Modelling Opinion Dynamics

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Abstract

We represent two mechanisms of opinion formation by kinetic equations involving both the exchange of opinion between individuals and the diffusion of information. The first one is a possible approach to the formation of choice in a society, following a linear Boltzmann equation where the background density is nothing but a fixed distribution of possible choices. In the second model we focus on the opinion formation in the presence of leaders, where the Boltzmann type equation based on a model predictive control formulation, is introduced and discussed. Notably, according to the leaders strategy, in order to achieve opinion consensus, a suitable functional is modified. Thanks to the receding horizon strategy, the minimization of suitable cost functional is able to embed into the binary interaction of the corresponding Boltzmann equation. Both models start from microscopic interaction among individuals and we arrive at a macroscopic description of the opinion formation process which is described by a system of Fokker-Planck type equation. The analytical results of this system give us an explicit expression of a stationary solution. Several numerical results show various effects of different model parameters in the first description and the robustness of the Boltzmann type control approach to illustrate different variations in the leaders’ strategy.
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1 Introduction

In recent years, opinion dynamics have attracted many authors to analyze and discuss this collective behavior. Not only to predict the human behavior on a social level, but also its application to politics, to forecast voters’ behavior during an election process or the public tendencies. Many started modeling the opinion formation by cellular automata \[3\]. In these numerical studies, the lattice points denote the agents (individuals) and society is considered as a graph, where the agents interact with their neighbors in an iterative way. Other attempts include approaching opinion dynamics through the use of the mean field equation \[2\]. In general, it is often very difficult to obtain an exact solution in a kinetic system with a lot of interacting particles. However, the means of mean field equation is applied by system of ordinary differential equations or partial differential equations of diffusion type that can replace microscopic interaction. As a result, under certain assumptions, the time asymptotic behavior of the distribution function can be obtained easily and it helps to describe the stationary solutions of the kinetic equation from which one can easily elaborate information on the behavior of opinion. In short, they can be treated analytically and helps to obtain a deeper understanding of the opinion dynamics. Furthermore, in a basic opinion model, the individuals’ opinion can change randomly, as well as the dynamics of the interaction between people. In order to describe it efficiently, two aspects of human behaviors have been emphasized to govern the interaction of opinion formation \[8\]. One of them is the compromise process, in which each pairs of agents reaches an agreement after exchanging opinions. It indicates the tendency of human behavior to settle conflicts within our society. The other is the self-thinking process, where individuals change their opinions in a random diffusive fashion. Individuals are possibly influenced by the global access of information, such as e-mail, web navigation, social media, etc. Consequently, a large variety of human behavior is included sufficiently by these aspects and they are mainly responsible for the large time behavior of the solution and reproduce explicit steady profiles. Similar diffusion equations to the mean field limit of the Ochrombel simplification of the Sznajd model \[2\], have also been applied, namely quasi-invariant opinion limit \[1\]. In a suitable scaling limit, a partial differential equation of Fokker-Planck
type can be derived which helps the explicit computation of the steady state. Later, different to these classical approaches, the opinion formation dynamic is approached in a constrained setting. The feedback control types have been discussed [4], [6]. The classical examples are: how a certain group of agents persuades voters to vote for a specific candidate or influence buyers towards a given good or asset. Based on a microscopic model, a Boltzmann type control optimal control has been developed to approach this idea. The approach is closely related to model predictive and instantaneous control techniques.

Following the models [1] and [2], we focus on modeling of opinion formation in a particular setting. Firstly, we represent the idea of the formation of choice. Borrowing from the kinetic theory of rarefied gases we introduce the basic model of opinion formation and additionally the agents have to choose from a finite set of possible options depending on the settings previously specified. As it will be explored, the collision kernel for the microscopic particle interactions are defined and in dependence on these rules, the quasi-invariant opinion limit is developed. Then the related Fokker-Planck equation is derived and subsequently, the steady state is obtained. In the next chapter, the concept of leadership is introduced, in which the strategies of leaders are described by a control term to influence other individuals toward a desire statement. In order to permit this, the main properties of the Boltzmann description are presented. Then the Fokker-Planck limit system is introduced which allows us to give an explicit solution of the steady profiles. In the final analysis, several numerical examples of these models are represented.
2 Choice Formation

2.1 Basic model of the opinion formation

In opinion dynamics, the basic model of opinion formation is borrowed from the kinetic equation in Physics. The kinetic model will be used to describe the evolution of the distribution of opinion respectively formation of choice in a society by means of microscopic interaction among their social agents (individuals), whose point of views can be changed and can be influenced by the others in a random fashion.

We label opinion with the variables $\omega = -1$ and $\omega = +1$ as the two extreme answers to the statement, i.e. “yes” or “no”, without reverse. Note that any intermediate value between the two extremes (zero excluded) means a partial agreement or disagreement, with a degree of conviction proportional to $|\omega|$. The value $\omega = 0$ means the agent is undecided and completely indifferent with respect to the question. In order to derive the kinetic equation which will be shown later by a Boltzmann type integro-differential equation, we introduce the density (or distribution function) $f = f(\omega, t)$ depending on $\omega \in I = [-1, 1]$ and the time $t \geq 0$. The integral

$$\int_{D} f(\omega, t) d\omega$$

represents the number of individuals with opinions included in $D \subset I$ where the opinions are defined on, at time $t > 0$. Note that, if we integrate the density function over the lateral bounds, that is always normalized to 1, it is

$$\int_{I} f(\omega, t) d\omega = 1$$

2.1.1 From opinion to choice

Normally the modelling of opinion formation in a multiagent society describes individual exchanging their viewpoints without any certain preferences. However in many
cases, such as political elections or the public opinion tendencies, the agents often make a decision or vote after the opinion formation. It is also possible that the corresponding individuals may adapt their opinions to the electoral result which may be completely different than their primary viewpoints. In this case, we will show the evolution of the distribution of opinions in a closed society around a well-defined statement by using the kinetic model.

According to Toscani’s model \[1\], a fixed distribution of possible choices \(M(\omega)\) is added, which is comparable with a fixed background of field particles. As it will be shown later, the choice formation of the population is driven by the collisions subjected by the background \(M(\omega)\). Consequently a single linear kinetic equation will govern the opinion distribution function. If we only consider opinions regarding a finite number \(N + 1\) as questions, the background \(M(\omega)\) is regarded as a mass- concentrated distribution.

\[
M(\omega) = \sum_{i=0}^{N} \alpha_i \delta(\omega - \bar{\omega}_i), \quad \sum_{i=0}^{N} \alpha_i = 1 \tag{2.3}
\]

As usual, \(\delta(\omega - \bar{\omega})\) is a Dirac delta function concentrated at the opinion \(\omega = \bar{\omega} \in I\). In (2.3) the parameter \(0 < \alpha_i < 1\) is the probability that one can choose the i-th possibility. The other parameters \(\bar{\omega}_i \in (-1,1)\) represent the stability of the i-th choice. This parameter, which reflects the social fact, that extremal opinions are more difficult to change, is assumed to play a decisive role in our model.

Although the background \(M(\omega)\) resembles a fixed background of field particles and can be altered in many ways, contrasting the classical model, this representation of equilibria shows that the possible choices are well defined and they are dominated in the given boundary \(I\) which allows us for a simple treatment of the underlying equation. In our model of opinion formation, for example in a problem between two choices, we are particularly interested in the mean value of the resulting steady state. This asymptotic state shows which one of the two possible options, reaches the prevalence of consensus. For instance, if the agents only have two options, it is obvious, that a background distribution is presented by the distribution of a two-valued random variable, where the two Dirac delta functions are localized at two points \(\omega_- < 0\) and \(\omega_+ > 0\) of the interval \(I\). In the symmetric case, we have \(\alpha_1 = \alpha_2 = 1/2\) respectively \(\omega_- = -\omega_+\), which is linked to the distribution of the random experiment of tossing a coin. According to the characteristic of these two parameters, each individual chooses between two options which are equally probable and stable. Aside from this, if we set \(\alpha_1 = 1/3, \alpha_2 = 2/3, \omega_- = -2/3\) and \(\omega_+ = 1/3\), we obtain a non-symmetric case with zero mean where the less probable choice is more stable than the other.
2.1.2 Exchange of opinions inside the population

Following the means of kinetic collision-like model [7], which describes the time evolution of the distribution of opinion among individuals in society, the model is based on the binary interaction.

Let us concentrate on microscopic interactions which are responsible for the configuration of the final macroscopic equilibrium. In comparison to the classical interactions between molecules which is familiar to people working in kinetic theory of rarefied gases [11], the bounds in the following interaction rules of the social interaction, cannot be violated. This ensure that −1 and 1 really are the extreme opinions. The lateral bound emphasizes the difference between the social interactions, where not all outcomes are permitted. When an arbitrary individual with opinion $\omega$ before the interaction exchanges with the background $M(\omega_*)$, the post-interaction $\omega'$ is given by

$$\omega' = \omega - \gamma P(|\omega - \omega_*|)(\omega - \omega_*) + \eta D(\omega^2),$$  \hspace{1cm} (2.4)

Note that, the opinions are not allowed to cross boundaries, otherwise, the interaction cannot take place. In (2.4) $\gamma$ is a constant such that $\gamma \in (0, 1/2)$ which describes the compromise propensity. The quantity $\eta \in B \subset \mathbb{R}$ with distribution $\Theta$ is a random variable with zero mean and variance $\sigma^2$. This characterizes the self-thinking process.

$$\int_{\mathbb{R}} \omega d\Theta(\omega), \quad \int_{\mathbb{R}} \omega^2 d\Theta(\omega)$$  \hspace{1cm} (2.5)

The functions $P(\cdot)$ and $D(\cdot)$ model the relevance of compromise and diffusion for a given opinion. The pre-interaction opinion increases (getting closer to $\omega_*$) when $\omega_* > \omega$ and vice versa in the opposite situation. The presence of both functions is linked to the hypothesis that the availability to the change of opinion is related to both opinion itself and to the distance between opinions. Both functions decrease, as soon as one gets closer to an extremal opinion. Again, this represents that the extremal opinions are more difficult to change. Moreover the functions $P(\cdot)$ and $D(\cdot)$ are assumed as decreasing functions with respect to $|\omega|$ and in addition $0 \leq P(|\omega - \omega_*|) \leq 1$ and $0 \leq D(\omega^2) \leq 1$. If the diffusion contribution ($\eta \equiv 0$) is absent, the definition of the post-interaction (2.4) is given by

$$\omega' = (1 - \gamma P(|\omega - \omega_*|))\omega + \gamma P(|\omega - \omega_*|)\omega_*$$
which implies
\[ |\omega'| \leq \max\{|\omega|, |\omega_*|\} \leq 1. \]

### 2.1.3 The Boltzmann equation

Once given the collision rule, the time evolution of the distribution function \( f \) is characterized by a linear collisional integral of Boltzmann type, it reads

\[
\frac{\partial}{\partial t} f = (Q(f, M)),
\]

(2.6)

where the interaction between the individuals and the field background of opinions generated the variation in time of the distribution \( f \). The collision integral \( Q \), which has the classical structure of the dissipative linear Boltzmann kernel of Maxwell type, can be viewed as the comparison of two parts: The first one is a gain term \( Q_+ \), which quantifies the exchange of opinion between individuals which give, after the interaction with background, the opinion \( \omega \); and the other is a loss term \( Q_- \), which quantifies the exchange of opinion where an individual with pre-collisional opinion \( \omega \) experience an interaction with the background. Following the standard methods of kinetic theory of binary interaction \cite{7} allowed us to recover the time evolution of \( f(\omega, t) \) as a balance bilinear gain and loss of opinion terms. However, it is apparent, that the existence of the pre-collisional pair which returns to post-collisional pair \((\omega, \omega_*)\) through a collision of Boltzmann type is not assured. In this case the collisional operator can be written in its weak form for a suitable regular test function.

