{array}{c} v^j \\ \frac{\partial}{\partial \nu} v^j - ikv^j \end{array} \right)_{\Gamma_+(\theta_j)} \tag{3.1}$$

is a good choice. Here, v^j is the same function as in (2.1). The system we have to solve is now

$$R_j(f) = \left(\begin{array}{c} g_+(\theta_j, \cdot) \\ \left(\frac{\partial g_+}{\partial \nu} - ikg_+\right)(\theta_j, \cdot) \end{array} \right) \quad , \qquad j = 1, \ldots, p \, . \tag{3.2}$$

Again $\partial g_+ / \partial \nu$ can be computed easily by solving the exterior Helmholtz equation.

We compute the operator $R_j'(f) : L_2(|x| < \rho) \to L_2(\Gamma_+(\theta_j))^2$ and its adjoint. Replacing v by $v + w$, f by $f + h$ in (2.4) and neglecting wh we obtain

$$\begin{aligned} \Delta w + 2ik\theta \cdot w + k^2 f w &= -k^2(1 + v)h \quad \text{in } \mathbb{R}^2 \, , \\ w = 0 \, , \quad \frac{\partial w}{\partial \nu} &= 0 \quad \text{on } \Gamma_-(\theta) \ . \end{aligned} \tag{3.3}$$

Let w^j be a low-pass filtered version of w for $\theta = \theta_j$. Then,

$$R_j'(f)h = \left(\begin{array}{c} w^j \\ \frac{\partial}{\partial \nu} w^j - ikw^j \end{array} \right)_{\Gamma_+(\theta_j)} \ . \tag{3.4}$$

In order to determine $R_j'(f)^* : L_2(\Gamma_+(\theta_j))^2 \to L_2(|x| < \rho)$ we take $g = \left(\begin{array}{c} g_0 \\ g_1 \end{array} \right) \in L_2(\Gamma_+(\theta_j))^2$ and compute

$$(R_j'(f)h, g) = (w^j, g_0)_{L_2(\Gamma_+(\theta_j))} + \left(\frac{\partial w^j}{\partial \nu} - ikw^j, g_1 \right)_{L_2(\Gamma_+(\theta_j))} \ .$$

Let z^j be a low-pass filtered version of the solution of the initial (better: final) value problem

$$\Delta z + 2ik\theta \cdot \nabla z + k^2 \overline{f} z = 0 \quad \text{in } \mathbb{R}^2 \, ,$$

$$z = -g_1 \ , \quad \frac{\partial z}{\partial \nu} - ikz = g_0 \quad \text{on} \quad \Gamma_+(\theta) \tag{3.5}$$

for $\theta = \theta_j$. Let $Q(\theta)$ be the strip between $\Gamma_-(\theta)$, $\Gamma_+(\theta)$. Green's second formula and an integration by parts yields for any w, z

$$\int\limits_{Q(\theta)} \left\{ (\Delta w + 2ik\theta \cdot \nabla w - k^2 f w)\overline{z} - w\overline{(\Delta z + 2ik\theta \cdot \nabla z - k^2 \overline{f} z)} \right\} dx$$

$$= \int\limits_{\Gamma_+(\theta) \cup \Gamma_-(\theta)} \left(w \frac{\partial \overline{z}}{\partial \nu} - \frac{\partial w}{\partial \nu} \overline{z} \right) ds + 2ik \left(\int\limits_{\Gamma_+(\theta)} w\overline{z} ds - \int\limits_{\Gamma_-(\theta)} w\overline{z} ds \right) .$$

For $w = w^j$, $z = z^j$, $\theta = \theta_j$ this reads

$$-k^2 \int\limits_{Q(\theta)} (1+v) h \overline{z} dx = \int\limits_{\Gamma_+(\theta)} \left\{ w \overline{\left(\frac{\partial z}{\partial \nu} - ikz \right)} - \left(\frac{\partial w}{\partial \nu} - ikw \right) \overline{z} \right\} ds \ ,$$

hence

$$\left(h, -k^2(1+\overline{v}) z \right)_{L_2(|x|<\rho)} = (w, g_0)_{L_2(\Gamma_+(\theta))} + \left(\frac{\partial w}{\partial \nu} - ikw, g_1 \right)_{L_2(\Gamma_+(\theta))} .$$

