# Sampling of functions with symmetries

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Let $f$ be a function on $\mathbb{R}^n$ which is sampled on the grid $L_1 = W_1\, \mathbb{Z}^n + w_1$, where $W_1$ is a nonsingular real $(n,n)$-matrix and $w_1$ an $n$-vector, the set-off. Assume that the support of $\hat{f}$ is contained in a set $K \subseteq \mathbb{R}^n$. Then, $f$ can be recovered exactly from the sampled values provided that

$$(K+\xi) \cap (K+\xi') = \phi \ , \qquad \xi, \xi' \in L_1^\perp \ , \ \xi \neq \xi' \tag{1}$$

where $L_1^\perp$ is the reciprocal lattice $2\pi W_1^{-T}\mathbb{Z}^n$ [3].

In this note we address the following question. Assume $f$ has the symmetry

$$f(Ax+a) = f(x)$$

for some real invertible $(n,n)$-matrix $A$ and $n$-vector $a$. Then, $f$ is known automatically on $L_2 = AL_1 + a = W_2\mathbb{Z}^n + w_2$ with $W_2 = AW_1$, $w_2 = Aw_1 + a$. Can we use this additional information on $f$ to reconstruct $f$ under weaker assumptions on $L_1$ than (1)? Below we give 3 examples. In the first two (the first one is trivial) the answer is yes. In the third one we don't know.

**Examples**

**1)** Let $n = 1$ and $K = [-\Omega, \Omega]$, i.e. $f$ is an $\Omega$- band-limited function. Let $A = -1$ and $a = 0$, i.e. $f$ is even. By Shannon's sampling theorem $f$ can be recovered from any of the grids $L = h\mathbb{Z} + w$ provided $h \leq \frac{\pi}{\Omega}$. Let $L_1 = 2h\mathbb{Z} + h/2$. Then, $L_2 = 2h\mathbb{Z} - h/2$, and

$$L_1 \cup L_2 = h\mathbb{Z} + h/2 \ ,$$

see Fig. 1. Thus it suffices to sample $f$ on $L_1$ which is twice as coarse as required by Shannon's theorem in the absence of symmetries.

If $f$, $g$ satisfy the conditions on $f$, then

$$\int fg\,dx = h \sum_{x\in L} f(x)g(x) = 2h \sum_{x\in L_1} f(x)g(x)\ .$$

Thus we can evaluate inner products by the trapeziodal rule on the coarse grid $L_1$.

**2)** Let $g = Rf$ be the Radon transform of a function $f$ in $\mathbb{R}^2$ which is essentially $\Omega$-band-limited. As a function of $\varphi$ and $s$, $g$ has the symmetries

$$g(\varphi+\pi, -s) = g(\varphi, s)\ ,$$

i.e.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad a = \begin{pmatrix} \pi \\ 0 \end{pmatrix}\ .$$

We sample $g$ on the grid

$$L_1 = \begin{array}{rcll} \varphi & = & \Delta\varphi \cdot j & , \quad j = 0, \dots, p-1\ , \\ s & = & \Delta s \cdot (\ell + 1/4) & , \quad \ell \in \mathbb{Z}\ . \end{array}$$

Here, $\Delta\varphi = 2\pi/p$ with $p$ odd. We call $L_1$ a grid with quarter set-off. We have

$$\begin{aligned} W_1 &= \begin{pmatrix} \Delta\varphi & 0 \\ 0 & \Delta s \end{pmatrix} \quad , \quad w_1 = \begin{pmatrix} 0 \\ \Delta s/4 \end{pmatrix} \quad , \\ W_2 &= \begin{pmatrix} \Delta\varphi & 0 \\ 0 & -\Delta s \end{pmatrix} \quad , \quad w_2 = \begin{pmatrix} \pi \\ -\Delta s/4 \end{pmatrix} \quad , \end{aligned}$$

i.e., since $p$ is odd and $g$ has period $2\pi$ in $\varphi$,

$$L_2 : \quad \begin{array}{rcll} \varphi & = & \Delta\varphi(j+1/2) & , \quad j = 0, \dots, p-1, \\ s & = & \Delta s(\ell - 1/4) & , \quad \ell \in \mathbb{Z}\ . \end{array}$$