The weak form of the collisional kernel is defined as

\[
(Q(f, M), \varphi) = \left\langle \int \varphi(\omega', \omega) f(\omega, t) M(\omega_*) d\omega d\omega_* \right\rangle
\]

(2.7)

In (2.7), as usual, \( \langle \cdot \rangle \) denotes the expectation value with respect to \( \eta \) with zero mean and \( \sigma^2 \) variance. Note the function \( f \) denotes the distribution of opinion and \( M \) the fixed background equilibrium distribution (2.3) from the previous section. The Boltzmann equation in weak form reads

\[
\frac{d}{dt} \int f(\omega, t) d\omega = (Q(f, M), \varphi).
\]

(2.8)
For $\varphi(\omega) = 1$, the collisional operator guarantees the conservation of the total number of opinions which is the only conserved quantity of the process. Consequently we can assume that $f(\omega, t = 0)$ is a probability density for the solution to remain a probability density at any subsequent time.

For $\varphi(\omega) = \omega$, we obtain the time evolution of the average opinion. Since $\langle \eta \rangle = 0$ we have

$$\frac{d}{dt} \int_I \omega f(\omega, t) d\omega = \left\langle \int_I (\omega' - \omega) f(\omega, t) M(\omega_s) d\omega d\omega_s \right\rangle$$

Substituting the definition of the post-interaction (2.4) into equation above we have,

$$-\gamma \int_I P(|\omega - \omega_s|)(\omega - \omega_s) f(\omega, t) M(\omega_s) d\omega d\omega_s =$$

$$-\gamma \int_I K(\omega) f(\omega, t) d\omega. \quad (2.9)$$

By using the convolution between the background and the non-local compromise function we conserve $K$ in (2.9), which is linear and it reads

$$K(\omega) = \int_I P(|\omega - \omega_s|)(\omega - \omega_s) M(\omega_s) d\omega_s. \quad (2.10)$$

The choice $P(|\omega - \omega_s|) = 1$ leads us to

$$K(\omega) = \omega - \int_I \omega_s M(\omega_s) d\omega_s = \omega - \langle \omega_s \rangle_M. \quad \text{(2.10)}$$

Substituting this value into (2.9) we have

$$\frac{d}{dt} \int_I \omega f(\omega, t) d\omega = -\gamma \int_I \omega - \langle \omega_s \rangle_M f(\omega, t) d\omega$$

This is explicitly solvable, since the integral of the density function is normalized to one (2.2), it reads

$$\int_I \omega f(\omega, t) d\omega = \int_I \omega f_0(\omega) d\omega e^{-\gamma t} + \langle \omega_s \rangle_M (1 - e^{-\gamma t}). \quad (2.11)$$

It shows that the mean value of solution converges exponentially to the mean value of the background. However, it is not solvable if $P$ is not constant.

Let us consider the case $\varphi(\omega) = \omega^2$, with regard to the definition of $\omega'$ (2.4), $\langle \eta \rangle = 0$
and \( \langle \eta^2 \rangle = \sigma^2 \), we obtain

\[
\frac{d}{dt} \int \omega^2 f(\omega, t) d\omega = \int ((\omega')^2 f(\omega, t) M(\omega_s) d\omega d\omega_s, \tag{2.12}
\]

and

\[
\gamma^2 \int_{I_s^2} P(|\omega - \omega_s|)^2 (\omega - \omega_s)^2 f(\omega) M(\omega_s) d\omega d\omega_s - 2\gamma \int_{I_s^2} P(|\omega - \omega_s|) (\omega - \omega_s) f(\omega) M(\omega_s) d\omega d\omega_s + \sigma^2 \int_{I_s^2} D(\omega)^2 f(\omega) d\omega = \gamma^2 \int J(\omega) f(\omega) d\omega - 2\gamma \int K(\omega) f(\omega) d\omega + \sigma^2 \int D(\omega)^2 f(\omega) d\omega.
\]

In (2.12) \( J(\omega) \) is defined as

\[
J(\omega) = \int_{I_s} P(|\omega - \omega_s|)^2 (\omega - \omega_s)^2 M(\omega_s) d\omega_s. \tag{2.13}
\]

If we assume \( P(|\omega - \omega_s|) = 1 \) and \( D(\omega)^2 = 1 - \omega^2 \), it leads to a close evolution equation for the second moment \( m_2 \)

\[
\frac{d}{dt} \int \omega^2 f(\omega, t) d\omega = -2(2\gamma + \sigma^2 - \gamma^2) \int \omega^2 f(\omega) d\omega + 2(\gamma - \gamma^2) \langle w_s \rangle_M \int \omega f(\omega) d\omega + \gamma^2 \langle w_s^2 \rangle_M + \sigma^2 \tag{2.14}
\]

In this case, as we mentioned, the expectation value of the solution converges to the mean value of the background \( \langle w_s \rangle_M \), and the equation above can be simplified. We define \( r = (2\gamma + \sigma^2 - \gamma^2) \), then

\[
\frac{d}{dt} \int \omega^2 f(\omega, t) d\omega = -\int \omega^2 f(\omega) d\omega + (2\gamma - \gamma^2) \langle w_s^2 \rangle_M + \sigma^2.
\]

As computation shows that the explicit solution of the above equation has the lowing structure

\[
\int \omega^2 f(\omega, t) d\omega = \int f_0(\omega) d\omega e^{-rt} + \frac{(2\gamma - \gamma^2) \langle w_s^2 \rangle_M + \sigma^2}{r} (1 - e^{-rt}), \tag{2.15}
\]
and the second moment \( m_2 \) of the solution converges to the limit value
\[
m_2 = \frac{2\gamma - \gamma^2}{2\gamma - \gamma^2 + \sigma^2} (w_\ast)_M + \frac{\sigma^2}{2\gamma - \gamma^2 + \sigma^2}.
\]

Since value of \( \langle w_\ast \rangle_M < 1 \), we conclude that the value of \( m_2 \) is always lower than the second moment of the background. It is even possible to solve the moments of the equation if we fix the function \( P(|\omega - \omega_\ast| = 1) \), but we remark that, the final value is not realistic. Indeed, as we showed the mean opinion is converging to the mean value of the background, which means, if \( \langle w_\ast \rangle_M = 0 \), the final opinion will have zero mean as well. For that reason no choice will be produced after binary interaction, in the absence of the diffusion contribution.

### 2.2 Fokker-Planck Modeling

In the previous section we have observed that it is difficult to obtain both analytic results on the large time behavior and the time evolution of the opinion density. The main idea to solve this problem is to replace the microscopic interaction by a unique averaged interaction, namely the moments of the opinion distribution function. Accordingly, not only can the mean opinion be obtained easier, but also the time asymptotic behavior of the opinion around the fixed possible choices. Following the kinetic theory this asymptotic limit of the Boltzmann model results in simplified models, generally of Fokker-Planck type. If \( \gamma \ll 1 \), we set
\[
\tau = \gamma t, \quad g(\omega, \tau) = f(\omega, t),
\]
which implies \( f_0(\omega) = g_0(\omega) \). Furthermore since we set \( \gamma \ll 0 \), we obtain the behavior of \( g(\omega, \tau) \) at any time \( \tau \) and coevally the behavior of \( f(\omega, t) \) at the large time \( t = \tau/\gamma \gg 1 \), namely the knowledge of the solution close to the steady state. If we set the rescaling (2.16) into the evolution equations 2.9 and 2.12, we obtain the evolution of moments for \( g(\omega, \tau) \),
\[
\frac{d}{d\tau} \int I \omega g(\omega, \tau) d\omega = - \int I K(\omega) g(\omega, \tau) d\omega,
\]
\[
\frac{d}{d\tau} \int I w^2 g(w, \tau) dw =
\]
\[ -2 \int_I K(\omega)\omega g(\omega, \tau) d\omega + \frac{\sigma^2}{\gamma} \int_I D(\omega^2) g(\omega, \tau) d\omega + R_\gamma(\tau); \quad (2.18) \]

We define

\[ R_\gamma(\tau) = \gamma \int_I J(\omega) g(\omega, \tau) d\omega. \]

If most of the interactions produce a very small exchange of opinion (which means \( \gamma \to 0 \)), and a small diffusion effects of external events \( \sigma^2 \to 0 \), with \( \sigma^2 / \gamma = \lambda \), the equation (2.18) is read in the limit

\[ \frac{d}{d\tau} \int_I \omega^2 g(\omega, \tau) d\omega = \]

\[ -2 \int_I K(\omega)\omega g(\omega, \tau) d\omega + \lambda \int_I D(\omega^2) g(\omega, \tau) d\omega. \quad (2.19) \]

This case represents that the value of the ratio \( \sigma^2 / \gamma \) is primarily responsible for the steady profile, which will be explained in the coming section. Moreover, this idea can be transferred to the linear Boltzmann equation (2.8). According to the collision rule (2.4), it reads

\[ \omega' - \omega = \gamma P(|\omega - \omega_*|)(\omega_* - \omega) + \eta D(\omega^2) \]

Then the Taylor expansion of \( \varphi \) up to second order around \( \omega \) leads to

\[ \varphi(\omega') - \varphi(\omega) = (\gamma P(|\omega - \omega_*|)(\omega_* - \omega) + \eta D(\omega^2)) \varphi'(\omega) + \frac{1}{2} (\gamma P(|\omega - \omega_*|)(\omega_* - \omega) + \eta D(\omega^2))^2 \varphi''(\tilde{\omega}), \]

where, for some \( 0 \leq \theta \leq 1 \)

\[ \tilde{\omega} = \theta \omega' + (1 - \theta) \omega. \]

In addition we substitute this expansion in the collision operator (2.7) and remark that, with the properties \( \langle \eta \rangle = 0 \) and \( \langle \eta^2 \rangle = \sigma^2 \), we obtain

\[ \frac{d}{d\tau} \int_I g(\omega) \varphi(\omega) d\omega = \]
\[
\int_{I} \left[ -K(\omega)\varphi' (\omega) + \frac{\sigma^2}{2\lambda} D(\omega^2)^2 \varphi''(\omega) \right] g(\omega, \tau) d\omega + R(\gamma, \sigma). \tag{2.20}
\]

As it will be presented later, the remainder \( R \) converges to zero as \( \gamma \to 0 \). However, some definitions are needed.

