Compensation of Modelling Errors in EEG Source Imaging using Bayesian Statistics

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A primary current is generated by the electrochemical processes in the excited nerve cells (Conversion of energy from chemical to electric form). Cellular (primary) currents due to synchronous activity of a large population of neurons with similar spatial orientation induce extracellular (secondary) currents that can be measured as extra-cranial electric potentials.

Assumption: The individual cells behave like electric current dipoles: \( j(x_i) = d_i \delta(x - x_i) \).
Estimation of the brain activity by non-invasive measurements along the scalp surface.

- Diagnostic tool in neurology, e.g., epileptic seizures, brain death and neuroscience.
- Estimate Cognitive States using EEG recordings.

EEG is relatively easy and inexpensive technique/device.

Image Source: Dr. R. Henson, Univ. of Cambridge
**EEG Macroscopic Mathematical Model**

**Definition (Poisson Equation with Neumann Condition)**

Let $\Omega \subset \mathbb{R}^k$ be the head domain and $\sigma(x)$ the conductivity of the tissues, where $x \in \Omega$. The neural activity $\nabla \cdot j(x): \Omega \to \mathbb{R}^k$ gives rise to potentials $u(x)$ within the domain. Under the quasi-static approximation of the Maxwell’s equations, we have the Poisson equation,

$$\nabla \cdot \sigma(x) \nabla u(x) = \nabla \cdot j(x)$$

with boundary conditions

$$\hat{n} \cdot \sigma(x) \nabla u(x) = 0 \quad x \in \partial \Omega \quad \text{(no-outward flux)}$$

$$\int_{x \in \partial \Omega} u(x) \; dS = 0 \quad \text{(fixed ground - Ensures uniqueness)}$$
Mapping between measurements and brain activity

The EEG observations are a scalar function $v(x)|_{\partial \Omega}$ defined as the voltage at a point $x \in \partial \Omega$ with respect to the reference electrode at $x_0$. In practice, $v = (v(x_1), v(x_2), \ldots, v(x_m))^T \in \mathbb{R}^m$ and the neural activity $j(x)$ is approximated as dipole sources $d(x) : \Omega \to \mathbb{R}^k$.

**Definition (EEG Mapping)**

The mapping from the source field to the $i^{th}$ boundary measurements can be written as

$$v_i = \int \kappa_i(\lambda, \sigma, x) \cdot d(x) \, dx,$$

where $\kappa_i(\lambda, \sigma, x)$ is a non-linear vector function that depends on

- the parameterizations of the geometry $\lambda$,
- tissue conductivity $\sigma$ and
- electrode locations.

Function $\kappa_i(\lambda, \sigma, x)$ is referred to as the lead field function.
Computational Model for EEG Source Imaging

In this work, we employ the Distributed Dipole Source Model (DSM) which relates linearly the EEG data \( v \) with the electrical brain activity\(^5\).

The brain domain is discretized and the individual cells behave like electric current dipoles.

**Definition (Computational Model)**

\[
\nu = \sum_{x_i \in \Omega} K_i(\lambda, \sigma) \cdot d_i + \xi = K(\lambda, \sigma)d + \xi
\]

\( \Omega \): discretized head

The potential \( \nu \in \mathbb{R}^m \) at \( m \) electrodes is a function of
- \( d_i \in \mathbb{R}^k \): dipole source at location \( x_i \);
- \( K_i(\lambda, \sigma) \in \mathbb{R}^{m \times k} \): mapping from dipole source at \( x_i \) to measurements;
- \( K(\lambda, \sigma) \in \mathbb{R}^{m \times kn} \): lead field matrix;
- \( d \in \mathbb{R}^{kn} \): dipole source distribution;
- \( \xi \in \mathbb{R}^m \): additive measurement noise \( \xi \sim \mathcal{N}(\xi_*, \Gamma_\xi) \).

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\(^5\)lead field function is estimated numerically using finite element method.
The Source Imaging Problem

Definition: Reconstruction of dipole source distribution in the bounded domain (brain) using EEG recording. The problem is linear and the solution requires the inversion of the mapping (lead field matrix $K$ where $m \ll kn$) but the problem is severely under-determined.

Difficulties:

a. Different source configurations give the same potential measurements.
b. $K$ needs to be accurate i.e. conductivities, geometry, electrode locations etc.
c. Instability of the solution to the measurement noise.

Prior information is required.

