A discrete Gelfand - Levitan theory

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1 Introduction

In the pioneering paper [5], Gelfand and Levitan introduced an elegant and explicit method for computing the potential of a Sturm-Liouville Operator from its eigenvalues and certain values of its eigenfunctions. It has been noted in Burridge [3] that the Gelfand-Levitan method is closely related to inversion methods for recovering the potential in a hyperbolic equation with focused initial state, thus giving a unified theory for the methods of Gelfand-Levitan, of Gopinath and Sondhi [8] and of Parijskij [11] and Blagoveshchenskij [1]; (see also Romanov [12, p. 39]), the latter ones going unnoticed in [3].

In the present note we analyse discrete analogues of these problems. The spacial differential operator of the Gelfand-Levitan theory is replaced by a symmetric tridiagonal matrix. It will turn out that in this setting the Gelfand-Levitan method reduces essentially to a Cholesky decomposition. The hyperbolic equation in the other theories is replaced by a recursion relation involving an arbitrary tridiagonal matrix, the essential step for the inversion being an LU-decomposition. In our approach, (approximate) transmutation operators as originally used in [5] (see Levitan [10] for a systematic study) play a paramount role.

We are aware of the fact that much work has been done on discrete inverse problems. We mention in particular Burridge [3, p. 514-537], Case and Kack [4], Bruckstein and Kailath [2], Landau [9], Gladwell and Willms [7]. Our approach differs from others in that it gives a unified treatment in terms of transmutation operators.

2 Transmutation matrices

Let T, T_0 be tridiagonal (n,n)-matrices, i.e.

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \gamma_1 & \alpha_2 & \beta_2 & & \\ & & \ddots & & \\ & & & & \beta_{n-1} \\ & & & \gamma_{n-1} & \alpha_n \end{pmatrix} , \quad T_0 = \begin{pmatrix} \alpha_1^0 & \beta_1^0 & & & \\ \gamma_1^0 & \alpha_2^0 & \beta_2^0 & & \\ & & \ddots & & \\ & & & & \beta_{n-1}^0 \\ & & & \gamma_{n-1}^0 & \alpha_n^0 \end{pmatrix} .$$

By a transmutation matrix for T, T_0 one usually means a (n,n)-matrix L with $T_0 L = LT$. We shall use (approximate) transmutation matrices for which this holds only in rows 1 through $n-1$. More precisely, with E the projection

$$E = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}$$

we define a transmutation L by

$$E\, T_0\, L \;=\; E\, L\, T \, . \tag{2.1}$$

Theorem 2.1 *Let $\beta_i^0 \neq 0$, $i = 1, \ldots, n-1$. Then there is a unique lower triangular transmutation $L = (\ell_{ij})$ whose $(1,1)$-entry is 1.*

Proof: With $\ell_1, \ldots, \ell_n$ the rows of L, (2.1) reads

$$\begin{aligned} \alpha_1^0 \ell_1 + \beta_1^0 \ell_2 \quad &= \quad \ell_1 T \, , \\ \gamma_{i-1}^0 \ell_{i-1} + \alpha_i^0 \ell_i + \beta_i^0 \ell_{i+1} \quad &= \quad \ell_i T \, , \quad i = 2, \ldots, n-1 \, . \end{aligned}$$

Since $\beta_i^0 \neq 0$ this recursion determines the ℓ_i uniquely once ℓ_1 is given. For $\ell_1 = (1, 0, \ldots, 0)$, ℓ_i is nonzero only in its first i components since T is tridiagonal.

$\square$

We remark that Theorem 2.1 holds also for lower Hessenberg matrices T_0, T.

3 The inverse eigenvalue problem

Now assume T to be symmetric with non-zero off-diagonal elements, i.e. $\gamma_i = \beta_i > 0, \quad i = 1, \ldots, n-1$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues and $x(\lambda_1), \ldots, x(\lambda_n)$ the normalized eigenvectors of T, i.e.

$$T\, x(\lambda_k) = \lambda_k x(\lambda_k)\ , \qquad k = 1, \ldots, n\ .$$

We consider the inverse problem: Determine T from $\lambda_1, \ldots, \lambda_n$ and $x_1(\lambda_1), \ldots, x_1(\lambda_n)$. It is well known that this problem can easily be solved by the Lanczos method.