We consider \( \mathcal{M}_0(A) \) to be the space of all probability measures taking values in \( A \subseteq \mathbb{R} \) and the space of all Borel probability measures of finite momentum of order \( p \) is defined by

\[
\mathcal{M}(A) = \left\{ \Theta \in \mathcal{M}_0 : \int_A |w|^p d\Theta(w) < +\infty, p \geq 0 \right\}, \tag{2.21}
\]

where provided with the topology of the weak convergence of the measures.

Furthermore, let \( \mathcal{F}(I) \) be the class of all real functions \( h \) on \( I \) such that \( h(\pm 1) = h'(\pm 1) = 0 \), and the \( m \)-th derivative of \( h \), \( h^m(v) \) is Hölder continuous of order \( \sigma \),

\[
\|h^m\|_\delta = \sup_{v \neq w} \frac{|h^m(v) - h^m(w)|}{|v - w|^{\delta}} < \infty. \tag{2.22}
\]

The integer \( m \) and the number \( 0 < \sigma \leq 1 \) are such that \( m + \delta = s \). In order to present the rate of decay of moments, we suppose that the probability density \( \eta \) is obtained from a given random variable \( Y \), with zero mean and unit variance, that like \( \eta \) belongs to \( \mathcal{M}_p(A) \). Consequently, \( \eta \) of variance \( \sigma^2 \) is the density of \( \sigma Y \) and by this assumption, we obtain for any \( p > 0 \) such that the \( p \)-th moments of \( Y \),

\[
\langle |\eta|^p \rangle = \langle |\sigma Y|^p \rangle = \sigma^p \langle |Y|^p \rangle.
\]

Let us prove the remainder in (2.20).

**Proof.** The remainder \( R(\gamma, \sigma) \) in (2.20) is

\[
R(\gamma, \sigma) = \frac{1}{2\gamma} \left\{ \int_{I^2} \left[ (\gamma P(|\omega - \omega_*)|) \omega_* - \omega) + \eta D(\omega^2)(\varphi''(\omega) - \varphi(\omega))g(\omega, \tau)M(\omega_*)d\omega_* d\omega \right] \right\} \tag{2.23}
\]
2 Choice Formation

Since $\varphi \in F_{2+\delta}(I)$, respectively $|\tilde{\omega} - \omega| = \theta|\omega' - \omega|$, we obtain

$$|\varphi''(\tilde{\omega}) - \varphi''(\omega)| \leq \|\varphi''\|\delta|\tilde{\omega} - \omega|^\delta \leq \|\varphi''\|\delta|\omega' - \omega|^\delta.$$  \tag{2.24}

For this purpose,

$$|R(\gamma, \sigma)| \leq \frac{\|\varphi''\|\delta}{2\gamma} \left( \int_{I^2} |\gamma P(|\omega - \omega_s|)(\omega_s - \omega) + \eta D(\omega^2)|^{2+\delta}g(\omega)M_\omega(\omega_s)d\omega_s d\omega \right).$$

Based on the inequality and with regard to the functions $P(\cdot)$, $D(\cdot)$ taking values in $[0, 1]$, we have

$$|\gamma P(|\omega - \omega_s|)(\omega_s - \omega) + \eta D(\omega^2)|^{2+\delta} \leq 2^{1+\delta}(\gamma P(|\omega - \omega_s|)(\omega_s - \omega)|^{2+\delta} + |\eta D(\omega^2)|^{2+\delta})$$

$$\leq 2^{2+\delta}\gamma^{1+\delta} + 2^{1+\delta}|\eta|^{2+\delta},$$

Finally we obtain the bound

$$|R(\gamma, \sigma)| \leq 2^{1+\delta}\|\varphi''\|\delta \left( \gamma^{1+\delta} + \frac{1}{2\gamma} \langle|\eta|^{2+\delta}\rangle \right).$$  \tag{2.25}

Since $\eta$ is a probability density with zero mean and $\lambda \gamma$ variance, and $\eta$ belongs to $M_{2+\alpha}$, for $\alpha > \delta$,

$$\langle|\eta|^{2+\delta}\rangle = \langle|\lambda \gamma Y|^{2+\delta}\rangle = (\lambda \gamma)^{1+\delta/2}\langle|Y|^{2+\delta}\rangle,$$

at which $\langle|Y|^{2+\delta}\rangle$ is bounded. Insert this equality in the inequality of $|R(\gamma, \sigma)|$ \tag{2.25}, we observe that as $\gamma$ and $\sigma$ both converge to zero in such a way $\sigma^2 = \lambda \gamma$, $R(\gamma, \sigma)$ converges to zero as well.

Applying the same line from [1], we have the following theorem.

**Theorem 1.1.** Let the probability density $f_0 \in M_0(I)$, and let the symmetric random variable $\eta$ in \tag{2.4} have a density in $M_{2+\alpha}$. Then, as $\gamma \to 0, \sigma \to 0$ in such a way that $\sigma^2 = \lambda \gamma$, for any $\varphi \in F_{2+\delta}(I)$ with $\delta < \alpha$ the weak solution on the Boltzmann equation for the scaled density $g_\gamma(\omega, \tau) = f(\omega, t)$, with $\tau = \gamma t$ converges up to
extraction of a subsequence, to a probability density $g(\omega, \tau)$, namely
\[
\lim_{\gamma \to 0} \int_I \varphi(w) g_\gamma(w, \tau) dw = \int_I \varphi g(w, \tau) dw.
\]

Then if we integrate (2.20) back by parts. We obtain a weak solution of Fokker-Planck equation
\[
\frac{\partial g}{\partial \tau} = \lambda \frac{\partial^2}{\partial w^2} (D(w^2)g) + \frac{\partial}{\partial w} (K(w)g).
\] (2.26)

In brief, the Fokker-Planck equation (2.26) describes the asymptotic behavior of the solution of the Boltzmann equation (2.8), as the parameters $\gamma \ll 1, \sigma \ll 1$, in a such way $\sigma^2 = \lambda \gamma$. Similar to physical phenomena, the Fokker-Planck equation is related to the drift-diffusion equation. Clearly, on the right hand side of the equation (2.26), the first term is the diffusion part, while the compromise is related to the drift. Note that, the drift depends on the convolution operator $K(\omega)$ and it is not explicitly solvable in general. For that reason, the steady state $g_\infty$ can only be evaluated for a particular choice of the two functions $P(\cdot), D(\cdot)$.

### 2.3 The steady state

In this section we analyze the asymptotic behavior of the Fokker-Planck equation which is derived by the Boltzmann equation, with an explicitly computable steady state and the effects of the varied model parameters by presenting different simulations for the steady profile of the Fokker-Planck model. Then the structure of the steady state indicates the formation of opinion result in the choice of the interaction dynamic.

In the previous section the solutions of the first-order differential equation, steady profiles $g_\infty(\omega)$, which are denoted the final distribution of opinion around the possible choices are presented, it reads
\[
\frac{d}{d\omega} (D(\omega^2)g) = -\frac{2}{\lambda} K(\omega) g
\] (2.27)

Subject to the mass constraint
\[
\int_{\mathbb{R}} g_\infty(\omega)d\omega = 1.
\] (2.28)
Moreover, in adaption to the realistic results, we will exclude solutions that do not satisfy the constraints on the lateral boundaries

\[ g_\infty(-1) = g_\infty(1) = 0. \] (2.29)

We remark that, the steady profile \( g_\infty \) of equation bases on the binary interactions involving both compromise and diffusion properties in exchanges between individuals which are described by the functions \( P(\cdot) \) and \( D(\cdot) \). Additionally, as we have described, the parameter \( \lambda \), which is the quotient between the compromise and diffusion parameters \( \lambda = \sigma^2/\gamma \), influence the right hand side of the equation (2.27). Depending on the values of \( \lambda \) it can lead to the primary question of describing the choice formation in a diffusion-dominated society \( (\sigma^2/\gamma \to \infty) \) or compromise dominated society \( (\sigma^2/\gamma \to 0) \). Indeed, a bad choice of \( \lambda \) will maintain violation of the constraint (2.29).

Note that, since the diffusion operator does not influence the evolution of both the mass and momentum, the final mean opinion will not be affected by the parameter \( \lambda \). As it will be shown in the last chapter, the choice of \( \lambda \) only affects the shape of the steady profile, the smaller the value of \( \lambda \) the sharper is the steady profile. To fulfill all the assumptions made in section 1.3., especially that the extremal opinions are more difficult to change, we propose the diffusion function \( D(\cdot) \) can be defined in the simulations by

\[ D(\omega) = (1 - \omega^2)^\alpha, \] (2.30)

which clearly is a decreasing function of \( \omega^2 \) with \( D(1) = 0 \). Furthermore the diffusion of opinions is more noticeable when the choice of \( \alpha \) is linked to the choice of the zone around the neutral choice point \( \omega = 0 \). The compromise function \( P(\cdot) \), which is related to the influence zone of Maxwellian opinion (2.3), can be chosen, for example:

\[ P(\omega) = e^{-r|\omega - \bar{\omega}|} \] (2.31)

or

\[ P(\omega) = \chi(|\omega - \bar{\omega}| \leq \rho). \] (2.32)

In both of these equations, \( \bar{\omega} \) is one of the points where the Maxwellian (2.3) is localized. The parameter \( \rho \) is a non-negative constant and \( \chi(A) \) is denoted as usual the indicator function of the set \( A \), with \( \chi(x) = 1 \) if \( x \in A \) and \( \chi(x) = 0 \), if \( x \notin A \). The difference between these two equation is, in the first one \( P \), although the distance of the two opinions is unlimited, one of the option in \( \bar{\omega} \) can attract the individual of opinion \( \omega \). In contrast to the second equation the option is limited to opinions which belongs to a
limited interval centered in $\tilde{\omega}$. In addition the parameter $\rho$ determines how intensive
the range of the interaction should be. Because of these characteristics both functions
are suitable for the long and short interaction between agents. In order to better test
the role of the different parameters, the simulations will be constricted to the case of
the formation of choice between two opinions. Additionally, the Maxwellian will be
given by,

$$M_2(\omega) = p\delta(\omega - \omega_-) + q\delta(\omega - \omega_+), \quad p + q = 1, \quad \omega_- < 0, \quad \omega_+ > 0. \quad (2.33)$$