Solving the EEG inverse problem $\Rightarrow$ selecting the solution that best matches the prior knowledge and explains the data.
Bayesian Approximation Error Approach (BAE) in the EEG Focal Source Imaging

Motivation:

- The EEG imaging problem is an ill-posed inverse problem and the estimates can be highly sensitive to modelling errors in addition to measurement noise.
- Thus, the reliability of the source imaging results depends on the accuracy of the computational model i.e. head geometry and tissue conductivity.
- Due to individual variations, the construction of separate head model is usually required for each individual.
- This model construction is a multidisciplinary task which needs CT/MRI-data, is expensive and takes time.
- If instead of the accurate model, we use a coarse (approximate) model then can we still reconstruct the correct source distribution?

Aim: In this feasibility study, we assess to which degree we can alleviate the source localization errors when a standard head model with a probabilistic model for the uncertainties of the head features is employed.
The probabilistic approach is called Bayesian Approximation Error Approach (BAE) and has been used in different inverse problems.

- Coarse meshes are needed to improve computational efficiency (e.g. speed). The coarse modelling results in model reduction errors which can be handled by the BAE.

- Certain features of the target are partly unknown (e.g. geometry, shape).
Bayesian Approximation Error

The idea is to use a simplified computational model and incorporate an additional error term to compensate the simplification.

Let the unknowns be \((\lambda, \sigma, \bar{d}, \xi)\) where \(\lambda\) and \(\sigma\) represent auxiliary uncertainties (i.e. geometry and conductivity), \(\xi\) is the additive noise and \(\bar{d}\) is an accurate presentation of the unknown of interest (dipole distribution). All the variables are considered as random.

Let

\[ v = \bar{K}(\lambda, \sigma)\bar{d} + \xi \in \mathbb{R}^m \]

denote an accurate model for the relationship between the measurements and the unknowns, where \(\xi\) is assumed to be mutually independent with \((\lambda, \sigma, \bar{d})\).
Approximate Computational Model and Modelling Errors

In the approximation error approach, instead of using the accurate mapping $(\lambda, \sigma, \bar{d}) \mapsto \bar{K}(\lambda, \sigma, \bar{d})$, we fix the random variables $(\lambda, \sigma) \leftarrow (\lambda_0, \sigma_0)$ and we use a computationally approximate model

$$d \mapsto K(\lambda_0, \sigma_0, d).$$

The primary unknown satisfies: $d = P\bar{d}$, where $P$ is typically a linear projection operator \(^6\).

\(^6\) In general: $d(x) = \bar{d}(\mathcal{T}(\mathcal{X}))$, where $\mathcal{T} : \bar{\Omega} \mapsto \Omega$ (Registration, wrapping techniques).
Thus, the computational model becomes

\[ \nu = K(\lambda_0, \sigma_0)d + [\tilde{K}(\lambda, \sigma)\tilde{d} - K(\lambda_0, \sigma_0)d] + \xi \]

\[ = K(\lambda_0, \sigma_0)d + \varepsilon + \xi, \]

where \( \varepsilon = \tilde{K}(\lambda, \sigma)\tilde{d} - K(\lambda_0, \sigma_0)d \) is the approximation error. The errors \( \xi \) and \( \varepsilon \) are considered as mutually independent.
Bayesian Inference

Definition (Approximate Computation Model)

\[ v = K(\lambda_0, \sigma_0) d + \varepsilon(\lambda, \sigma, d) + \xi \]

Before the statistical inference of \( d \) (e.g. MAP estimate), the additive errors \((\varepsilon, \xi)\) are pre-marginalized using Gaussian approximation.

From Bayes’ theorem, the posterior distribution is

\[ \pi(d|v) \propto \pi(v|d) \pi(d) \]

and the likelihood is

\[
\pi(v|d) = \int \int \int \int \pi(v, \lambda, \sigma, \varepsilon, \xi|d) \, d\lambda \, d\sigma \, d\varepsilon \, d\xi \\
= \int \int \pi(v|\lambda, \sigma, \varepsilon, \xi, d) \left[ \int \int \pi(\varepsilon, \xi|\lambda, \sigma) \pi(\lambda, \sigma|d) \, d\lambda \, d\sigma \right] \, d\varepsilon \, d\xi \\
= \int \int \delta(v - K(\lambda_0, \sigma_0) d - \varepsilon - \xi) \left[ \pi_\xi(\xi) \pi_\varepsilon|d(\varepsilon|d) \right] \, d\varepsilon \, d\xi...
\]
Marginalization over Additive Errors

The likelihood is a convolution integral$^7$ over $\varepsilon$.

$$\pi(\nu|d) = \int \pi_\xi(\nu - K(\lambda_0, \sigma_0 - \varepsilon)) \pi_\varepsilon|d(\varepsilon|d) \, d\varepsilon.$$  

If we set $\nu = \varepsilon + \xi$, then the likelihood becomes

$$\pi(\nu|d) = \pi_{\nu}|d(\nu - K(\lambda_0, \sigma_0)|d)$$

where the subscript $\nu$ is used to clarify that $\pi_{\nu}|d$ is the conditional probability density of variable $\nu$.