Combining $L_1$ and $L_2$ yields the grid $L$ from Fig. 2, namely

$$L : \quad \begin{array}{rcll} \varphi & = & \frac{\Delta\varphi}{2} j & , \quad j = 0, \dots, 2p-1\ , \\ s & = & \Delta s(\ell + 1/4) & , \quad \ell \in \mathbb{Z}\ , \quad \ell + j \text{ even}\ . \end{array}$$

$L$ is the interlaced grid suggested by Cormack [1] and Rattey-Lindgren [6]. The sampling requirements for the grids $L_1$, $L_2$, $L$ are as follows. Assume that $f(x) = 0$ for $|x| > \rho$ and that $f$ is essentially $\Omega$-band-limited. $f$ can be recovered reliably from the values of $g$ on $L_1$ provided that

$$\Delta\varphi \leq \frac{\pi}{\Omega\rho} \quad , \quad \Delta s \leq \frac{\pi}{\Omega} \, .$$

This is a classical result in tomography, see e.g. [4].

Thus the required number of data is

$$\frac{2\pi}{\Delta\varphi} \cdot \frac{2\rho}{\Delta s} = \frac{4}{\pi}\rho^2\Omega^2 \, .$$

For the interlaced grid $L$ the sampling requirements are

$$\frac{\Delta\varphi}{2} \leq \frac{\pi}{\Omega\rho} \quad , \quad \frac{\Delta s}{2} \leq \frac{\pi}{\Omega} \, ,$$

leading to $\frac{1}{\pi}\rho^2\Omega^2$ data points [2]. This is only 1/4 of the gridpoints of $L_1$ or of any translate thereof. Note that we can do with $\frac{2}{\pi}\rho^2\Omega^2$ data points simply by choosing $p$ even and the grid $L_1$ without the set-off. Then, by the symmetry of $g$, the scanning can be restricted to $j < p/2$ leaving us with $\frac{2}{\pi}\rho^2\Omega^2$ data points.

It is clear that inner products can be evaluated on $L_1$ exactly as in example 1, with $h$ replaced by $\Delta\varphi\Delta s$.

**3)** Now let $g$ be the fan beam transform of $f$, i.e.

$$g(\beta, \alpha) = (Rf)(\beta + \alpha, r \sin \alpha)$$

where $r \geq \rho$ and $f(x) = 0$ for $|x| \geq \rho$. $g$ is invariant under the affine transformation with

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \quad , \quad a = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

which, incidentally, is involutory.

We sample $g$ on the grid

$$L_1 : \quad \begin{array}{rcll} \beta & = & \frac{2\pi}{p} j & , \quad j = 0, \ldots, p-1 \\ \alpha & = & \frac{2\pi}{p} a(\ell + c) & , \quad \ell \in \mathbb{Z} \, . \end{array}$$

Here, $c$ is the detector set-off. The parameter $a$ can be chosen freely, but sampling theory [5] suggest the value $\rho/(r+\rho)$, which, for commercial scanners, is rather small, typically 0.20. In our notation,

$$\begin{array}{rclcrcl} W_1 & = & \frac{2\pi}{p}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} & , & w_1 & = & \frac{2\pi}{p}\begin{pmatrix} 0 \\ ac \end{pmatrix} \quad , \\ W_2 & = & \frac{2\pi}{p}\begin{pmatrix} 1 & 2a \\ 0 & -a \end{pmatrix} & , & w_2 & = & \frac{2\pi}{p}\begin{pmatrix} 2ac+p/2 \\ -ac \end{pmatrix} \quad . \end{array}$$

Thus we obtain for $L_2$

$$L_2 : \quad \begin{array}{rclcl} \beta & = & \frac{2\pi}{p}(j+2a\ell+2ac+p/2) & , & j=0,\dots,p-1 \\ \alpha & = & -\frac{2\pi}{p}a(\ell+c) & , & \ell\in\mathbb{Z}\ . \end{array}$$

The combined grid $L = L_1 \cup L_2$ may now be quite complicated, and it is not clear how to deal with sampling on $L$. At least we want to avoid redundancy, i.e. we want to avoid that $L_1$, $L_2$ represent the same lines. It is easily seen that redundancy occurs exactly in the following cases:

$$\begin{array}{llllclcl} \text{(i)} & p & \text{even} & , & 2a & \text{integer} & , & c=0 \quad , \\ \text{(ii)} & p & \text{even} & , & a & \text{integer} & , & c=1/2 \quad , \\ \text{(iii)} & p & \text{odd} & , & a-\frac{1}{2} & \text{integer} & , & c=1/2 \quad . \end{array}$$

Since the conditions on $a$ are not satisfied for commercial scanners, redundancy will not occur in practice.

