A constructive solution to the inverse scattering problem of the wave equation at one frequency and plane wave irradiation

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Summary: This is a tutorial paper reporting essentially on a method suggested by [2], [3] and [1]. The conclusion is that the method is useless for numerical purposes.

1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 3$, and let $\text{supp}(q) \subseteq \Omega$. Let $k \geq 0$ be a fixed wave number. For each $\Theta \in S^{n-1}$ let u_Θ be the solution to the reduced wave equation

$$(\Delta + k^2 + q)u = 0 \tag{1}$$

such that

$$u_\Theta = e^{ikx\cdot\Theta} + w_\Theta$$

where w_Θ satisfies Sommerfelds radiation condition. The problem is to recover q from the knowledge of u_Θ outside Ω.

2 The NSU method

An essential ingredient of the Nachmann - Sylvester - Uhlmann method is a certain - non-physical - solution ψ_ζ of (1) depending on the parameter $\zeta \in C^n$.

Proposition 1: Let $\zeta^2 (= \zeta \cdot \zeta) = k^2$, $Im\zeta \neq 0$ and $\delta > 1/2$. Then there is a unique solution ψ_ζ of (1) such that

$$\psi_\zeta = e^{ix\cdot\zeta}(1 + v_\zeta)$$

where $v_\zeta \in L_{2,-\delta}$ $(= L_2(\mathbf{R}^n, (1+x^2)^{-\delta}))$ and

$$\|v_\zeta\|_{L_{2,-\delta}} \leq \frac{c}{|\zeta|} \quad .$$

Proof: With $\psi_\zeta = e^{ix\zeta}(1 + v_\zeta)$, (1) assumes the form

$$\Delta v_\zeta + 2i\zeta \cdot \bigtriangledown v_\zeta = -q(1 + v_\zeta) \quad .$$

In [2] it is shown that for a sufficiently large and $\delta > \frac{1}{2}$ there is a constant $c(\delta, a)$ such that for $Im\zeta \neq 0$ and $|\zeta| \geq a$

$$\Delta v + 2i\zeta \cdot \bigtriangledown v = f$$

has a unique solution for each $f \in L_{2,\delta}$, and

$$\|v\|_{L_{2,-\delta}} \leq \frac{c(\delta, a)}{|\zeta|} \|f\|_{L_{2,\delta}} \quad . \tag{2}$$

It follows that v_ζ satisfies

$$\begin{aligned} \|v_\zeta\|_{L_{2,-\delta}} &\leq \tfrac{c(\delta,a)}{|\zeta|} \|q(1 + v_\zeta)\|_{L_{2,\delta}} \\ &\leq \tfrac{c(\delta,a)}{|\zeta|} \left\{ \|q\|_{L_{2,\delta}} + \|q(1+x^2)^\delta\|_{L_\infty(\Omega)} \|v_\zeta\|_{L_{2,-\delta}} \right\} \; . \end{aligned}$$

For $|\zeta| > c(\delta, a)\|q(1 + x^2)^\delta\|_{L_\infty(\Omega)}$ the estimate on v_ζ follows.

$\square$

The NSU method starts out from Green‘s formula

$$\begin{aligned} & \int\limits_\Omega \left\{ (\Delta + k^2 + q)\psi \cdot \varphi - \psi \cdot (\Delta + k^2 + q)\varphi \right\} dx \\ = & \int\limits_{\partial\Omega} \left(\frac{\partial\psi}{\partial\nu} \cdot \varphi - \psi \frac{\partial\varphi}{\partial\nu} \right) d\sigma(x) \; . \end{aligned}$$

With $\psi = \psi_\zeta$, $\varphi = e^{-ix\cdot(\zeta+\xi)}$, $\xi \in \mathbf{R}^n$, $(\xi+\zeta)^2 = k^2$, this reads

$$-\int_\Omega \psi_\zeta q e^{-ix(\zeta+\xi)} dx = t(\xi,\zeta)$$

where

$$t(\xi,\zeta) = \int_{\partial\Omega} \left(\frac{\partial \psi_\zeta}{\partial \nu} + i(\xi+\zeta)\cdot \nu \psi_\zeta \right) e^{-ix\cdot(\zeta+\xi)} d\sigma(x) \quad .$$

Making use of Proposition 1 we get

$$-\int_\Omega (1+v_\zeta) q e^{-ix\cdot\xi} dx = t(\xi,\zeta) \ .$$

As an immediate consequence we obtain

Theorem 2.1 *The Fourier-transform $\hat{q}$ of q can be obtained from t by*

$$\hat{q}(\xi) = -(2\pi)^{-n/2} \lim_{\substack{\zeta^2 = k^2 \\ \xi^2 + 2\zeta\cdot\xi = 0 \\ |\zeta|\to\infty}} t(\xi,\zeta) \quad .$$

We are left with the problem of determining t.

