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15.10.2019

Übungen zur Vorlesung

Partielle Differentialgleichungen 2

WiSe 19/20, Blatt 2

Übung: Donnerstag, 24.10.2019

Aufgabe 1
Investigate the following functions with respect to integrability and calculate, if applicable, their Bochner-Integral (take here into account Pettis' Theorem):

(a) $f : [0, \frac{1}{2}] \to l^1(\mathbb{N}), f(t) := (t^n)_{n\in\mathbb{N}}$.

(b) $g : [0, 1] \to L^p(0, 1), g(t) := \chi_{(0,t)}$ for $1 \le p \le \infty$.
Hinweis. Distinguish the cases $p < \infty$ and $p = \infty$!

(c) $h : [0, 1] \to L^2(0, 2\pi), h(t) := x \mapsto t \cdot x$.

In the following let X be a Banach space. A function $f : [a, b] \to X$ is differentiable at $t \in (a, b)$ if the limit

$$f'(t) := \lim_{h\to 0} \frac{1}{h}(f(t + h) - f(t))$$

exists. Analogously to the case $X = \mathbb{R}$, we define the normed spaces $C^r([a, b], X), r \in \mathbb{N}_0 \cup \{\infty\}$.

Aufgabe 2
Let $f_n \to f$ in $L^p(0, T; X)$ as $n \to \infty$. Show that there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} f_{n_k}(t) = f(t)$$

for almost all $t \in [0, T]$.

Aufgabe 3

(a) Let $f \in C^1([0, T]; X)$ and $0 \le r \le s \le T$. Show that

$$f(s) - f(r) = \int_r^s f'(t)\mathrm{d}t := \int \chi_{[r,s]} f'(t)\mathrm{d}t.$$

(b) Conclude that $C^1([0, T]; X) \hookrightarrow W^{1,p}(0, T; X)$ for all $p \in [1, \infty]$.

Aufgabe 4
Prove the following theorem:
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(0, T; X)$ and $f : [0, T] \to X$ be such that

$$\lim_{n\to\infty} f_n(t) = f(t)$$

for almost all $t \in [0, T]$. Furthermore, assume that there exists $g \in L^1(0, T; \mathbb{R})$ such that $||f_n(t)||_X \le g(t)$ for all $n \in \mathbb{N}, t \in [0, T]$. Then, $f \in L^1(0, T; X)$ and

$$\lim_{n\to\infty} \int f_n \mathrm{d}t = \int f \mathrm{d}t.$$

Aufgabe 5
Prove the following statement:
Assume that

$$\begin{cases} u_k \rightharpoonup u & \text{in } L^2(0,T;H_0^1(\Omega)), \\ u_k' \rightharpoonup v & \text{in } L^2(0,T;H^{-1}(\Omega)). \end{cases}$$

Prove that $v = u'$.
Hinweis. Let $\phi \in C_c^1(0,T), w \in H_0^1(\Omega)$. Then

$$\int_0^T \langle u_k', \phi w \rangle \mathrm{d}t = -\int_0^T \langle u_k, \phi' w \rangle \mathrm{d}t.$$