For the case of multiple choices, the same code will be used for the simulation.
To illustrate the role of the various parameters, we set for example $\alpha = 1/2$ in (2.30),
and $r = 0$ in (2.31). We obtain $D(\omega^2) = 1 - \omega^2$ and $P(\omega) = 1$. As described before
$K(\omega) = \omega - \langle w_\ast \rangle_M$ and the steady state profile $g_\infty$ is a solution to

$$\frac{d}{d\omega}((1 - \omega^2)g) = -\frac{2}{\lambda}(\omega - m)g. \quad (2.34)$$

Here, $m$ is the expectation value of the Maxwellian, respectively $m = \langle w_\ast \rangle_M = p\omega_- + q\omega_+$, excluded $\omega_-$ and $\omega_+$ are the extreme opinions, which implies $-1 < m < 1$. If we
integrate the equation above, it reads,

$$g_\infty(\omega) = C(1 + \omega)^{\frac{1+m}{\lambda}}(1 - \omega)^{\frac{1-m}{\lambda}}. \quad (2.35)$$

With attention to the constraints of $m$, the function $g_\infty$ can be integrated on the
interval $I$. Hence, it leads to the conclusion, that for a suitable choice of the constant
$C$, the mass constraint (2.28) can be always satisfied. However, the constraints (2.29)
only are satisfied if

$$\min \left\{ \frac{1+m}{\lambda} - 1, \frac{1-m}{\lambda} - 1 \right\} > 0. \quad (2.36)$$
3 Leadership

The previous model is presented on the hypothesis that all individuals in the society play the same role with respect to the exchange of information. However, there is a highlighted group of agents that affects the opinion formation dynamic, namely the leaders’ population. Following Lazarsfeld’s theory [5], interpersonal communication is much more influential than direct media effects. In particular, the opinion leaders are active media users that select, interpret, modify, facilitate, and transmit the information, and in contrast to the other group, the followers’ population is more passive and ductile. Furthermore, one of the typical characteristics of leaders is described by the public individuation, which denotes people who feel the urge to differentiate themselves and act unique from the other people. Certainly, the other characteristic features of opinion leaders are: High confidence, high self-esteem, and the ability to withstand criticism. The main emphasis of the following model will be the control problem in which mentioning how the collective behavior corresponds to the formation process of opinion consensus. Based on a concept of a hierarchical leadership, the leaders aim at controlling the followers. In order to influence the followers effectively, a control strategy is added through a suitable cost function.

This chapter is organized as follows: In the first part we deal with the microscopic model of the leader strategy in the leader-follower interactions and afterwards following the ideas of [4], a Boltzmann type optimal control is approached; later on, the main properties of the kinetic model are studied. In this section the idea that the leaders’ control strategy may lead the followers’ opinion toward the desired state is emphasized. Finally with the aid of so-called quasi invariant opinion limit, the Fokker-Planck equation corresponding to the model is derived.
3.1 Microscopic models of opinion control through leaders

Similar to the first chapter, the binary exchange of information is essentially responsible for the interaction in opinion formation. Different to that classical approach, we are particularly interested in the opinion formation process of a followers’ population controlled by the action of a leaders’ population. In order to succeed it, the leaders use an applicable control strategy which is based on the interplay between the leaders’ desired state and the need to keep it close to the mean opinion of followers’ population. In this section, a differential system describing the evolution of the population of leaders and followers is introduced. Later we present a binary interaction model for the same dynamics, to show the relation between these two descriptions.

3.1.1 Microscopic modeling

Let us suppose there are two different groups of individuals, the followers and the leaders. Each evolution of a population, there are $N_L$ leaders and $N_F$ followers. We label the opinions $\omega_i, \tilde{\omega}_k \in I = [-1, 1]$, for $i = 1, \ldots, N_F$ and $k = 1, \ldots, N_L$ and these opinions can change over time according to,

$$\dot{\omega}_i = \frac{1}{N_F} \sum_{j=1}^{N_F} P(\omega_i, \omega_j)(\omega_j - \omega_i) + \frac{1}{N_L} \sum_{h=1}^{N_L} S(\omega_i, \tilde{\omega}_h)(\tilde{\omega}_h - \omega_i),$$

(3.1)

while $\omega_i(0) = \omega_{i,0}$ and

$$\dot{\tilde{\omega}}_k = \frac{1}{N_L} \sum_{h=1}^{N_L} R(\tilde{\omega}_k, \tilde{\omega}_h)(\tilde{\omega}_h - \tilde{\omega}_k) + u \quad \tilde{\omega}_k(0) = \tilde{\omega}_{k,0},$$

(3.2)

The functions $P(\cdot, \cdot), S(\cdot, \cdot)$ and $R(\cdot, \cdot)$ model the local relevance of compromise for a given opinion. To ensure that the post-opinions remain in interval $I$, they take value in $[0, 1]$. The strategy of the leaders is characterized by the control term $u$ which is given by the solution of the following optimal control problem over a certain time horizon $T$

$$u = \arg\min_u \{ J(u, \omega, \tilde{\omega}) \},$$

(3.3)
where

\[ J(u, \omega, \tilde{\omega}) = \frac{1}{2} \int_0^T \left\{ \frac{\psi}{N_L} \sum_{h=1}^{N_L} (\tilde{\omega}_h - \omega_d) + \frac{\mu}{N_L} \sum_{h=1}^{N_L} (\tilde{\omega}_h - m_F)^2 \right\} ds + \int_0^T \frac{\nu}{2} u^2 ds \quad (3.4) \]

Note that, \( \omega \) is the vector with followers’ opinions and \( \tilde{\omega} \) with the leaders’ opinion. The value \( \omega_d \) is the leaders’ desired opinion and \( m_F \) is the mean opinion of the followers’ population at time \( t \geq 0 \), which is given by

\[ m_F = \frac{1}{N_F} \sum_{j=1}^{N_F} \omega_j. \]

Additionally, the importance of the control \( u \) in the general dynamic is represented by a regularization parameter \( \nu > 0 \), which weakens the control action for a large value \( \nu \) and vice versa.

As it will be presented later, if we split the time intervals of length \( \Delta t \) and let \( t^n = \Delta tn. \)

The optimal control problem can be drafted as a constrained minimization problem for \( u^n, \omega^n \) and \( \tilde{\omega}^n \) in term of

\[ \min \{ J(u^n, \omega^n, \tilde{\omega}^n) \} \quad (3.5) \]

subject to (3.1) and (3.2).

As usual, it is difficult to solve this problem, especially for nonlinear constraints and a non-convex functional. For this reason henceforth we assume that sufficient regularity condition of (3.5) in such a manner that any minimizer \( u \equiv u^n \) to problem (3.3) fulfills the necessary first order optimality conditions. For more details about necessary and sufficient optimality conditions, we refer to [12].

For this purpose the control strategy of the leaders’ group is assessed by the interplay between two positive parameters \( \psi \) and \( \mu \) such that \( \psi + \mu = 1 \). The first parameter is responsible for minimizing the distance related to the desire state \( \omega_d \) (radical behavior) and the second, for minimizing the distance pertaining to the followers’ mean opinion (populistic behavior). Consequently, the aim that the leader control the followers’ opinion interaction, is illustrated by the function \( S(\cdot) \) and the followers’ mean opinion in (3.4) describes how the followers affect the leaders’ strategy. In order to avoid the complexity of the case when the number of the agents is large, this optimization problem is approached by the Boltzmann type optimal control [4], which describes the microscopic binary model predictive control of (3.1)–(3.2).
3.1.2 Instantaneous binary opinion exchange through leaders

To derive a computable control $u$ at any time $t$, we adapt the idea of the moving instantaneous control. Ordinarily, the solution in (3.1)-(3.3) will be suboptimal. As a consequence we apply in the following the model predictive control framework which is called receding horizon strategy or instantaneous control in the engineering literature. This then allows us to express the control as a feedback of the state variables. Firstly, we split the time interval $[0, T]$ in $M$ time intervals of length $\Delta t$ and let $t^n = \Delta tn$. In addition, the control $u$ is assumed as constant piecewise to solve the optimal control problem in each time interval until we reach $n\Delta t = T$. Briefly, at the beginning (3.1) and (3.2) are approximated by the discretized binary dynamics.

\[
\begin{align*}
\omega_{n+1}^i &= \omega_n^i + \alpha P(\omega_n^i, \omega_n^j)(\omega_n^j - \omega_n^i) + \alpha S(\omega_n^i - \tilde{\omega}_l, \omega_n^l - \omega_n^i) \\
\omega_{n+1}^j &= \omega_n^j + \alpha P(\omega_n^j, \omega_n^i)(\omega_n^i - \omega_n^j) + \alpha S(\omega_n^i, \tilde{\omega}_l, \omega_n^j - \omega_n^i)
\end{align*}
\]

in which $\alpha = \Delta t/2$, $i$, and $j$ are the indices of the two interacting followers, $l$ is the index of an arbitrary leader, $h$ and $k$ denote the indices of the two interacting leaders. The control variable $u$ is determined by the solution of the optimization problem, which is

\[
\begin{align*}
u^n &= \arg\min \{ J(u^n, \omega^n, \tilde{\omega}^n) \} \\
J(u^n, \omega^n, \tilde{\omega}^n) &= \alpha \left( \frac{\psi}{2} \sum_{p=\{k,h\}} (\tilde{\omega}_p - \omega_d)^2 + \frac{\mu}{2} \sum_{p=\{k,h\}} (\tilde{\omega}_p - m_F^n)^2 + v(u^n)^2 \right).
\end{align*}
\]

Following the idea in [4], to solve the minimization problem in (3.8), the control variable $u^n$ is explicitly computed by using a standard Lagrange multipliers approach. Then the feedback control is given by

\[
2\alpha u^n = -\sum_{p=\{k,h\}} \frac{2\alpha^2}{v} \left[ \psi(\tilde{\omega}_p^{n+1} - \omega_d) + \mu(\tilde{\omega}_p^{n+1} - m_F^{n+1}) \right].
\]
We remark that, the constrained binary interaction (3.7) is implicitly defined, because $u^n$ in (3.10) depends on the post-interaction opinion. However, it can be explicitly defined as well. To have a fully explicit expression, $m^n_{F}^{n+1}$ is approximated by $m^n_{F}$, it reads

$$2\alpha u^n = - \sum_{p=(k,h)} \frac{\beta}{2} \left[ \psi(\tilde{\omega}_p^n - \omega_d) + \mu(\tilde{\omega}_p^n - m^n_F) \right] - \frac{\alpha \beta}{2} (R(\tilde{\omega}_k^n, \tilde{\omega}_h^n) - R(\tilde{\omega}_h^n, \tilde{\omega}_k^n))(\tilde{\omega}_k^n - \tilde{\omega}_h^n),$$