In commercial scanners, a detector set-off is available, at least as an option. So the existence of a parameter $c$ such that the sampling properties of $L$ are considerably better than those of $L_1$ would be of immediate practical interest.

The conditions (i), (ii), (iii) characterize complete redundancy, i.e. each of the lines is represented exactly twice. A common way to prevent even partial redundancy is the quarter set-off, i.e. $c = 1/4$. In that case, $L$ has the form

$$L : \quad \begin{array}{rclcl} \beta & = & \frac{2\pi}{p}(j+\gamma_\ell) & , & j=0,\dots,p-1 \\ \alpha & = & \frac{\pi a}{p}a(\ell+1/2) & , & \end{array}$$

where $\gamma_\ell$ is the non-integer part of $2a\ell + \frac{p+a}{2}$. Except for this shift, $L$ is a fan-beam geometry with $p$ sources and the detector spacing $\pi a/p$ (as opposed to $2\pi a/p$ for $L_1$). Thus one is tempted to take as sampling conditions for $L$ those of [5] for such a geometry, i.e.

$$\frac{2\pi}{p} \leq \frac{r+\rho}{\rho}\frac{\pi}{r\Omega} \quad , \quad \frac{\pi a}{p} \leq \frac{\pi}{r\Omega} \, ,$$

allowing for a detector spacing twice as large as for $L_1$. However, this conclusion is questionable since the analysis of [5] is based on the rather special shape of the essential support of $\hat{g}$.

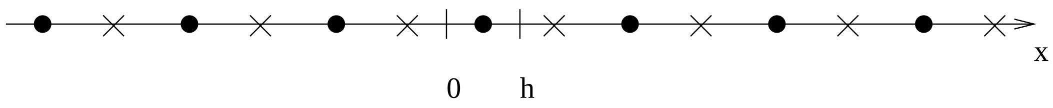


**Fig. 1:** The grids $L_1(\cdot)$, $L_2(*)$ for example 1. They combine to yield the grid $L$ of stepsize $h$.

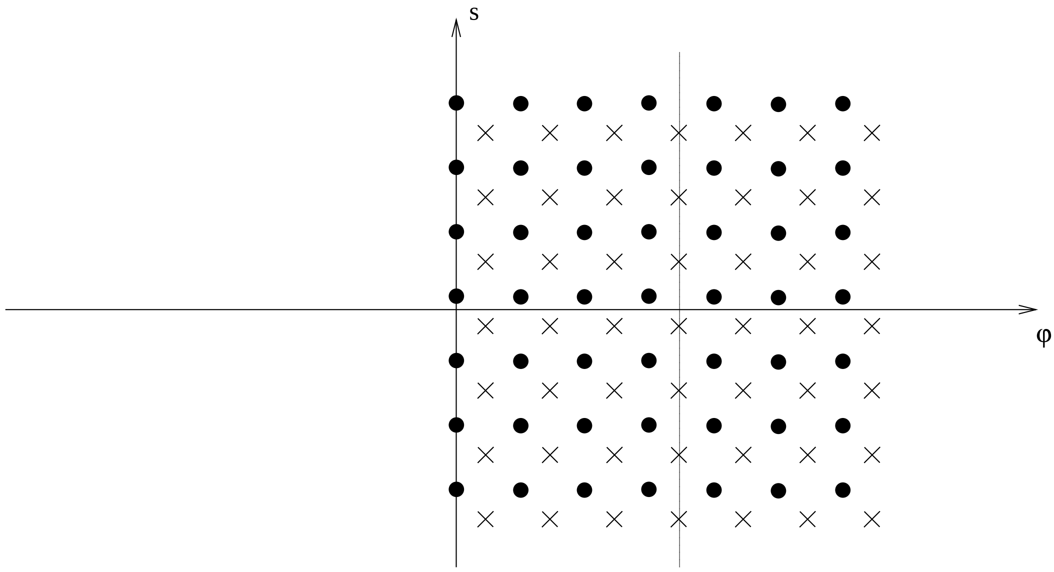


**Fig. 2:** The grids $L_1(\cdot)$, $L_2(*)$ for example 2 for $p = 7$. They combine to yield the interlaced grid $L$. The dotted line is at $\varphi = \pi$. Due to the oddness of $p$ it is lying exactly between neighbouring gridpoints of $L_1$.

# References

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