This means that

$$R_j'(f)^* g = -k^2(1+\overline{v}) z^j \ . \tag{3.6}$$

We may convince ourselves that $R_j(f)$ from (3.1) contains, at least within the Born approximation, enough information about f to determine it up to spatial frequencies $2k$. For this we look at the operator $R_j'(0)$. Solving (3.3) for $f = 0$ yields according to (2.6)

$$\hat{w}(t\theta + \sigma\theta^\perp) = -\frac{k^2}{\kappa} \int\limits_{-\rho}^{t} e^{-ik(t-t')} \sin(\kappa(t-t')) \hat{h}(t'\theta + \sigma\theta^\perp) dt' \ .$$

As in (2.6), $\hat{w}$, $\hat{h}$ denote the 1D Fourier transforms of $w(t\theta + s\theta^\perp)$, $h(t'\theta + s\theta^\perp)$ with respect to s. Observing that $\frac{\partial w}{\partial \nu} = -\frac{\partial w}{\partial t}$ on $\Gamma_+(\theta)$ yields

$$(R_j'(0)h)^\wedge(\sigma) = k^2 \int\limits_{-\rho}^{\rho} e^{-ik(\rho-t)} \begin{pmatrix} \frac{-1}{\kappa} \sin \kappa(\rho - t) \\ \cos \kappa(\rho - t) \end{pmatrix} \hat{h}(t\theta + \sigma\theta^\perp) dt \ . \tag{3.7}$$

Inserting here the 2D Fourier transform of h, again denoted by $\hat{h}$, we obtain in a way completely analogous to (2.7)

$$(R_j'(0)h)^\wedge(\sigma) = k^2 \frac{1}{2}(2\pi)^{1/2} e^{-ik\rho} \cdot$$

$$\begin{pmatrix} \frac{i}{\kappa}(e^{i\kappa\rho}\hat{h}((\kappa - k)\theta_j + \sigma\theta_j^\perp) & - & e^{-i\kappa\rho}\hat{h}((-\kappa - k)\theta_j + \sigma\theta_j^\perp)) \\ e^{i\kappa\rho}\hat{h}((\kappa - k)\theta_j + \sigma\theta_j^\perp) & + & e^{-i\kappa\rho}\hat{h}((-\kappa - k)\theta_j + \sigma\theta_j^\perp) \end{pmatrix} .$$

Let $K : L_2(\Gamma_+(\theta)) \to L_2(\Gamma_+(\theta))$ be defined by $(Kg)^\wedge = \kappa\hat{g}$. Then,

$$((\pm iK, I)R_j'(0)h)^\wedge(\sigma) = (2\pi)^{1/2} k^2 e^{-i(k\pm\kappa)\rho}\hat{h}((\mp\kappa - k)\theta_j + \sigma\theta_j^\perp) . \quad (3.8)$$

This shows that $R_j'(0)h$ determines $\hat{h}(\xi)$ for $|\xi| \le 2k$, with $+$ corresponding to the backscatter and $-$ to the transmitted field.

4 Implementation of the extended PBP method

There are two ways of implementing ART for (3.2). In the first one we use exactly (2.3) with R_j as in (3.1). For S_j we take the operator $R'_j(0)R'_j(0)^*$. For $R'_j(0)$ we already found in (3.7) a handy expression. $R'_j(0)^*$ is determined essentially in the same way. Solving (3.5) for $f = 0$ by taking Fourier transforms in $\Gamma_+(\theta)$ yields

$$\hat{z}(t\theta + \sigma\theta^\perp) = \left(\hat{g}_0(\sigma)\frac{\sin(\kappa(\rho - t))}{\kappa} - \hat{g}_1(\sigma)\cos(\kappa(\rho - t))\right) e^{ik(\rho - t)} ,$$

hence, according to (3.6),

$$(R'_j(0)^* g)^\wedge(t\theta + \sigma\theta^\perp) =$$
$$-k^2 \left(\hat{g}_0(\sigma)\frac{\sin(\kappa(\rho - t))}{\kappa} - \hat{g}_1(\sigma)\cos(\kappa(\rho - t))\right) e^{ik(\rho - t)} .$$