(3.11)

where the parameter $\beta$ is defined as

$$\beta = \frac{4\alpha^2}{v + 4\alpha^2},$$

(3.12)

### 3.1.3 Boltzmann type control

In the previous model, the evolution of the distribution of opinions is described by a kinetic equation of Boltzmann type. Comparatively, we borrowed the Boltzmann dynamic adapted to the above instantaneous control formulation. Firstly, we introduce a density distribution of followers $f_F$ and leaders $f_L$. Again the corresponding opinion $\omega, \tilde{\omega}$ take values in $I$ and time $t \geq 0$. The followers’ density is normalized to 1, it is

$$\int_I f_F(\omega,t)d\omega = 1,$$

while the leader’s density is

$$\int_I f_L(\tilde{\omega},t)d\tilde{\omega} = \rho \leq 1.$$

We remark that, the Boltzmann- like collision operator which will be presented later, are derived by standards methods of kinetic theory, considering that the change in time of $f_F(\omega,t)$ and $f_L(\tilde{\omega},t)$ due to binary interaction and the leaders’ strategy depend on a balance between the gain and loss of individuals with opinions $\omega, \tilde{\omega}$. 
3.1.4 Binary constrained interactions dynamic

Let us consider the pairwise opinions of two followers \((\omega, v)\) and opinions of the leaders \((\tilde{\omega}, \tilde{v})\). The post- interactions between the two populations are described by three dynamics, the interaction between two followers, the interaction between leaders and between follower and leader. Firstly the post- interaction opinions \((\tilde{\omega}, \tilde{v})\) of two leaders are given by

\[
\begin{align*}
\tilde{\omega}^* &= \tilde{\omega} + \alpha R(\tilde{\omega}, \tilde{v})(\tilde{v} - \tilde{\omega}) + 2\alpha u + \tilde{\theta}_1 \tilde{D}(\tilde{\omega}) \\
\tilde{v}^* &= \tilde{v} + \alpha R(\tilde{v}, \tilde{\omega})(\tilde{\omega} - \tilde{v}) + 2\alpha u + \tilde{\theta}_2 \tilde{D}(\tilde{v})
\end{align*}
\] (3.13)

and the feedback control is defined as

\[
2\alpha u = -\frac{\beta}{2} [\psi((\tilde{\omega} - \omega_d) + (\tilde{v} - \omega_d)) + \mu((\tilde{\omega} - m_F) + (\tilde{v} - m_F))] \\
-\frac{\alpha \beta}{2} (R(\tilde{\omega}, \tilde{v}) - R(\tilde{v}, \tilde{\omega}))(\tilde{v} - \tilde{\omega}),
\] (3.14)

We remark that \(m_F(t)\) is the average opinion of the follower population, it reads

\[
m_F(t) = \int_I f_F(\omega, t)\omega d\omega
\] (3.15)

In (3.13) we added a control term from the previous section in the binary interactions. Moreover the binary interaction includes a noise term which is characterized by the random variables \(\tilde{\theta}_1\) and \(\tilde{\theta}_2\) with the identical distribution of zero mean and finite variance \(\tilde{\sigma}^2\). The noise term is assessed by the function \(0 \leq \tilde{D}(\cdot) \leq 1\), which represents the local relevance of diffusion for a given opinion. Identical to the previous chapter, the post-interaction between follower and follower is given by

\[
\begin{align*}
\omega^* &= \omega + \alpha P(\omega, v)(v - \omega) + \theta_1 D(\omega), \\
v^* &= v + \alpha P(v, \omega)(\omega - v) + \theta_2 D(v),
\end{align*}
\] (3.16)

Correspondingly the diffusion function \(D(\cdot)\) and the diffusion variables \(\theta_1, \theta_2\) have the same properties. At last the interaction between leader with the opinion \(\tilde{v}\) and follower with the opinion \(\omega\) is described for every individual from the leaders’ group, it reads

\[
\begin{align*}
\omega^{**} &= \omega + \alpha S(\omega, \tilde{v})(\tilde{v} - \omega) + \hat{D}(\omega) \\
\tilde{v}^{**} &= \tilde{v}
\end{align*}
\] (3.17)
Again, $\hat{\theta}$ is a random variable with zero mean and variance $\hat{\sigma}^2$ and $0 \leq \hat{D}(\cdot) \leq 1$. Note that in all three dynamics, we have to ensure the individuals’ opinions do not cross the boundaries, namely $I = [-1, 1]$. In the absence of the diffusion contribution ($\tilde{\theta}_1, \tilde{\theta}_2 \equiv 0$), the binary interaction for the leaders implies that $|\tilde{\omega}^* - \tilde{v}^*|$ is a contraction if $\alpha \leq 1/2$ and

$$|\tilde{\omega}^* - \tilde{v}^*| = |(\tilde{\omega} - \tilde{v}) - \alpha(\tilde{\omega} - \tilde{v})(R(\tilde{\omega}, \tilde{v}) + R(\tilde{\omega}, \tilde{v}))| \leq |1 - 2\alpha||\tilde{\omega} - \tilde{v}|.$$  

To ensure the bounds for the leaders’ interaction (3.13) is not violated, the following sufficient conditions is given below.

**Proposition 2.1**

Let $r, d_+$ and $d_-$ be defined as follows

$$r = \min_{\bar{v}, \bar{\omega} \in I} [R(\bar{v}, \bar{\omega})], \quad d_+ = \min_{\bar{\omega} \in I} \left[ \frac{1}{\hat{D}(\bar{\omega})}, \hat{D}(\bar{\omega}) \neq 0 \right] \quad (3.18)$$

If $\bar{v}, \bar{\omega} \in I$ then $\bar{v}^*, \bar{\omega}^* \in I$ if the following conditions hold

$$\alpha r \geq \frac{\beta}{2}, \quad d_-(1 - \frac{\beta}{2}) \leq \hat{\theta}_i \leq d_+(1 - \frac{\beta}{2}), \quad i = 1, 2. \quad (3.19)$$

**Proof.** The proof follows the same arguments used in [4]. We will proceed in two subsequent steps: first we consider the case of interaction of the leaders without noise and then, the case including the noise action. Let us define the following quantity

$$\gamma = \alpha(1 - \frac{\beta}{2})R(\bar{\omega}, \bar{v}) + \frac{\alpha\beta}{2}R(\bar{v}, \bar{\omega}), \quad (3.20)$$

where $0 \leq \beta \leq 1/2$ by definition.

Thus its relation to (3.13) in the absence of noise can be rewritten as

$$\bar{\omega}^* = \left(1 - \gamma - \frac{\beta}{2}(\psi + \mu)\right)\bar{\omega} + \left(\gamma - \frac{\beta}{2}(\psi + \mu)\right)\bar{v} + \beta(\psi\omega_d + \mu m_F), \quad (3.21)$$

remarking that $\psi + \mu = 1$. Therefore it is sufficient that the following bounds are satisfied

$$\frac{\beta}{2} \leq \gamma \leq 1 - \frac{\beta}{2} \quad (3.22)$$
to have a convex combination of $\omega, v$ and $\omega_d$. From equation (3.20), by the assumption on $R(\tilde{\omega}, \tilde{v})$, we have $\alpha r \leq \gamma \leq \alpha$. So the bound equation requires that $\alpha r \geq \beta/2$, which is the first assumption (3.19).

Let us consider now the presence of noise, and we obtain

$$\tilde{\omega}^* = \left(1 - \gamma - \frac{\beta}{2}(\psi + \mu)\right)\tilde{\omega} + \left(\gamma - \frac{\beta}{2}(\psi + \mu)\right)\tilde{v} + \beta(\psi \omega_d + \mu m_F) + \tilde{\theta}_1 \tilde{D}(\tilde{\omega}),$$  
(3.23)

which implies the following inequalities

$$\tilde{\omega}^* \leq \left(1 - \gamma - \frac{\beta}{2}(\psi + \mu)\right)\tilde{\omega} + \left(\gamma - \frac{\beta}{2}(\psi + \mu)\right)\tilde{v} + \beta(\psi \omega_d + \mu m_F) + \tilde{\theta}_1 \tilde{D}(\tilde{\omega}) \leq \left(1 - \gamma - \frac{\beta}{2}(\psi + \mu)\right)\tilde{\omega} + \left(\gamma + \frac{\beta}{2}(\psi + \mu)\right)\tilde{v} + \tilde{\theta}_1 \tilde{D}(\tilde{\omega}).$$  
(3.24)

Finally, the last relation is bounded by one if

$$\tilde{\theta}_1 \leq (1 - \gamma - \frac{\beta}{2}) \frac{1 - \tilde{\omega}}{\tilde{D}(\tilde{\omega})}, \quad \tilde{D}(\tilde{\omega}) \neq 0,$$
(3.25)

which yields the second condition in (3.19). The same results apply for the post interaction opinion $\tilde{v}^*$ as well.

Finally, since $|\omega| \leq 1$ which leads to $|v - \omega| \leq 1$, $0 \leq P(\cdot, \cdot) \leq 1$, in the absence of diffusion, the binary interaction between followers (3.16) always take place within the interval.

Let us consider the case for the interaction between follower and leader (3.17), the post-interaction opinion $\omega^{**}$ only takes value in the interval $I$ if the following condition is satisfied.

