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Übungen zur Vorlesung

Partielle Differentialgleichungen 2

WiSe 19/20, Blatt 4

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Aufgabe 1

Show that there exists at most one smooth solution of this initial boundary value problem for the telegraph equation

$$\begin{cases} \partial_{tt}u + d\partial_t u - \partial_{xx}u = f & \text{in } (0,1)\times(0,T), \\ u = 0 & \text{on } (\{0\}\times[0,T])\cup(\{1\}\times[0,T]), \\ u = g,\ \partial_t u = h & \text{on } (0,1)\times\{t=0\}. \end{cases}$$

Here, d is a constant.

Aufgabe 2

Can you here the shape of a drum?

Let $U \subset \mathbb{R}^n$ be a bounded domain, $T > 0$ and define on U a linear uniformly elliptic differential operator by

$$Lw(x) := -\mathrm{div}\left(A(x)^T \nabla w(x)\right) + c(x)w(x),$$

where $a_{ij}, c \in L^\infty(U)$, $A(x) = (a_{ij}(x)) \in \mathbb{R}^n$ symmetric, $c(x) \geq 0$ for all $x \in U$. Consider the homogeneous hyperbolic boundary value problem

$$\begin{cases} \partial_{tt}u + Lu = 0 & \text{in } U\times(0,T), \\ u = 0 & \text{on } \partial U\times[0,T], \\ u = u_0,\ \partial_t u = u_1 & \text{on } U\times\{t=0\}, \end{cases} \tag{1}$$

where $u_0 \in H_0^1(U)$, $u_1 \in L^2(U)$.

(a) Argue that a weak solution of (1) can be expanded in terms of eigenfunctions of L (similar to Sheet 3, Exercise 2) and write down the corresponding representation.

(b) Consider now the wave-equation with $L = -\Delta$. Determine a basis of $L^2(U)$ of $H_0^1(U)$-eigenfunctions of $-\Delta$ for bounded rectangles

$$U = (\alpha_1, \beta_1)\times\cdots\times(\alpha_n,\beta_n) \subset \mathbb{R}^n,$$

where $\alpha_i < \beta_i$, $\alpha_i, \beta_i \in \mathbb{R}$.
Hinweis. Start with $n = 1$ and argue that in higher dimensions eigenfunctions are of the form $w(x_1,\ldots,x_n) = w_1(x_1)\cdots w_n(x_n)$.

(c) For $R \in \mathbb{R}$ denote by $N_U(R)$ the cardinality of eigenvalues $\lambda \leq R$ of $-\Delta$ on U. Prove the Weyl formula for rectangles $U \subset \mathbb{R}^n$

$$|U| = 2\pi^n \lim_{R\to\infty} \frac{N_U(R)}{R^{n/2}}.$$

(d) Answer the question from the title of the exercise.

Aufgabe 3
Consider the conservation law in one space dimension:

$$\begin{cases} \partial_t u + \partial_x F(u) = 0 & \text{in } \mathbb{R} \times (0, +\infty), \\ u = g & \text{on } \mathbb{R} \times \{0\}, \end{cases} \tag{2}$$

where $F \in C^1(\mathbb{R})$ and $g \in C^1(\mathbb{R})$. Show that if u is a smooth integral solution of (2), i.e. u is an integral solution and $u \in C^1(\mathbb{R}\times[0,+\infty))$, then u is a classical solution, i.e. $u \in C^1(\mathbb{R}\times(0,+\infty))\cap C(\mathbb{R}\times[0,+\infty))$ solves (2) pointwise.