The Lanczos algorithm computes for a symmetric matrix A a symmetric tridiagonal matrix T and a unitary matrix U such that

$$TU = UA \tag{3.1}$$

in the following way. With $u_1, \ldots, u_n$ the rows of U, (3.1) reads

$$\begin{array}{lllll} \alpha_1 u_1 + \beta_1 u_2 & & & = & u_1 A \\ \beta_1 u_1 + \alpha_2 u_2 & +\beta_2 u_3 & & = & u_2 A \\ & \ddots & & & \\ & & \beta_{n-1} u_{n-1} + \alpha_n u_n & = & u_n A \end{array}$$

Now let u_1 be arbitrary. Then, from equation 1,

$$\begin{aligned} \alpha_1 &= u_1 A u_1^*\ , \\ \tilde{u}_2 &= u_1 A - \alpha_1 u_1\ , \quad \beta_1 = \|\tilde{u}_2\| \quad \text{(euclidean norm)} \\ u_2 &= \tilde{u}_2 / \beta_1\ . \end{aligned}$$

Thus α_1, β_1, u_1, u_2 are determined. Likewise, from equation 2,

$$\begin{aligned} \alpha_2 &= u_2 A u_2^* \\ \tilde{u}_3 &= u_2 A - \beta_1 u_1 - \alpha_2 u_2\ , \quad \beta_2 = \|\tilde{u}_3\| \\ u_3 &= \tilde{u}_3 / \beta_2\ , \end{aligned}$$

yielding β_2, u_3. Proceeding in this fashion in equations up to and including $n-1$ we obtain $\alpha_1, \ldots, \alpha_{n-1}$, $\beta_1, \ldots, \beta_{n-1}$ and $u_1 \ldots u_n$. α_n is obtained from equation n.

In order to solve our inverse problem we simply apply the Lanczos method to the matrix $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. The same problem can be solved by the Gelfand-Levitan method in the following way.

Let T_0 be an arbitrary known symmetric tridiagonal matrix, subject only to the condition $\beta_i^0 \neq 0$, $i = 1, \ldots, n-1$. We introduce solutions $\varphi(\lambda)$, $\varphi^0(\lambda) \in \mathbf{R}^n$ of

$$\begin{aligned}
\varphi_1(\lambda) &= 1 \,, \\
\alpha_1\varphi_1(\lambda) + \beta_1\varphi_2(\lambda) &= \lambda\varphi_1(\lambda) \,, \\
\beta_{i-1}\varphi_{i-1}(\lambda) + \alpha_i\varphi_i(\lambda) + \beta_i\varphi_{i+1}(\lambda) &= \lambda\varphi_i(\lambda) \,, \quad i = 2, \ldots, n-1
\end{aligned}$$

and correspondingly for φ^0 using T_0 instead of T. In other words, $\varphi(\lambda)$, $\varphi^0(\lambda)$ satisfy

$$ET\varphi(\lambda) = \lambda E\varphi(\lambda) \,, \quad \varphi_1(\lambda) = 1 \tag{3.2}$$

$$ET_0\varphi^0(\lambda) = \lambda E\varphi^0(\lambda) \,, \; \varphi_1^0(\lambda) = 1 \,. \tag{3.3}$$

Note that for $\lambda = \lambda_k$, $\varphi(\lambda_k)$ is an eigenvector of T, hence

$$T\varphi(\lambda_k) = \lambda_k\varphi(\lambda_k) \,, \quad k = 1, \ldots, n \,, \tag{3.4}$$

i.e. the projection operator E can be dropped in that case.

Theorem 3.1 *Let $\beta_i^0 \neq 0$, $i = 1, \ldots, n-1$ and let L be the transmutation of Theorem 2.1 Then,*

$$\varphi^0(\lambda_k) = L\varphi(\lambda_k) \,, \quad k = 1, \ldots, n \,.$$

Proof: Let $\psi^0 = L\varphi(\lambda_k)$. Then, because of (2.1), (3.4),

$$ET_0\psi^0 = ET_0L\varphi(\lambda_k) = ELT\varphi(\lambda_k) = \lambda_k EL\varphi(\lambda_k) = \lambda_k E\psi^0 \,.$$

Since $\ell_{11} = 1$, $\psi_1^0 = 1$. Thus ψ^0 and $\varphi^0(\lambda_k)$ both satisfy (3.3) with $\lambda = \lambda_k$. Because of $\beta_i^0 \neq 0$, (3.3) is uniquely solvable. Hence $\psi^0 = \varphi^0(\lambda_k)$.