**Proposition 2.2.**

Let $K_-$ and $K_+$ be defined as follows

$$K_\pm = \min_{\omega \in I} \left[ \frac{1 \mp \omega}{\tilde{D}(\omega)}, \tilde{D}(w) \neq 0 \right].$$  
(3.26)

If $\omega \in I$ then $\omega^{**} \in I$ if the following conditions hold

$$(1 - \alpha)K_- \leq \tilde{\theta} \leq (1 - \alpha)K_+, \quad i = 1, 2.$$  
(3.27)
Proof. We consider the upper bound at \( \hat{v} = 1 \). To ensure that the opinions do not violate the bounds, we need,

\[
\omega^{**} = \omega + \alpha S(\omega, \hat{v})(\hat{v} - \omega) + \hat{\theta} \hat{D}(\omega) \leq 1.
\]  

(3.28)

Clearly, the worst case is \( \hat{v} = 1 \), where we have to ensure

\[
\hat{\theta} \hat{D}(\omega) \leq 1 - \omega - \alpha(1 - \omega)
\]

\[
= 1 - \omega + \alpha(\omega - 1) = (1 - \omega)(1 - \alpha).
\]  

(3.29)

If we consider the definition of \( K_{\pm} \), it suffices the inequalities from (3.27) by computation. \( \square \)

### 3.1.5 Boltzmann description

Along the same line as Chapter 1, we recover the time evolution of the distribution of opinion \( f \) as a balance between bilinear gain and loss of opinion terms, described in weak form by the integro- differential equation of Boltzmann type. First of all we consider the distribution function of followers \( f_F(\omega, t) \)

\[
\frac{d}{dt} \int_I \varphi(\omega)f_F(\omega, t)d\omega = (Q_F(f_F, f_F), \varphi) + (Q_{FL}(f_L, f_F), \varphi) 
\]

(3.30)

where

\[
(Q_F(f_F, f_F), \varphi) = \left\langle \int_{I^2} B^F_{int}(\varphi(\omega^*) - \varphi(\omega))f_F(\omega, t)f_F(v, t)d\omega dv \right\rangle
\]

(3.31)

and

\[
(Q_{FL}(f_F, f_L), \varphi) = \left\langle \int_{I^2} B^{FL}_{int}(\varphi(\omega^{**}) - \varphi(\omega))f_F(\omega, t)f_L(\hat{v}, t)d\omega dv \right\rangle
\]

(3.32)

As usual, in (3.31) and (3.32) \( \langle \cdot \rangle \) denotes the expectation with respect to the random variables \( \theta_1, \theta_2 \) and \( \hat{\theta} \). The interaction kernels \( B^F_{int} > 0 \) and \( B^{FL}_{int} > 0 \) are parameters, which govern the probability that an exchange of opinions can occur, the interaction
kernels are given by

\[ B^F_{int} = B^F_{int}(\omega, v, \theta_1, \theta_2) = \eta_F \chi(|\omega^*| \leq 1) \chi(|v^*| \leq 1) \]  
(3.33)

\[ B^{FL}_{int} = B^{FL}_{int}(\omega, \tilde{v}, \tilde{\theta}) = \eta_{FL} \chi(|\omega^{**}| \leq 1) \chi(|\tilde{v}| \leq 1), \]

where \( \eta_F, \eta_{FL} \) are positive constant relaxation rates and \( \chi(\cdot) \) is the indicator function. Hence, the presence of the interaction kernel ensures that the post-interaction opinions belong to the correct interval \( I \). Let us assume that \( |\omega^*|, |\omega^{**}| \leq 1 \), the interaction dynamic of \( f_F \) can be described by the following Boltzmann operators

\[ (Q_F(f_F, f_F), \phi) = \eta_F \left\langle \int_{I^2} (\phi(\omega^*) - \phi(\omega)) f_F(\omega, t) f_F(v, t) d\omega dv \right \rangle \]  
(3.34)

\[ (Q_{FL}(f_F, f_L), \phi) = \eta_{FL} \left\langle \int_{I^2} (\phi(\omega^{**}) - \phi(\omega)) f_F(\omega, t) f_L(\tilde{v}, t) d\omega d\tilde{v} \right \rangle. \]  
(3.35)

Let us choose \( \phi = \omega \) in the Boltzmann equation in the weak form for followers (3.30), this particular choice gives the evolution of the average opinion of followers \( m_F(t) \), we have

\[ \frac{d}{dt} m_F(t) = \frac{\eta_F}{2} \left[ \int_{I^2} (\omega^* + v^* - \omega - v) f_F(\omega, t) f_F(v, t) d\omega dv \right] \]

\[ + \eta_{FL} \int_{I^2} (\omega^{**} - \omega) f_F(\omega, t) f_L(\tilde{v}, t) d\omega d\tilde{v}. \]  
(3.36)

In this case, since the mean value of the random variables \( \theta_1, \theta_2 \) (3.16) is zero, if we set the definitions from (3.16) in the equation above, we have

\[ \frac{d}{dt} m_F(t) = \frac{\eta_F}{2} \alpha \int_{I^2} (v - \omega)(P(\omega, v) - P(v, \omega)) f_F(\omega, t) f_F(v, t) d\omega dv \]

\[ + \eta_{FL} \alpha \int_{I^2} S(\omega, \tilde{v})(\tilde{v}, \omega) f_F(\omega, t) f_L(\tilde{v}, t) d\omega d\tilde{v}. \]  
(3.37)

Since \( P \) is assumed to be symmetric \( P(\omega, v) = P(v, \omega) \) respectively \( S \equiv 1 \), it leads us to a simplified equation for the time evolution of \( m_F \)

\[ \frac{d}{dt} m_F(t) = \tilde{\eta}_{FL} \alpha (m_L(t) - m_F(t)) \]  
(3.38)
where we define

\[ \tilde{\eta}_{FL} = \rho \eta_{FL}, \quad m_L(t) = \frac{1}{\rho} \int \tilde{\omega} f_L(\tilde{\omega}, t) d\tilde{\omega} \]

We obtain the evolution equation for \( m_L(t) \) with the comparative arguments. For the dynamic of \( f_L(\tilde{\omega}, t) \), an integro- differential equation of Boltzmann type in weak form is given by

\[
\frac{d}{dt} \int \phi(\tilde{\omega} f_L(\tilde{\omega}, t)) d\tilde{\omega} = (Q_L(f_L, f_L), \phi) \quad (3.39)
\]

where

\[
(Q_L(f_L, f_L), \phi) = \left\langle \int_{I^2} B_{int}(\phi(\tilde{\omega}^*) - \phi(\tilde{\omega}^*)) f_L(\tilde{\omega}, t) f_L(\tilde{v}, t) d\tilde{\omega} d\tilde{v} \right\rangle. \quad (3.40)
\]

Likewise, \( \langle \cdot \rangle \) denotes the expectation with respect to the random variables \( \tilde{\theta}_1, \tilde{\theta}_2 \) and the kernel \( B_{int} \) is related to the details of the binary interaction and it is defined as

\[
B_{int} = B_{int}(\tilde{\omega}, \tilde{v}, \tilde{\theta}_1, \tilde{\theta}_2) = \eta_L \chi(|\tilde{\omega}^*| \leq 1) \chi(|\tilde{v}^*| \leq 1), \quad (3.41)
\]

where \( \eta_L > 0 \) is a constant rate and \( \chi(\cdot) \) as usual an indicator function. If \( \varphi = \tilde{\omega} \), it follows

\[
\frac{d}{dt} \int I \phi(\tilde{\omega} f_L(\tilde{\omega}, t)) d\tilde{\omega} = \eta_L \left\langle \int_{I^2} (\tilde{\omega}^* - \tilde{\omega}) f_L(\tilde{\omega}, t) f_L(\tilde{v}, t) d\tilde{\omega} d\tilde{v} \right\rangle, \quad (3.42)
\]

then setting the definitions of the opinion dynamic between leader and follower (3.17) in (3.42), we have

\[
\frac{d}{dt} \int I \tilde{\omega} f_L(\tilde{\omega}, t) d\tilde{\omega} = \frac{\eta_L}{2} \left\langle \int_{I^2} (\tilde{\omega}^* + \tilde{\omega}^* - \tilde{\omega} - \tilde{\omega}) f_L(\tilde{\omega}, t) f_L(\tilde{v}, t) d\tilde{\omega} d\tilde{v} \right\rangle \quad (3.43)
\]

Considering the \( \langle \tilde{\theta}_i \rangle = 0 \) with \( i = 1, 2 \) we obtain

\[
\frac{d}{dt} m_L(t) = \eta_L \alpha (1 - \beta) \frac{1}{\rho} \int J^2 (R(\tilde{\omega}, \tilde{v}) - R(\tilde{\omega}, \tilde{\omega})) \tilde{\omega} f_L(\tilde{\omega}, t) f_L(\tilde{v}, t) d\tilde{\omega} d\tilde{v} \]

\[
+ \tilde{\eta}_L \beta (\omega_d - m_L(t)) + \tilde{\eta}_L \beta \mu (m_F(t) - m_L(t)), \quad (3.44)
\]

where \( \tilde{\eta}_L = \rho \eta_L \).
If \( R(\tilde{\omega}, \tilde{v}) \) is supposed to be symmetric, the evolution equation becomes
\[
\frac{d}{dt} m_L(t) = \tilde{\eta}_{L} \phi \beta (\omega_d - m_L(t)) + \tilde{\eta}_{L} \mu \beta (m_F(t) - m_L(t)).
\] (3.45)

The time evolution of \( m_F \) (3.38) and \( m_L \) (3.45) are explicitly solvable, if the assumptions for \( m_F \) still hold, we have
\[
\begin{align*}
\left\{ \begin{array}{l}
m_L(t) = C_1 \exp\{-|\lambda_1|t\} + C_2 \exp\{-|\lambda_2|t\} + \omega_d \\
m_F(t) = C_1 (1 + \frac{\lambda_1}{\beta \tilde{\eta}_{L}} \exp\{-|\lambda_1|t\}) + C_2 (1 + \frac{\lambda_2}{\beta \tilde{\eta}_{L}}) \exp\{-|\lambda_2|t\} + \omega_d 
\end{array} \right.
\end{align*}
\] (3.46)

where \( C_1, C_2 \) depend on the initial data \( m_F(0), m_L(0) \) and they are defined as
\[
C_1 = -\frac{1}{\lambda_1 - \lambda_2} ((\beta \tilde{\eta}_L m_L(0) + \lambda_1) m_L(0) - \mu \beta \tilde{\eta}_L (0) - (\lambda_2 + \beta \tilde{\eta}_L \psi) \omega_d)
\] (3.47)

\[
C_2 = \frac{1}{\lambda_1 - \lambda_2} ((\beta \tilde{\eta}_L m_L(0) + \lambda_1) m_L(0) - \mu \beta \tilde{\eta}_L m_F(0) - (\lambda_1 + \beta \tilde{\eta}_L \psi) \omega_d)
\] (3.48)

with
\[
\lambda_{1,2} = -\frac{1}{2} (\alpha \tilde{\eta}_{FL} + \beta \tilde{\eta}_{L}) \pm \frac{1}{2} \sqrt{(\alpha \tilde{\eta}_{FL} + \beta \tilde{\eta}_{L})^2 - 4 \psi \alpha \beta \tilde{\eta}_L \tilde{\eta}_{FL}}.
\] (3.49)

In order to ensure the contribution of the initial averages, \( m_L(0), m_F(0) \) disappears as soon as time increases and the mean opinions of leaders and followers converge towards the desired state \( \omega_d \), the parameters \( \lambda_1, \lambda_2 \) are always assumed to be negative.