From this and (3.7) we obtain for the operator S_j in Fourier space

$$(S_j g)^\wedge(\sigma) = (R'_j(0)R'^*_j(0)g)^\wedge(\sigma)$$

$$= -k^4 \int_{-\rho}^{\rho} \begin{pmatrix} -\frac{1}{\kappa}\sin(\kappa(\rho - t)) \\ \cos(\kappa(\rho - t)) \end{pmatrix} \left(\hat{g}_0(\sigma)\frac{\sin(\kappa(\rho - t))}{\kappa} - \hat{g}_1(\sigma)\cos(\kappa(\rho - t))\right) dt$$

$$= k^4 \begin{pmatrix} c_1/\kappa^2 & , & c_2 \\ c_2 & , & c_3 \end{pmatrix} \begin{pmatrix} \hat{g}_0(\sigma) \\ \hat{g}_1(\sigma) \end{pmatrix} , \tag{4.1}$$

$$c_1 = \int_{-\rho}^{\rho} \sin^2(\kappa(\rho - t))dt \ , \quad c_3 = \int_{-\rho}^{\rho} \cos^2(\kappa(\rho - t))dt$$

$$c_2 = \frac{-1}{\kappa}\int_{-\rho}^{\rho} \cos(\kappa(\rho - t))\sin(\kappa(\rho - t))dt$$

Introducing $\alpha = 2\rho\kappa$ be we obtain

$$
\begin{aligned}
c_1 &= \rho\left(1-\frac{\cos\alpha\sin\alpha}{\alpha}\right)\ , \quad c_3=\rho\left(1+\frac{\cos\alpha\sin\alpha}{\alpha}\right)\ , \\
c_2 &= -2\rho^2\left(\frac{\sin\alpha}{\alpha}\right)^2\ .
\end{aligned} \tag{4.2}
$$

The operator S_j^{-1} in (2.3) in Fourier space is the inverse of (4.1), i.e.

$$
\begin{aligned}
S_j^{-1} &= \frac{1}{\Delta k^4}\begin{pmatrix} c_3 & -c_2 \\ -c_2 & c_1/\kappa^2 \end{pmatrix}\ , \\
\Delta &= \frac{c_1c_3}{\kappa^2}-c_2^2=\frac{\rho^2}{\kappa^2}\left(1-\left(\frac{\sin\alpha}{\kappa}\right)^2\right)\ .
\end{aligned} \tag{4.3}
$$

With (4.3), (3.6) all the operations in (2.3) are known.

Alternatively we may work with the operator

$$
\mathcal{K}\ \ R_j\ , \qquad \mathcal{K}=\begin{pmatrix} iK & I \\ -iK & I \end{pmatrix}
$$

instead of R_j. Then, (2.3) reads

$$
f_j=f_{j-1}-\omega R_j'^{*}(f_{j-1})\mathcal{K}^*(\mathcal{K}S_j\mathcal{K}^*)^{-1}\mathcal{K}(R_jf_{j-1}-g_j)
$$

with S_j as above. From (4.1) we obtain

$$
\begin{aligned}
(\mathcal{K}S_j\mathcal{K}^*g)^\wedge &= \begin{pmatrix} i\kappa & , & 1 \\ -i\kappa & & 1 \end{pmatrix} k^4 \begin{pmatrix} c_1/\kappa^2 & , & c_2 \\ c_2 & , & c_3 \end{pmatrix}\ \begin{pmatrix} -i\kappa & i\kappa \\ 1 & 1 \end{pmatrix}\hat{g} \\
&= 2\rho k^4\begin{pmatrix} 1 & , & c \\ \overline{c} & , & 1 \end{pmatrix}\ \hat{g}\ , \quad c=\frac{\sin\alpha}{\alpha}e^{-i\alpha}\ .
\end{aligned}
$$