$\square$

Now we come to the core of the Gelfand-Levitan method. We introduce the matrices

$$\Phi = (\varphi(\lambda_1), \ldots, \varphi(\lambda_n)) \ , \quad \Phi_0 = (\varphi^0(\lambda_1), \ldots, \varphi^0(\lambda_n)) \ ,$$

$$P = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_m \end{pmatrix} \ , \quad p_k = \|\varphi(\lambda_k)\| \quad \text{(euclidean norm)} \ .$$

Assume that $x_1(\lambda_k) > 0$, $k = 1, \ldots, n$. Then, $\varphi(\lambda_k) = \frac{1}{x_1(\lambda_k)}\, x(\lambda_k)$, hence $p_k = \frac{1}{x_1(\lambda_k)}$ and

$$\Phi = XP \quad , \qquad X = (x(\lambda_1), \ldots, x(\lambda_n)) \ .$$

Since $XX^t = I$,

$$\Phi P^{-2} \Phi^t = I \ . \tag{3.5}$$

Theorem 3.1 means that

$$\Phi_0 = L\Phi \ . \tag{3.6}$$

Inserting this into (3.5) yields

$$\begin{aligned} &L^{-1}\Phi_0 P^{-2} \Phi_0^t L^{-t} = I \ , \\ &LL^t = \Phi_0 P^{-2} \Phi_0^t \ . \end{aligned}$$

Since Φ_0, P are known from the data, L can be computed by a Cholesky decomposition of $\Phi_0 P^{-2} \Phi_0^t$. Once L is determined, T can be computed from

$$T = \Phi \Lambda \Phi^{-1} = L^{-1} \Phi_0 \Lambda \Phi_0^{-1} L \ , \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \ .$$

This solves the inverse eigenvalue problem.

4 The inverse evolution problem

Now let $T = T_0 + Q$ with T_0 known and Q a diagonal matrix whose entries we denote by $q_1, \dots, q_n$. Let u^ℓ be a solution to

$$u^{\ell+1} = Tu^\ell \ , \quad \ell = 0, 1, \dots \ , \qquad u^0 = e_1 \tag{4.1}$$

with e_1 the first unit vector. We consider the inverse problem of recovering Q from

$$u_1^\ell = g^\ell \ , \quad \ell = 0, \dots, 2n-1 \ .$$

Again we make use of the transmutation L from Theorem 2.1. Multiplying (4.1) with EL we obtain from (2.1)

$$ELu^{\ell+1} = ELTu^\ell = ET_0Lu^\ell \ .$$

Thus the vectors $z^\ell = Lu^\ell$ satisfy

$$Ez^{\ell+1} = ET_0 z^\ell \ , \qquad \ell = 0, 1, \dots \ , \tag{4.2}$$

i.e. $z^{\ell+1} = T_0 z^\ell$ in components $1, \dots, n-1$. Since T_0 is tridiagonal we can show that for $k = 0, \dots, n-1$

$$z_i^{\ell+k} = (T_0^k z^\ell)_i \ , \qquad i = 1, \dots, n-k \ . \tag{4.3}$$

We use induction with respect to k. The case $k = 0$ is obvious. Assume that (4.3) is correct for some k with $0 \le k < n-1$. From (4.2) we get

$$z_i^{\ell+k+1} = (T_0 z^{\ell+k})_i \ , \qquad i = 1, \dots, n-1 \ .$$

For the evaluation of $(T_0 z^{\ell+k})_i$, $i = 1, \dots, n-k-1$, we need only the first $n-k$ components of $z^{\ell+k}$ since T_0 is tridiagonal. Thus, by the induction hypothesis,

$$z_i^{\ell+k+1} = (T_0 T_0^k z^\ell)_i = (T_0^{k+1} z^\ell)_i \ , \quad i = 1, \dots, n-k-1 \ .$$

This is (4.3) for $k+1$. Hence (4.3) is established.

Since the first row of L is $(1, 0, \dots, 0)$ we also have

$$z_1^\ell = g^\ell \ , \qquad \ell = 0, 1, \dots \ .$$

This combines with (4.3) to yield, for $\ell = 0, 1, \ldots, n-1$,

$$g^{\ell+k} = (T_0^k z^\ell)_1 \ , \qquad k = 0, 1, \ldots, n-1 \ .$$

Introducing the row vector

$$(u_0^k)^t = e_1^t T_0^k$$

we have

$$g^{\ell+k} = (u_0^k)^t z^\ell \ , \qquad k = 0, \ldots, n-1 \tag{4.4}$$

and

$$u_0^{k+1} = T_0^t u_0^k \ , \quad k = 0, 1, \ldots, \quad u_0^0 = e_1 \ . \tag{4.5}$$