The posterior is

$$\pi(d|\nu) \propto \pi_{\nu}|d(\nu - K(\lambda_0, \sigma_0)|d) \pi(d).$$

$^7$Remember: $\pi_{\nu}(\nu) = \int \pi_\xi(\nu - \varepsilon) \pi_\varepsilon(\varepsilon) \, d\varepsilon$ for $\nu = \varepsilon + \xi$
In the Bayesian approximation error approach, $\pi_\xi(\xi)$ and $\pi(\varepsilon, d)$ are approximated with Gaussian distributions.

**Approximations:**

- $\xi \sim \mathcal{N}(\xi_*, \Gamma_\xi)$
- $\tilde{\pi}(\varepsilon, d) \propto \exp\left(-\frac{1}{2} \left(\begin{array}{c} d - d_* \\ \varepsilon - \varepsilon_* \end{array}\right)^T \left(\begin{array}{cc} \Gamma_d & \Gamma_{d\varepsilon} \\ \Gamma_{\varepsilon d} & \Gamma_\varepsilon \end{array}\right)^{-1} \left(\begin{array}{c} d - d_* \\ \varepsilon - \varepsilon_* \end{array}\right)\right)$

Thus, the approximate conditional $\tilde{\pi}_{\varepsilon|d}(\cdot)$

$$\varepsilon|d \sim \mathcal{N}(\varepsilon_*|d, \Gamma_{\varepsilon|d}).$$

And the approximation error statistics are

$$\varepsilon_*|d = \varepsilon_* + \Gamma_{\varepsilon d}\Gamma_d^{-1}(d - d_*),$$

$$\Gamma_{\varepsilon|d} = \Gamma_\varepsilon - \Gamma_{\varepsilon d}\Gamma_d^{-1}\Gamma_{d\varepsilon}.$$
Approximate likelihood and Posterior

Under the previous approximation, the total error $\nu|d = \varepsilon|d + \xi$ has distribution

$$\nu|d \sim \mathcal{N}(\nu_*|d, \Gamma_{\nu|d})$$

where $\nu_*|d = \varepsilon_*|d + \xi_*$ and $\Gamma_{\nu|d} = \Gamma_{\varepsilon|d} + \Gamma_{\xi}$.

The likelihood model is approximated by

$$\tilde{\pi}(\nu|d) = \tilde{\pi}_{\nu|d}(\nu - K(\lambda_0, \sigma_0)|d)$$

$$\propto \exp \left( -\frac{1}{2} (\nu - K(\lambda_0, \sigma_0) d - \nu_*|d))^T \Gamma_{\nu|d}^{-1} (\nu - K(\lambda_0, \sigma_0) d - \nu_*|d) \right)$$
Finally, the posterior $\pi(d|\nu)$ is

$$\pi(d|\nu) \propto \tilde{\pi}(\nu|d) \pi(d)$$

MAP estimate:

$$\hat{d} := \arg \max_d \pi(d|\nu) = \min_d \| L_\nu (\nu - K(\lambda_0, \sigma_0)d - \nu_*|d) \|_2^2 + 2 \ln(\pi(d))$$

where $L_\nu^T L_\nu = \Gamma^{-1}_{\nu|d}$ (Cholesky factorization).
Summary: Bayesian Approximation Error Approach