Hence the operator $(\mathcal{K}S_j\mathcal{K}^*)^{-1}$ in Fourier space is

$$
\frac{1}{2\rho k^4}\begin{pmatrix} 1 & , & c \\ \overline{c} & , & 1 \end{pmatrix}^{-1}=\frac{1}{2\rho k^4(1-c\overline{c})}\ \begin{pmatrix} 1 & , & -c \\ -\overline{c} & , & 1 \end{pmatrix}\ .
$$

In the second implementation we consider the high and low frequency parts

$$
R_{j,\pm}(f)=(\pm iK,I)R_j(f)
$$

separately, i.e. we apply ART to the systems of $2p$ equations

$$R_{j,\pm}(f) = g_{j,\pm} \quad , \qquad j = 1, \ldots, p \tag{4.4}$$

where

$$g_{j,\pm} = \pm iKg_+(\theta_j, \cdot) + \left(\frac{\partial g_+}{\partial \nu} - ikg_+ \right)(\theta_j, \cdot) \; .$$

The adjoint of $R'_{j,\pm}(f)$ is

$$R'_{j,\pm}(f)^* = R'_j(f)^* \begin{pmatrix} \mp iK \\ I \end{pmatrix} \; .$$

Hence the ART update for (4.4) is

$$f_j = f_{j-1} - \omega R'^*_j(f_{j-1}) \begin{pmatrix} \mp iK \\ I \end{pmatrix} S^{-1}_{j,\pm}(R_{j,\pm}(f_{j-1}) - g_{j,\pm}) \; .$$

A good choice for $S_{j,\pm}$ is

$$\begin{aligned} S_{j,\pm} &= R'_{j,\pm}(0) R'^*_{j,\pm}(0) \\ &= (\pm iK, I) S_j \begin{pmatrix} \mp iK \\ I \end{pmatrix} \end{aligned}$$

with S_j from (4.1). In Fourier space this operator is

$$\begin{aligned} (S_{j,\pm} g)^\wedge(\sigma) &= (\mp i\kappa, 1) k^4 \begin{pmatrix} c_1/\kappa^2 & , & c_2 \\ c_2 & , & c_3 \end{pmatrix} \begin{pmatrix} \mp i\kappa \\ 1 \end{pmatrix} \hat{g}(\sigma) \\ &= 2\rho k^4 \hat{g}(\sigma) \end{aligned}$$

as is easily seen from (4.2). Thus the update simply reads

$$f_j = f_{j-1} - \frac{\omega}{2\rho k^4} R'_j(f_{j-1})^* \begin{pmatrix} \mp iK \\ I \end{pmatrix} (R_{j,\pm}(f_{j-1}) - g_{j,\pm}) \; .$$

Note that this iteration is more time consuming than the first one since $R'_j(f_{j-1})^*$ has to be evaluated twice as often for one complete sweep.

5 Numerical example

We consider the function

$$f(x) = 0.2\,(1 + \cos(11x_1) + \sin(11x_2)) \ , \quad |x_i| \le \pi$$

which has been reconstructed in [9] succesfully from the scattering operator of that paper at frequencies $k = 1, 2, \ldots, 11$. According to the theoretical resolution (see §1) f should be computable from the scattered field for some k with $2k \le 11$. In fact the reconstruction in [9] for $k \le 6$ is almost perfect. We note that the condition for the Born approximation to be valid is $|Rf| \ll 2\pi/k$. This means that approximately $0.2 \cdot 2\pi \ll 2\pi/k$ or $k \ll 5$. Thus the Born approximation can only be used for $k = 1, 2, 3$, say, but not for $k \ge 4$.