With the (n,n)-matrices

$$Z = (z^0, \ldots, z^{n-1}) \ , \ G = \begin{pmatrix} g^0 & g^1 & \ldots & g^{n-1} \\ \vdots & \vdots & & \\ g^{n-1} & g^n & \ldots & g^{2n-2} \end{pmatrix} \ , \ U_0 = (u_0^0, \ldots, u_0^{n-1})$$

(4.4) simply reads

$$G = U_0^t Z \ .$$

Note that U_0 is upper triangular with diagonal elements

$$\prod_{i=1}^{k} \beta_i^0 \ , \qquad k = 0, \ldots, n-1 \ .$$

Thus U_0 is invertible provided that $\beta_i^0 \neq 0, \ \ i = 1, \ldots, n-1$. Hence

$$Z = U_0^{-t} G$$

which determines Z by the data.

The relations $z^\ell = Lu^\ell$ can be written as

$$Z = LU \ , \qquad U = (u^0, \ldots, u^{n-1}) \ .$$

We finally obtain

$$LU = U_0^{-t} G \ .$$

Since T is tridiagonal and $u^0 = e_1$, U is upper triangular, its diagonal entries being

$$\prod_{i=1}^{k} \gamma_i = \prod_{i=1}^{k} \gamma_i^0 \ , \qquad k = 0, \ldots, n-1 \ .$$

Thus L, U can be determined simply by doing an LU-decomposition on the matrix $U_0^{-t}G$, with the diagonal of U being known.

Once L, U are known there is a variety of ways to find Q. For instance we can compute u^n from $Lu^n = z^n$ and

$$\begin{pmatrix} g^n \\ \vdots \\ g^{2n-1} \end{pmatrix} = U_0^t \, z^n \, ,$$

which is (4.4) for $\ell = n$. Then,

$$u^{\ell+1} = T_0 u^\ell + \begin{pmatrix} q_1 u_1^\ell \\ \vdots \\ q_{\ell+1} u_{\ell+1}^\ell \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \qquad \ell = 0, \ldots, n-1$$

from which $q_1, \ldots, q_n$ can be computed recursively since $u_{\ell+1}^\ell \neq 0$, $\ell = 0, \ldots, n-1$. This solves the inverse evolution problem.

5 A second order inverse evolution problem

Let T be as in the preceeding section, and let u^ℓ be a solution to

$$\begin{aligned} &u^{\ell+1} - 2u^\ell + u^{\ell-1} = Tu^\ell \ , \quad \ell = 0,1,2,\dots \\ &u^{-1} = 0 \ , \quad u^0 = e_1 \ . \end{aligned} \tag{5.1}$$

We consider the inverse problem of recovering the diagonal matrix Q in $T = T_0 + Q$ from

$$u_1^\ell = g^\ell \ , \qquad \ell = 0,1,\dots,2n \ .$$

Again we put $z^\ell = Lu^\ell$ with L from Theorem 2.1. We obtain

$$\begin{aligned} Ez^{\ell+1} - 2Ez^\ell + Ez^{\ell-1} = ET_0 z^\ell \ , \quad \ell = 0,1,2,\dots \\ z^{-1} = 0 \ , \qquad z^0 = Le_1 \ . \end{aligned}$$

With $S = 2I + T_0$ the recursion can be written as

$$z_i^{\ell+1} = (Sz^\ell)_i - z_i^{\ell-1} \ , \qquad i = 1,\dots,n-1 \ . \tag{5.2}$$

We claim that for $k = 0,\dots,n-1$ and $\ell = 0,1,2,\dots$

$$z_i^{\ell+k} = (S_k z^\ell)_i - (S_{k-1} z^{\ell-1})_i \ , \quad i = 1,\dots,n-k \tag{5.3}$$

where

$$S_{-1} = 0 \ , \quad S_0 = I \ , \quad S_{k+1} = SS_k - S_{k-1} \ , \quad k = 0,\dots,n-1 \ .$$

We prove (5.3) by induction with respect to k. The case $k = 0$ is obvious. Assume (5.3) to be correct up to some k with $0 \le k < n-1$. From (5.2) we get

$$z_i^{\ell+k+1} = (Sz^{\ell+k})_i - z_i^{\ell+k-1} \ , \quad i = 1,\dots,n-k-1 \ .$$

Since S is tridiagonal, only components 1 through $n-k$ of $z^{\ell+k}$ enter $(Sz^{\ell+k})_i$ for $i < n-k$. Hence, by the induction hypothesis,

$$\begin{aligned} z_i^{\ell+k+1} &= (S(S_k z^\ell - S_{k-1} z^{\ell-1}))_i - (S_{k-1} z^\ell - S_{k-2} z^{\ell-1})_i \\ &= (S_{k+1} z^\ell)_i - (S_k z^{\ell-1})_i \ , \quad i = 1,\dots,n-k-1 \ . \end{aligned}$$

This is (5.3) with k replaced by $k+1$. Thus (5.3) is established.