- The computational model is: \( v = K(\lambda_0, \sigma_0)d + \varepsilon + \xi \).
- The posterior density from Bayes law is \( \pi(d|v) \propto \pi(v|d) \pi(d) \).
- Gaussian approximations of the errors.
- So, the likelihood model is approximated by
  \[
  \pi(v|d) \approx \pi_{\nu|d}(v - K(\lambda_0, \sigma_0)d|d)
  \]
  where \( \nu = \varepsilon + \xi \) and \( \nu|d \sim \mathcal{N}(\nu_*|d, \Gamma_{\nu|d}) \).
- The mean \( \nu_*|d = \varepsilon_*|d + \xi_* \) and covariance matrix \( \Gamma_{\nu|d} = \Gamma_{\varepsilon|d} + \Gamma_{\xi} \).
- The approximation error statistics are given by
  \[
  \varepsilon_*|d = \varepsilon_* + \Gamma_{\varepsilon d} \Gamma_d^{-1}(d - d_*)
  \]
  \[
  \Gamma_{\varepsilon|d} = \Gamma_{\varepsilon} - \Gamma_{\varepsilon d} \Gamma_d^{-1} \Gamma_{d\varepsilon}.
  \]
- MAP:
  \[
  \hat{d} := \text{arg max}_d \pi(d|v) = \min_d \|L_{\nu}(v - K(\lambda_0, \sigma_0)d - \nu_*|d)\|^2 + 2\ln(\pi(d))
  \]
Computation of Approximation Error statistics

1. The mean \((\varepsilon_*|d)\) and covariance \((\Gamma_{\varepsilon}|d)\) of the approximation error can be computed through Markov chain Monte Carlo (MCMC) simulations.

2. A set of (training) accurate models \(\bar{K}_i(\lambda, \sigma)\) and a set of simulated dipole distributions, \((\bar{d}^{(1)}, \ldots, \bar{d}^{N})\) are required.

3. The difference between EEG data from an accurate head model and standard model was estimated, i.e. \(\varepsilon^{(i)} = \bar{K}_i(\lambda, \sigma)\bar{d}^{(i)} - K(\lambda_0, \sigma_0)(P\bar{d}^{(i)})\).

4. Sample mean and covariance

\[
\varepsilon_*|d = \frac{\sum_{i=1}^{N} \varepsilon^{(i)}}{N}
\]
\[
\Gamma_{\varepsilon}|d = \frac{\sum_{i=1}^{N} (\varepsilon^{(i)} - \varepsilon_*|d)(\varepsilon^{(i)} - \varepsilon_*|d)^T}{N - 1}
\]
Sparse/focal Source Reconstructions

The $L_{1,2}$ norm -prior was used to focalize the sources, and depth weights were used to compensate the depth bias:

$$p(d) \propto \exp(\alpha/2 \sum_{i=1}^{n} w_i \|d_i\|),$$

where $\|d_i\| = (d_{xi}^2 + d_{yi}^2)^{1/2}$ and $[w_1, \ldots, w_N] = \text{diag}(K(KK^T)^{-1}K)$.

Maximum a posteriori (MAP) estimates.

Compare Results:

Accurate head model:

$$\hat{d}_{ACC} := \min_{d} \|L_\xi(v - K(\lambda_{acc}, \sigma_{acc})d - \xi_*)\|^2 + \alpha \sum_{i=1}^{n} w_i \|d_i\|$$

Standard model:

$$\hat{d}_{ST} := \min_{d} \|L_\xi(v - K(\lambda_0, \sigma_0)d - \xi_*)\|^2 + \alpha \sum_{i=1}^{n_0} w_i \|d_i\|$$

Standard model with Bayesian approximation error statistics:

$$\hat{d}_{BAE} := \min_{d} \|L_\nu(v - K(\lambda_0, \sigma_0)d - \nu_{*|d})\|^2 + \alpha \sum_{i=1}^{n_0} w_i \|d_i\|$$
Single Source Reconstructions - No Noise

To simulate the data (potential measurements), an accurate forward was used. Three different computational models were used in the inversion: a) Accurate geometry/conductivity model, b) standard geometry and c) standard geometry with the approximation error statistics.
Reconstructions in the Presence of Noise

A single source case.
The inverse models were:

a) accurate geometry/conductivity model, b) standard geometry and c) standard geometry with error statistics.
Conclusions

Inverse problems are (very) sensitive to measurement and modelling errors.

This means that errors and uncertainties shift the posterior probability mass away from the actual values.

Modelling all the uncertainties as unknowns and carrying out inference for all of them using MCMC is extremely time consuming.

With the Bayesian approximation error approach (BAE), the modelling errors are included as additive errors in the likelihood model, and thus they can be formally marginalized before the inference.

The statistics of the modelling errors can be pre-computed and used in the inference stage.

EEG and BAE:
In this study, we showed that a simplified head model with the approximation error statistics can improve the source reconstruction and give comparable results to those obtained with the accurate head model. This makes it possible that a standardized head model(s) could be used in EEG.