3 The Dirichlet - to - Neumann map

This is an operator, denoted by Λ_q, which acts on suitable function spaces on $\partial\Omega$. For a function f on $\partial\Omega$, $\Lambda_q f$ is defined as follows: Solve (1) in Ω with Dirichlet boundary values f on $\partial\Omega$ and put $\Lambda_q f = \frac{\partial u}{\partial \nu}$ on $\partial\Omega$.

Knowing Λ_q we can reconstruct q. In fact, since $q = 0$ in $\Omega' = \mathbf{R}^n \setminus \Omega$, ψ_ζ satisfies

$$\begin{aligned} (\Delta + k^2)\psi_\zeta &= 0 \qquad \text{in} \quad \Omega' \\ \frac{\partial \psi_\zeta}{\partial \nu} &= \Lambda_q \psi_\zeta \quad \text{on} \quad \partial\Omega' \\ \psi_\zeta &= e^{ix\cdot\zeta}(1+v_\zeta), \quad v_\zeta \in L_{2,-\delta} \quad . \end{aligned} \tag{3}$$

This is an exterior Helmholtz problem with a non-standard condition at ∞ and with an operator boundary condition. It can be solved, preferably by boundary equation methods, once Λ_q is given. Then, t can be found, whence q by Theorem 2.1.

In order to find Λ_q from the plane-wave scattering data we represent a function f on $\partial\Omega$ as

$$f(x) = \int_{S^{n-1}} u_\Theta(x) w(\Theta) d\Theta \ , \quad x \in \partial\Omega \quad . \tag{4}$$

Then,

$$(\Lambda_q f)(x) = \int_{S^{n-1}} \frac{\partial}{\partial \nu} u_\Theta(x) w(\Theta) d\Theta \quad .$$

Since $u_\Theta(x)$ is known for $x \notin \Omega$ and $\Theta \in S^{n-1}$ this determines Λ_q.

4 Numerical considerations

The method requires 3 steps:

i) Determine Λ_q from $u_\Theta(x),\ x \notin \Omega,\ \Theta \in S^{n-1}$.

ii) For each $\xi \in \mathbf{R}^n$ find a sequence (ζ) of points $\zeta \in \mathbf{C}^n$ with $\zeta^2 = k^2$, $\xi^2 + 2\zeta \cdot \xi = 0$, $|\zeta| \to \infty$ and determine ψ_ζ as the solution of the exterior Helmholtz problem (3).

iii) Compute $t(\xi, \zeta)$ from ψ_ζ and $\hat{q}$ from t. Do an inverse Fourier transform to obtain q.

Step (i) calls for the solution of the integral equation (4). In the - trivial - case $q = 0$ we have $u_\Theta(x) = e^{ik\Theta \cdot x}$, hence (4) reads

$$f(x) = \int_{S^{n-1}} e^{ik\Theta \cdot x} w(\Theta) d\Theta, \quad x \in \partial\Omega \quad .$$

For Ω the unit ball, $\partial\Omega = S^{n-1}$, this equation can be solved by spherical harmonics: Let

$$f = \sum_{\ell,m} f_{\ell m} Y_{\ell m} \quad , \qquad w = \sum_{\ell,m} w_{\ell m} Y_{\ell m} \quad .$$

Since

$$\begin{aligned}\int_{S^{n-1}} e^{ik\Theta\cdot x} Y_{\ell m}(\Theta) d\Theta &= (2\pi)^{n/2} i^\ell k^{(2-n)/2} J_{\ell+(n-2)/2}(k) Y_{\ell m}(\tfrac{x}{|x|}) \\ &= c_\ell(k) Y_{\ell m}(\tfrac{x}{|x|})\end{aligned}$$

the solution is

$$w_{\ell m} = \frac{f_{\ell m}}{c_\ell(k)} \quad .$$

Unfortunately, $c_\ell(k) \ll 1$ if $\ell > k$. This means that $w_{\ell m}$ is poorly determined for $\ell > k$. Thus, Λ_q can be determined stably only on the subspace $\langle Y_{\ell m} : \ell \leq k \rangle$ of $L_2(\partial\Omega)$. This does not come as a surprise since we expect the resolution in an inverse scattering experiment with wave number k to be of the order k.

On the other hand, in step (ii), we have to apply Λ_q to the high frequency function ψ_ζ ($|\zeta|$ is large!). Thus the information on Λ_q we get from step (i) is not sufficient for step (ii). All we can expect is an $O(\frac{1}{k})$ - approximation to q. But this can be achieved much easier by other methods. Besides, the computational cost of step (ii) is probably prohibitive, unless one finds expedient asymptotic methods for solving (3) for $|\zeta|$ large.

References

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[2] Nachmann, A.I. (1988). Reconstructions from boundary measurements. *Annals of Mathematics* **128**, 531-576.

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