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08.10.2019

Übungen zur Vorlesung

Partielle Differentialgleichungen 2

WiSe 19/20, Blatt 1

Übung: Donnerstag, 17.10.2019

Aufgabe 1

Consider the Hilbert space l^2 defined by

$$l^2 = \left\{ x = (x_k)_{k=1}^{\infty} : x_k \in \mathbb{R} \ \ \forall k \in \mathbb{N}, \ \sum_{k=1}^{\infty} x_k^2 < +\infty \right\}$$

with scalar product

$$(x, y) = \sum_{k=1}^{\infty} x_k y_k.$$

(a) Define by

$$(e^n)_k = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise} \end{cases}$$

a sequence of orthonormal vectors. Show that $A = \{e^n\}_{n\in\mathbb{N}}$ satisfies $\overline{\text{span}(A)} = l^2$.

(b) Consider the sequence $x^n = ne^n$ for $n \in \mathbb{N}$. Show that $\lim_{n\to\infty}(x^n, e^m) = 0$ for all $m \in \mathbb{N}$, but that x^n does not converge weakly.

Aufgabe 2

Let $1 < p < \infty$ and let $\varphi \in C_c^\infty(\mathbb{R}^n), \varphi \neq 0$.

(a) Fix $\alpha > 0$ and define $\varphi_j^\alpha(x) = j^\alpha \varphi(jx)$. Check for which $\alpha > 0$ the sequence $\{\varphi_j^\alpha\}_j$ converges weakly (resp. strongly) in $L^p(\mathbb{R}^n)$ as $j \to \infty$ to some function φ_∞^α.

(b) Fix $\tau \in \mathbb{R}^n, \tau \neq 0$, and define $\varphi_j^\tau(x) = \varphi(x + j\tau)$. Does the sequence $\{\varphi_j^\tau\}_j$ converge weakly (resp. strongly) in $L^p(\mathbb{R}^n)$ as $j \to \infty$ to some function φ_∞^τ?

Aufgabe 3

Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $\{u_j\}_j \subset W^{1,p}(\Omega)$ be such that

$$\sup_{j\in\mathbb{N}} ||u_j||_{W^{1,p}(\Omega)} < +\infty.$$

Prove, upon extracting a subsequence, that there exists $u \in W^{1,p}(\Omega)$ such that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega)$ as $j \to \infty$.

Aufgabe 4

Let $\Omega \subset \mathbb{R}^n$ be open. We denote by $H^{-1}(\Omega)$ the dual space of the Hilbert space $H_0^1(\Omega)$ (which we do *not* identify with itself!), equipped with the operator norm.

(a) Let $f^0, f^1, \dots, f^n \in L^2(\Omega)$. Prove that the application

$$f(u) = \int_\Omega \Big(f^0 u + \sum_{k=1}^{n} f^k \partial_k u \Big) \,\mathrm{d}x, u \in H_0^1(\Omega) \tag{1}$$

defines a linear and continuous operator $f \in H^{-1}(\Omega)$.

(b) Let $f \in H^{-1}(\Omega)$. Show that there exist $f^0, f^1, \dots, f^n \in L^2(\Omega)$, such that f is given by (1). *Hinweis.* Use the Riesz' Representation Theorem.

(c) Prove that the operator norm on $H^{-1}(\Omega)$ can be written equivalently in the following form:

$$||f||_{H^{-1}(\Omega)} = \inf \left\{ \left(\sum_{k=0}^{n} ||f^k||^2_{L^2(\Omega)} \right)^{1/2} : f^0, f^1, \dots, f^n \in L^2(\Omega) \right\}.$$