We did the reconstruction using the first version of the extended PBP algorithm with $k = 6.1$. This is only slightly larger than the smallest value 5.5 for which f can be recovered. The result is displayed in Fig. 1. The reconstruction after 6 sweeps with $\omega = 0.2$ is almost perfect (with the exception of a Gibbs phenomenon close to the boundary of the object, and some disturbances in the vanishing imaginary part). As initial approximation we used the function f without the cosine and sine terms. The computing time on a SUN ultra 1 was 1 minute per sweep. We used 128 directions and we worked on a 129×129 grid corresponding to a square of sidelength 10.

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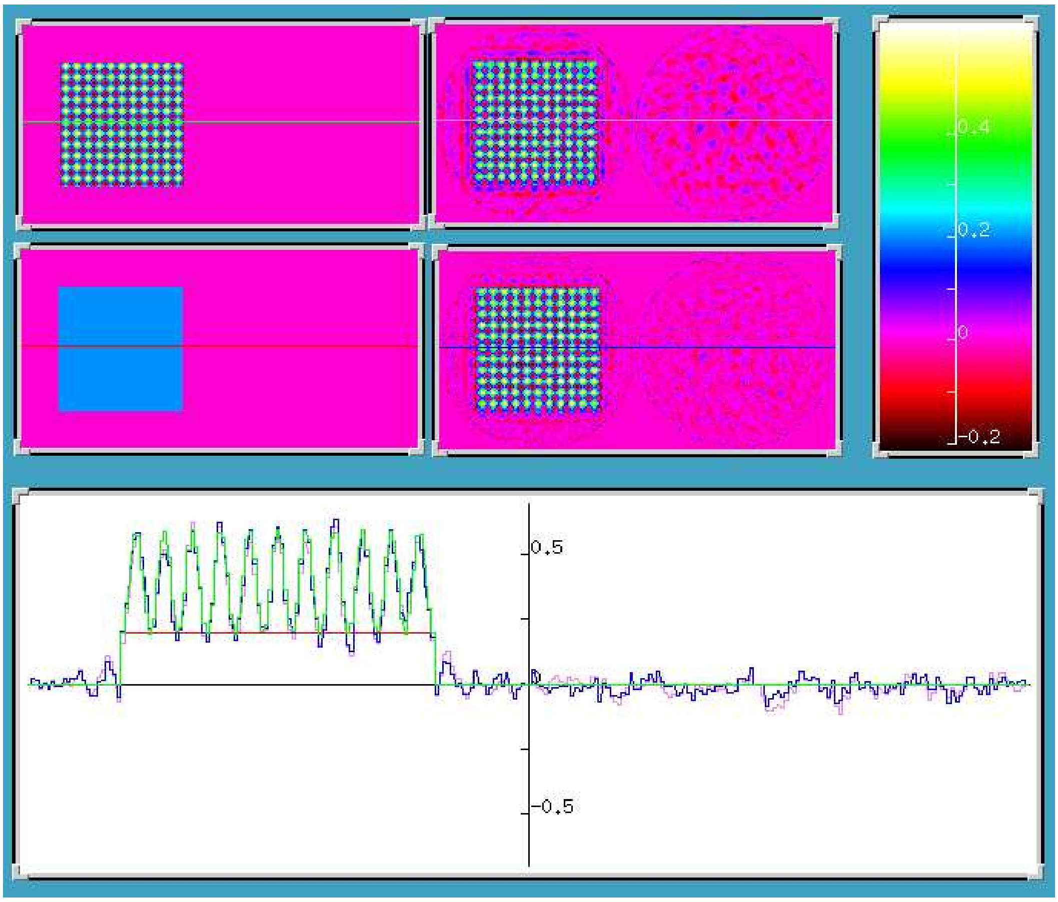

Figure Captions

Top left	:	Original on a 129×129 grid.
Middle left	:	Initial approximation.
Top right	:	Reconstruction after 6 sweeps.
Middle right	:	Reconstruction after 18 sweeps.
Bottom	:	Horizontal cross sections.
		Real part left, imaginary part right.

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