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Übungen zur Vorlesung

Partielle Differentialgleichungen 2

WiSe 19/20, Blatt 5

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Aufgabe 1

Show that

$$u(x,t) = \begin{cases} 1 & \text{for } x \le t < 1 \text{ or } t \ge 1, x \le \frac{t+1}{2}, \\ \frac{1-x}{1-t} & \text{for } 0 \le t < 1, t \le x \le 1, \\ 0 & \text{for } 0 \le t < 1 \le x \text{ or } t \ge 1, x > \frac{t+1}{2} \end{cases}$$

is an integral solution to the Burgers equation $\partial_t u + u \partial_x u = 0$ in $\mathbb{R} \times [0, +\infty)$.

Aufgabe 2

Show that

$$u(x,t) = \begin{cases} 0 & \text{for } x \le \frac{t}{2}, \\ 1 & \text{for } x > \frac{t}{2} \end{cases} \qquad \text{and} \qquad \tilde{u}(x,t) = \begin{cases} 0 & \text{for } x \le 0, \\ \frac{x}{t} & \text{for } 0 < x \le t, \\ 1 & \text{for } x > t \end{cases}$$

are two different integral solutions to the Burgers equation $\partial_t u + u \partial_x u = 0$ in $\mathbb{R} \times [0, +\infty)$ with the same initial conditions.

Aufgabe 3

Show the following properties for entropy/entropy-flux pairs:

(a) For $F, \Phi \in C^2(\mathbb{R})$ convex and $\Psi : \mathbb{R} \to \mathbb{R}$ such that $\Phi' F' = \Psi'$, where $F(0) = \Phi(0) = \Psi(0) = 0$, there holds

$$u\Psi(u) \ge F(u)\Phi(u) \text{ for } u \ge 0.$$

(b) Show that in general equality in (a) does not hold.

(c) Show that the inequality in (a) for integral solutions

$$u(x,t) = \begin{cases} c & \text{for } x \le s(t), \\ 0 & \text{for } x > s(t), \end{cases} \qquad c > 0,\ s \in C^1([0, +\infty)),$$

for the equation

$$\begin{cases} \partial_t u + \partial_x F(u) = 0 & \text{in } \mathbb{R} \times (0, +\infty), \\ u = g & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

follows from

$$\partial_t \Phi(u) + \partial_x \Psi(u) \le 0.$$

Recall: This non-negativity condition is to be understood in an integral sense, i.e.,

$$\int_0^\infty \int_{\mathbb{R}} \left(\Phi(u)\partial_t v + \Psi(u)\partial_x v \right) \mathrm{d}x \, \mathrm{d}t \ge 0, \quad \forall\, v \in C_c^\infty(\mathbb{R} \times (0, +\infty)), v \ge 0.$$