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Übungen zur Vorlesung

# Partielle Differentialgleichungen 2

WiSe 19/20, Blatt 8

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## Aufgabe 1

(Non-uniqueness of the representative of a polyconvex function) Consider the function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by

$$f(A) = |A|^2 + \det A.$$

Find two convex functions $g_1, g_2 : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that $g_1 \neq g_2$ and

$$f(A) = g_1(A, \det A) = g_2(A, \det A).$$

## Aufgabe 2

Consider the functional $F : \mathcal{A} \to [0, +\infty]$ defined by

$$F(u) = \int_0^1 f(u')\,\mathrm{d}x,$$

where $\mathcal{A} := \{u \in W^{1,2}(0,1) : u(0) = a, u(1) = b\}$, for $a, b \in \mathbb{R}$, $a \neq b$ and

$$f(\xi) = \begin{cases} \xi^2 & |\xi| \leq 1, \\ 1 & |\xi| > 1. \end{cases}$$

(a) Determine $\inf_{\mathcal{A}} F$ in dependence of $a$ and $b$ and show that the minimum is not attained.

(b) Show that minimizing sequences are not bounded in $W^{1,2}$ and therefore (even after extraction of a subsequence) do not converge weakly in $W^{1,2}$.

(c) Find a minimizing sequence $\{u_n\}_n \subset \mathcal{A}$ and $u \in L^2(0,1)$ such that $u_n \to u$. How does $u$ look like?

## Aufgabe 3

Let $F : \mathcal{A} \to \overline{\mathbb{R}}$ be defined by

$$F(u) = \int_U \frac{1}{2}|\nabla u|^2 - f u\,\mathrm{d}x,$$

where $f \in L^2(U)$ and

$$\mathcal{A} = \{u \in W_0^{1,2}(U) : |\nabla u| \leq 1 \text{ for almost all } x \in U\}.$$

(a) Show that there exists a unique minimizer $u_0 \in \mathcal{A}$ of $F$.

(b) Prove

$$\int_U \nabla u_0 \cdot \nabla(u - u_0)\mathrm{d}x \geq \int_U f(u - u_0)\mathrm{d}x$$

for all $u \in \mathcal{A}$.

**Aufgabe 4**
Let $U \subset \mathbb{R}^d$ be open, let $1 < p < \infty$, and let $p'$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let $u \in L^p(U)$ and let $\{u_j\}_j \subset L^p(U)$ be a sequence such that

$$u_j \rightharpoonup u \text{ in } L^p(U).$$

(a) Let $v \in L^{p'}(U)$ and let $\{v_j\}_j \subset L^{p'}(U)$ be a sequence such that

$$v_j \to v \text{ in } L^{p'}(U).$$

Prove that $u_j v_j \rightharpoonup uv$ in $L^1(U)$. Show that if only $v_j \rightharpoonup v$ in $L^{p'}(U)$, then the statement is in general not true.

(b) Let $v \in L^\infty(U)$ and let $\{v_j\}_j \subset L^\infty(U)$ be a sequence such that

$$v_j \to v \text{ in } L^\infty(U).$$

Prove that $u_j v_j \rightharpoonup uv$ in $L^p(U)$.