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Übungen zur Vorlesung

Partielle Differentialgleichungen 2

WiSe 19/20, Blatt 7

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Aufgabe 1

Consider the functional $F : \mathcal{A} \to [0, +\infty]$ defined by

$$F(u) = \int_a^b ((u')^2 - u^2)\,\mathrm{d}x,$$

where $\mathcal{A} := W_0^{1,2}(a, b)$.

(a) Prove the existence of a minimizer in the case $b - a \leq \pi$.

(b) Show that, in the case $b - a = \pi$, the minimum is not unique.

Hinweis: Use *Wirtinger's inequality* in $W_0^{1,2}(a, b)$ (Poincaré inequality in one dimension with optimal constant):

$$\|u\|_{L^2(a,b)} \leq \frac{b-a}{\pi} \|u'\|_{L^2(a,b)}.$$

Aufgabe 2

Let $\Omega \subset \mathbb{R}^2$ be open, bounded with Lipschitz boundary. We have seen that, if $p > 2$, then

$$u^j \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^2) \implies \det \nabla u^j \rightharpoonup \det \nabla u \in L^{p/2}(\Omega).$$

Show that the result is, in general, false if $p = 2$. To achieve this, construct a sequence $\{u_j\}_j \subset W^{1,2}(\Omega)$ and $u \in W^{1,2}(\Omega)$ such that

$$u_j \rightharpoonup u \text{ in } W^{1,2}(\Omega), \text{ but } \det \nabla u^j \not\rightharpoonup \det \nabla u \text{ in } L^1(\Omega).$$

Aufgabe 3

It can be non-trivial to determine whether a function is polyconvex or not. As an example, consider

$$W(F) = \begin{cases} 1 + |F|^2 & \rho(F) \geq 1, \\ 2\rho(F) - 2|\det(F)|, & \rho(F) < 1, \end{cases}$$

where

$$\rho(F) = \sqrt{|F|^2 + 2|\det(F)|}.$$

Hinweis. Show that $W(F) = g(F, \det(F))$, where

$$g(F, t) = \max_{\alpha = \pm 1} \left\{ f\left(\sqrt{|F|^2 + 2\alpha \det(F)} \right) - 2\alpha t \right\}$$

in which

$$f(t) = \begin{cases} 1 + t^2 & t \geq 1, \\ 2t & t < 1. \end{cases}$$

Then, check that g is a convex function of F and t.