Let us now fix \( \varphi(\omega) = \omega^2 \). We obtain
\[
\begin{align*}
E_F(t) &= \int_I \omega^2 f_F(\omega, t) d\omega, \\
E_L(t) &= \frac{1}{\rho} \int_I \tilde{\omega}^2 f_L(\tilde{\omega}, t) d\tilde{\omega}.
\end{align*}
\]

In the case of the followers group from (3.30), it reads
\[
\frac{d}{dt} E_F(t) = \frac{\eta_F}{2} \left( \int_I (\omega^*)^2 + (v^*)^2 - \omega^2 - v^2 f_F(\omega, t) f_F(v, t) d\omega dv \right)
\]
\[
+ \eta_{FL} \left( \int_I (\omega^{**})^2 - \omega^2 f_F(\omega, t) f_L(v, t) d\omega dv \right).
\] (3.50)

Bearing in mind that the random variables have zero mean and variance \( \sigma^2 \). Further-
more for simplification we choose $P \equiv S \equiv 1$, if we consider the definitions from (3.16) and (3.17) into the equation above, we have

\[
\frac{d}{dt} E_L(t) = 2\eta_F \alpha (\alpha - 1) (E_F(t) - m_L^2(t)) + \tilde{\eta}_F \alpha^2 (E_L + E_F - 2m_L(t)m_F(t)) \\
+ 2\alpha \tilde{\eta}_F (m_F(t)m_L(t) - E_F(t)) + \eta_F \sigma^2 \int_I D^2(\omega) f_F(\omega, t) d\omega \\
+ \tilde{\eta}_F \sigma^2 \int_I \hat{D}^2(\omega) f_L(\omega, t) d\omega.
\]

(3.51)

Now for the leaders group, we set the testfunction $\varphi(\tilde{\omega}) = \tilde{\omega}^2$ in (3.39). Thanks to (3.13) and if $R \equiv 1$, we have

\[
\frac{d}{dt} E_L(t) = \frac{\eta_L}{2} \varrho \left\{ \int_I ((\tilde{\omega}^*)^2 + (\tilde{v}^*)^2 - \tilde{\omega}^2 - \tilde{v}^2) f_L(\tilde{\omega}, t) f_L(\tilde{v}, t) d\tilde{\omega} d\tilde{v} \right\} \\
= \tilde{\eta}_L \left[ 2\alpha (\alpha - 1) (E_L(t) - m_L^2(t)) - \frac{\beta}{2} (2 - \beta) (E_L(t) + m_L^2(t)) \\
+ 2\beta (1 - \beta) (\psi \omega_d + \mu m_F(t)) m_L(t) + \beta^2 (\psi \omega_d) \\
+ \mu m_F(t)^2 + \sigma^2 \int_I \hat{D}^2(\tilde{\omega}) f_L(\tilde{\omega}, t) d\tilde{\omega} \right].
\]

(3.52)

While in the limit $t \to \infty$, $m_F(t)$ and $m_L(t)$ converge to $\omega_d$. In absence of diffusion, we obtain that $E_F(t), E_L(t)$ converge exponentially to $\omega_d^2$. For that reason if we consider the quantities

\[
\int_I f_F(\omega, t)(\omega - \omega_d)^2 d\omega = E_F(t) + \omega_d^2 - 2m_F(t)\omega_d \\
\frac{1}{\rho} \int_I f_L(\tilde{\omega}, t)(\tilde{\omega} - \omega_d)^2 d\tilde{\omega} = E_L(t) + \omega_d^2 - 2m_L(t)\omega_d
\]

(3.53)

they converge to zero as $t \to \infty$. Consequently the steady state solution have the form of a Dirac delta centered in the desired opinion state.

\[
f_\infty(\omega) = \delta(\omega - \omega_d), \quad f_\infty(\tilde{\omega}) = \delta(\tilde{\omega} - \omega_d).
\]
3.2 Fokker-Planck limit system

In order to obtain simple analytic results on the large time behavior of the kinetic equation (3.30), a similar strategy to chapter 1 can be applied. We will obtain from the kinetic model a Fokker-Planck equation as the limit of the quasi invariant opinion. In this case we rescale the interaction frequencies $\eta_L, \eta_F, \eta_{FL}$, the propensity strength $\alpha$, the diffusion variances $\tilde{\sigma}^2, \sigma^2, \hat{\sigma}^2$ and the action of the control $\nu$ at the same time. Consequently, it leads to a certain asymptotic which maintain memory of the microscopic interaction.

If we define the parameter $\varepsilon > 0$ and we set

$$\alpha = \varepsilon, \quad \nu = \varepsilon\kappa, \quad \sigma^2 = \varepsilon\xi^2, \quad \hat{\sigma}^2 = \varepsilon\xi^2, \quad \tilde{\sigma}^2 = \varepsilon\xi^2,$$

$$\eta_F = \frac{1}{c_F\varepsilon}, \quad \eta_{FL} = \frac{1}{c_{FL}\varepsilon}, \quad \eta_L = \frac{1}{c_L\varepsilon}, \quad \beta = \frac{4\varepsilon}{\kappa + 4\varepsilon}.$$

We consider now the situation in which the interactions produce a small exchange of opinion, while at the same time as $\varepsilon \to 0$, the main properties of the kinetic system remain at a macroscopic level. To make these possible, we observe now the evolution of the scaled first two moments for $S \equiv 1$ and the assumptions that $P, R$ are symmetric. If we rescale the evolution of the first moment equations (3.38) and (3.45), it reads

$$\begin{cases}
\frac{d}{dt}m_F(t) = \varepsilon \frac{1}{c_{FL}\varepsilon}(m_L(t) - m_F(t)) \\
\frac{d}{dt}m_L(t) = \psi \frac{4\varepsilon}{c_L\varepsilon \kappa + 4\varepsilon} (\omega_d - m_L(t)) + \frac{\mu}{c_L\varepsilon \kappa + 4\varepsilon} (m_F(t) - m_L(t))
\end{cases}$$

(3.55)

Now letting $\varepsilon \to 0$, it follows that

$$\begin{cases}
\frac{d}{dt}m_F(t) = \frac{\rho}{c_{FL}}(m_L(t) - m_F(t)) \\
\frac{d}{dt}m_L(t) = \frac{4\rho}{c_L\kappa} [\psi (\omega_d - m_L(t)) + \mu (m_F(t) - m_L(t))].
\end{cases}$$

(3.56)
For the second moment equations (3.51) and (3.52), we have

\[
\frac{d}{dt} E_F(t) = (\varepsilon - 1) \frac{2}{c_F} (E_F(t) - m_F^2(t)) + \frac{\varepsilon \rho}{c_{FL}} (E_L(t) + E_F(t) - 2m_L(t)m_F(t)) \\
+ \frac{2\rho}{c_{FL}} (m_F(t)m_L(t) - E_F(t)) + \frac{\varsigma^2}{c_F} \int D^2(\omega) f_F(\omega, t) d\omega \\
+ \frac{\zeta^2 \rho}{c_{FL}} \int \hat{D}^2(\omega) f_F(\omega, t) d\omega.
\]

(3.57)

\[
\frac{d}{dt} E_L(t) = \frac{\rho}{c_L \varepsilon} \left[ 2\varepsilon (E_L(t) - m_L^2(t)) - \frac{2\varepsilon}{\kappa + 4\varepsilon} (2 - \frac{4\varepsilon}{\kappa + 4\varepsilon}) (E_L(t) + m_L^2(t)) \\
+ \frac{8\varepsilon}{\kappa + 4\varepsilon} (1 - \frac{4\varepsilon}{\kappa + 4\varepsilon}) (\psi \omega_d + \mu m_F(t)m_L(t)) \\
+ (\frac{4\varepsilon}{\kappa + 4\varepsilon})^2 (\psi \omega_d + \mu m_F(t))^2 + \tilde{\sigma}^2 \int \hat{D}^2(\hat{\omega}) f_L(\hat{\omega}, t) d\hat{\omega} \right].
\]

(3.58)

As \( \varepsilon \to 0 \), it reads

\[
\frac{d}{dt} E_F(t) = - \frac{2}{c_F} (E_F(t) - m_F^2(t)) + \frac{2\rho}{c_{FL}} (m_L(t)m_F(t) - E_F(t)) \\
+ \frac{\zeta^2}{c_F} \int D^2(\omega) f_F(\omega, t) d\omega + \frac{\zeta^2 \rho}{c_{FL}} \int \hat{D}^2(\omega) f_F(\omega, t) d\omega.
\]

(3.59)

\[
\frac{d}{dt} E_L(t) = - \frac{2\rho}{c_L} (E_L(t) - m_L^2(t)) - \frac{4\rho}{c_{L\kappa}} (E_L(t) + m_L^2(t)) \\
+ \frac{8\rho}{c_{L\kappa}} (\psi \omega_d + \mu m_F(t)m_L(t)) + \frac{\zeta^2 \rho}{c_L} \int \hat{D}^2(\hat{\omega}) f_L(\hat{\omega}, t) d\hat{\omega}.
\]

As it is shown, we are able to investigate the behavior of the first two moments of the solution by rescaling the parameters (3.54), following the same arguments in Chapter 1, we can derive the Fokker-Planck equation.
Let us consider the scaled equation (3.30) is given by

\[
\frac{d}{dt} \int_I \varphi(\omega) f_F(\omega, t) d\omega = \frac{1}{c_{FL}} \int_I \left( \int_I (\varphi(\omega^*) - \varphi(\omega)) f_F(\omega, t) f_F(v, t) d\omega dv \right) + \frac{1}{c_{FL}} \int_{I^2} \left( \int_I (\varphi(\omega^{**}) - \varphi(\omega)) f_F(\omega, t) f_L(\tilde{\omega}, t) d\omega d\tilde{\omega} \right).
\]

(3.60)

Furthermore we consider the Taylor expansion of \( \varphi \) around \( \omega \) up to second order

\[
\varphi(\omega^*) - \varphi(\omega) = (\omega^* - \omega) \varphi'(\omega) + \frac{1}{2} (\omega^* - \omega)^2 \varphi''(\omega)
\]

\[
\varphi(\omega^{**}) - \varphi(\omega) = (\omega^{**} - \omega) \varphi'(\omega) + \frac{1}{2} (\omega^{**} - \omega)^2 \varphi''(\hat{\omega})
\]

note that for some \( 0 \leq \nu_1, \nu_2 \leq 1 \)

\[
\varpi = \tilde{\theta}_1 \omega^* + (1 - \tilde{\theta}_1) \omega, \quad \hat{\omega} = \tilde{\theta}_2 \omega^{**} + (1 - \tilde{\theta}_2) \omega.
\]