We use (5.3) for $i = 1$ only, yielding for $\ell = 1, 2, \ldots$

$$g^{\ell+k} = (S_k z^\ell)_1 - (S_{k-1} z^{\ell-1})_1 \ , \quad k = 0, \ldots, n-1 \ .$$

Introducing the row vectors

$$(u_0^k)^t = e_1^t S_k \quad , \qquad k = -1, \ldots, n-1$$

we have

$$g^{\ell+k} = (u_0^k)^t z^\ell - (u_0^{k-1})^t z^{\ell-1} \ , \quad k = 0, \ldots, n-1$$

and, since S, S_k commute,

$$\begin{aligned} (u_0^{k+1})^t &= e_1^t S_{k+1} = e_1^t (S S_k - S_{k-1}) \\ &= e_1^t (S_k S - S_{k-1}) = (u_0^k)^t S - (u_0^{k-1})^t \ . \end{aligned}$$

Thus u_0^k is the analogue to u^k with T replaced by T_0^t:

$$\begin{aligned} & u_0^{k+1} - 2u_0^k + u_0^{k-1} = T_0^t u^k \qquad k = 0, 1, 2, \ldots \\ & u_0^{-1} = 0 \ , \quad u_0^0 = e_1 \ . \end{aligned}$$

This shows that the matrix $U_0 = (u_0^0, \ldots, u_0^{n-1})$ is upper triangular with diagonal elements

$$\prod_{i=0}^{k} \beta_i^0 \quad , \qquad k = 0, \ldots, n-1 \ .$$

It follows that the systems

$$\begin{pmatrix} g^\ell \\ \vdots \\ g^{\ell+n-1} \end{pmatrix} = U_0 z^\ell - U_{-1} z^{\ell-1} \ , \quad \ell = 1, \ldots, n$$

where $U_{-1} = (u_0^{-1}, \ldots, u_0^{n-2})$ are uniquely solvable for z^ℓ provided that $\beta_i^0 \neq 0, \quad i = 1, \ldots, n-1$.

Given that $z^{-1} = 0$, we can determine the matrix $Z = (z^0, \ldots, z^{n-1})$ from our data. The matrix $U = (u^0, \ldots, u^{n-1})$ is upper triangular with diagonal elements

$$\prod_{i=1}^{k-1} \gamma_i = \prod_{i=1}^{k-1} \gamma_i^0 \ , \qquad k = 1, \ldots, n \ .$$

Thus computing L, U amounts to doing an LU-Decomposition on the matrix Z,

$$Z = LU \ ,$$

with the diagonal of U being known. Once L, U are known, Q is computed very much in the same fashion as in the preceding section.

6 Links to other work

It has been noticed in Landau [9] that the inverse eigenvalue problem is intimately related to orthogonal polynomials. Our treatment in section 2 can also be interpreted that way.

To begin with it is clear that $\varphi_k(\lambda)$, $\varphi_k^0(\lambda)$ are polynomials of degree $k-1$ in λ. Thus (3.5) simply states that the polynomials φ_k, $k = 1, \dots, n$, are the orthonormal polynomials with respect to the scalar product

$$(p, q) = \sum_{k=1}^{n} p_k^{-2} p(\lambda_k) q(\lambda_k) \tag{6.1}$$

in the space of polynomials of degree $n-1$. Writing φ_k^0 in terms of the orthogonal polynomials we get

$$\varphi_\ell^0 = \sum_{k=1}^{\ell} \ell_{\ell k} \varphi_k$$

or

$$\Phi_0 = L\Phi$$

with a lower triangular matrix L. This is (3.6). Thus solving the inverse eigenvalue problem, i.e. finding L, boils down to computing the orthogonal polynomials with respect to the scalar product (6.1).

The approach of Gladwell and Willms [7] is essentially equivalent to our treatment of the inverse eigenvalue problems, except that we made the role of transmutation more explicit. For instance, equation (10) of [7] is precisely our relation (3.6) in the form $\Phi = L^{-1}\Phi_0$.

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