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Übungen zur Vorlesung

Partielle Differentialgleichungen 2

WiSe 19/20, Blatt 3

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Aufgabe 1

Prove that there is at most one smooth solution of this initial boundary-value problem for the heat equation with Neumann boundary conditions:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } U_T \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\}, \end{cases}$$

where $\frac{\partial u}{\partial \nu} = \nabla \cdot \nu$ stands for the directional derivative with respect the unit normal ν to the boundary of U.

Hinweis. Use Green's identities.

Aufgabe 2

Assume that u is a smooth solution of

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } U \times (0,\infty) \\ u = 0 & \text{on } \partial U \times [0,\infty) \\ u = g & \text{on } U \times \{t = 0\}. \end{cases}$$

Prove the exponential decay estimate:

$$\|u(\cdot,t)\|_{L^2(U)} \leq e^{-\lambda_1 t}\|g\|_{L^2(U)} \quad (t \geq 0),$$

where $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ (with zero boundary conditions) on U.

Hinweis. For the proof, you may use that there exists an orthonormal basis of $L^2(U)$ of eigenfunctions of the Laplacian. Namely, there exists $(w_k)_k \subset H_0^1(U)$ orthonormal basis of $L^2(U)$ with

$$\Delta w_k = \lambda_k w_k,$$

where $0 < \lambda_1 \leq \lambda_2 \leq \dots$ with $\lambda_k \to \infty$ for $k \to \infty$.

Aufgabe 3

Let us consider Galerkin's method for the Poisson's equation: suppose $f \in L^2(U)$ and assume that $u_m = \sum_{k=1}^m d_m^k w_k$ solves

$$\int_U Du_m \cdot Dw_k \, dx = \int_U f \cdot w_k \, dx$$

for $k = 1, \dots, m$. Show that a subsequence of $(u_m)_{m=1}^\infty$ converges weakly in $H_0^1(U)$ to the weak solution u of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Aufgabe 4
Suppose H is a Hilbert space and

$$u_k \rightharpoonup u \quad \textit{in } L^2(0,T;H).$$

Suppose further that we have the uniform bounds

$$\operatorname*{ess\,sup}_{0\leq t\leq T} \|u_k(t)\| \leq C \quad (k = 1,\dots),$$

for some constant $C > 0$. Prove

$$\operatorname*{ess\,sup}_{0\leq t\leq T} \|u(t)\| \leq C.$$

Hinweis. If $v \in H$ and $0 \leq a \leq b \leq T$, we have

$$\int_a^b (v, u_k)\, dt \leq C\|v\||b-a|.$$