Then, using the binary interactions (3.16)-(3.17), we insert this expansion into the interaction integral (3.60). As the parameter \( \varepsilon \to 0 \), it leads to

\[
\frac{d}{dt} \int_I \varphi f_F(t) d\omega = \frac{1}{c_F} \left[ \int_I \int I^2 P(\omega, v) (v.\omega) \varphi'(\omega) f_F(\omega, t) f_F(v, t) d\omega dv \right] + \frac{1}{c_{FL}} \left[ \int_I \int_{I^2} S(\omega, \tilde{\omega}) (\tilde{\omega} - \omega) \varphi'(\omega) f_F(\omega) f_L(\tilde{\omega}) d\omega d\tilde{\omega} \right] + \frac{1}{2 c_F} \int_I \varphi''(\omega) D^2(\omega) f_F(\omega, t) d\omega + \frac{1}{2 c_F} \int_I \hat{\varphi}''(\hat{\omega}) \hat{D}^2(\hat{\omega}) f_F(\omega, t) d\omega.
\]

(3.62)

When we integrate back by parts the last expression, the Fokker-Planck equation for the followers’ opinion distribution is read as

\[
\frac{\partial f_F}{\partial t} + \frac{\partial}{\partial \omega} \left( \frac{1}{c_F} K_F[f_F](\omega) + \frac{1}{c_{FL}} K_{FL}[f_L](\omega) \right) f_F(\omega) = \frac{\zeta^2}{2 c_F} \left( \frac{\varphi''(\omega)}{c_F} \hat{D}^2(\hat{\omega}) + \frac{\varphi''(\omega)}{c_{FL}} \hat{D}^2(\hat{\omega}) \right) f_F(\omega)
\]

(3.63)
where

\[ K_F[f_F](\omega) = \int_I P(\omega, v)(v - \omega)f_F(v, t)dv, \]  
\[ K_{FL}[f_L](\omega) = \int_I S(\omega, \tilde{\omega})(\tilde{\omega} - \omega)f_L(\tilde{\omega})d\tilde{\omega}. \]  

(3.64)

A likewise procedure for the leaders’ opinion distribution, we obtain

\[ \frac{\partial f_L}{\partial t} + \frac{\partial}{\partial \tilde{\omega}} \left( \frac{\rho}{c_L} H[f_L](\tilde{\omega}) + \frac{1}{c_L} K_L[f_L](\tilde{\omega}) \right) f_L(\tilde{\omega}) = \]  
\[ \frac{1}{2} \frac{\xi^2 \rho}{c_L} \frac{\partial^2}{\partial \tilde{\omega}^2} \hat{D}(\tilde{\omega}) f_L(\tilde{\omega}) \]  

(3.65)

where

\[ K[f_L](\tilde{\omega}) = \int_I R(\tilde{\omega}, \bar{v})(\bar{v} - \tilde{\omega})f_L(\bar{v}, t)d\bar{v} \]  

(3.66)

and

\[ H[f_L](\tilde{\omega}) = \frac{2\psi}{\kappa} (\tilde{\omega} + m(t) - 2\omega_d) \frac{2\mu}{\kappa} (\tilde{\omega} + m(t) - 2m_F(t)). \]  

(3.67)

### 3.3 Stationary solutions of the Fokker-Planck opinion model

Finally we analyze in some different cases of the interaction dynamic in (3.63) and (3.65), which it is possible to derive a Fokker-Planck equation with an explicitly computable steady states. For simplification we are forced to suppose \( P \equiv S \equiv R \equiv 1 \).

Additionally we assume the diffusion functions are given by the following structure,

\[ D(\omega) = \hat{D}(\omega) = \hat{D}(\omega) = 1 - \omega^2. \]  

(3.68)
The steady state of equations (3.63) and (3.65) are solutions of

\[
\left( \frac{1}{c_F} (m_F - \omega) + \frac{\rho}{c_{FL}} (m_L - \omega) \right) f_{F,\infty} = \frac{1}{2} \left( \xi^2 + \frac{\xi^2 \rho}{c_{FL}} \right) \frac{\partial}{\partial \omega} D^2(\omega) f_{F,\infty},
\]

(3.69)

\[
\left( \frac{2\psi}{\kappa} [\tilde{\omega} - 2\omega_d - m_L] + \frac{2\mu}{\kappa} [\tilde{\omega} - 2m_F + m_L] \right) f_{L,\infty} = \frac{1}{2} \frac{\xi^2 \rho}{c_L} \frac{\partial}{\partial \omega} g_{L,\infty},
\]

(3.70)

Let the time \( t \to \infty \) by dint of (3.56), the followers and the leaders’ mean opinion \( m_F \) and \( m_L \) relax to the desired opinion \( \omega_d \). It is

\[
\left( \frac{1}{c_F} + \frac{\rho}{c_{FL}} (\omega_d - \omega) \right) f_{F,\infty} = \frac{1}{2} \left( \xi^2 + \frac{\xi^2 \rho}{c_{FL}} \right) \frac{\partial}{\partial \omega} D^2(\omega) f_{F,\infty},
\]

(3.70)

that is

\[
\left( \frac{1}{c_F} + \frac{\rho}{c_{FL}} \right) (\omega_d - \omega) \frac{g_F}{D^2(\omega)} = \frac{1}{2} \left( \xi^2 + \frac{\xi^2 \rho}{c_{FL}} \right) \frac{\partial}{\partial \omega} g_F,
\]

(3.71)

where \( g_F = D^2(\omega) f_{F,\infty} \). Under the assumptions of the diffusion function and compromise functions are constant, we have

\[
g_{F,\infty} = a_F \exp \left\{ - \frac{2}{b_F} \int_0^\omega \frac{z - \omega_d}{(1 - z^2)^2} \mathrm{d}z \right\},
\]

(3.72)

\[
b_F = \frac{\xi^2 c_{FL} + \xi^2 c_F \rho}{c_{FL} + c_F \rho}
\]

where \( a_F \) is a normalization constant such that \( \int_I g_{F,\infty} \mathrm{d}\omega = 1 \). Then we obtain

\[
f_{F,\infty} = \frac{a_F}{(1 - \omega^2)^2} \exp \left\{ - \frac{2}{b_F} \int_0^\omega \frac{z - \omega_d}{(1 - z^2)^2} \mathrm{d}z \right\}.
\]

(3.73)

Following the same strategy, the steady state of equation (3.65) is a solution of

\[
- \left( \frac{2\psi}{\kappa} + \frac{2\mu}{\kappa} \right) (\omega_d - \tilde{\omega}) \frac{g_{L,\infty}}{D^2(\tilde{\omega})} = \frac{1}{2} \frac{\xi^2 \rho}{c_L} \frac{\partial}{\partial \omega} g_{L,\infty},
\]

(3.74)
with $g_{L,\infty} = f_{L,\infty} D^2(\omega)$. The solution of the differential equation above is

$$g_{L,\infty} = a_L \exp \left\{ - \frac{2}{b_L} \int_0^{\tilde{\omega}} \left( \frac{z - \omega_d}{(1 - z^2)^2} \right) dz \right\}, \quad b_L = \frac{\zeta \rho \kappa}{2c_L (\psi + \mu)}, \quad (3.75)$$

where $a_L$ is defined such that the mass of $g_{L,\infty}$ is equal to $\rho$. Finally the steady state is

$$f_{L,\infty} = \frac{a_L}{(1 - \tilde{\omega}^2)^2} \exp \left\{ - \frac{2}{b_L} \int_0^{\tilde{\omega}} \left( \frac{z - \omega_d}{(1 - z^2)^2} \right) dz \right\}. \quad (3.76)$$
4 Numerical examples

In the last part various numerical experiments based on the two previous models are shown. At first, we illustrate the influence of the different parameters in the formation of choice presented in the steady state. Later we focus on the Boltzmann type control model; how the different changes in control strategies affect the opinion dynamics. Finally we attempt to compare these two models by lowering the control term $u$ in the previous chapter to zero.

4.1 Choice Formation: Different roles of the model parameters

In order to present the different roles of the model parameters, the steady states $g_\infty$ depending on different parameters, are solved numerically and are plotted in various Figures. Again, $g_\infty$ is a solution of

$$\frac{d}{d\omega} (D(\omega^2)^2 g(w)) = -\frac{2}{\lambda} K(\omega) g(w)$$

Let us firstly examine the role of the diffusion function $D(\omega^2)$ depending on the values of the exponent $\alpha$ in (2.30). In Figure 4.1 the short-range interaction with $\rho = 0.5$ and $\lambda = 0.003$ are represented. Clearly, the steady state is more peaked with a larger value of the parameter $\alpha$. Also, by applying a more peaked $D$ leads to more attractiveness of the stable opinion ($\omega_+ = 0.6$).

In Figure 4.2, the influence of the diffusion parameter $\lambda$ is highlighted. As we can see, a small variation of $\lambda$ generates a more defined and sharpened shape of the steady profile. Note that, we consider the parameter $\alpha = 1/2$ for the diffusion function (2.30).

In a diffusion-dominated society ($\lambda = \sigma^2/\gamma \to \infty$), the steady profile tends to lose its dependence on the Maxwellian equilibrium. In brief, the different opinions are destroyed by self-thinking which is too pronounced.
Figure 4.1: The different roles of the diffusion function depending on the values of the exponent $\alpha$.

Figure 4.2: The influence of diffusion parameter $\lambda$.

The influence of the short-range interactions are represented in Figure 4.3. At first we consider the indicator function (2.31) with different values of the interaction interval $\rho$ and in an asymmetric case for two possible choices. It is evident that the choice of a small value $\rho$ drives the opinion to stay around the more stable choice. In the other case, the opinion locates roughly between two options, since the intensity of the interaction is more tolerant.
Figure 4.3: $P(\omega) = \chi(\omega - \omega \leq \rho)$ with $\rho = 0.5$ (left) and $\rho = 0.8$ (right), $\lambda = 0.003, p = q = 0.5$.

Figure 4.4: steady profile $g$ with regard to the choices of Maxwellian localization $\omega_- = -0.8$ and $\omega_+ = 0.8, p = q = 0.5$ (left) and $p = 0.6, q = 0.4$ (right)

In the next Figure 4.4, the influence of the range of interaction is more emphasized. In this case the short- and long-range interaction functions in (2.31) and (2.35) are compared. While in the case of the short-range interaction the two options are well separated; in the case of long-range interaction the steady profile is completely changed. It is located nearly on the neutral opinion with a unique maximum. In the case of an asymmetric distribution of opinion for the Maxwellian equilibrium, the difference between the two is more intense. Obviously, a bigger value of the parameter $r$ leads to a situation similar to the short-range interaction function. Above all, the various numerical examples show the importance of the diffusion constant and the range of interaction in the formation of choice, as well as how they are strongly responsible for the final steady profiles.
4.2 Leadership: Validity of the Boltzmann type control approach

In this section we report the different results of a numerical simulation of the Boltzmann type control model. A Monte Carlo method for the Boltzmann model as described in [7] has been used for the simulations. In order to compute the kinetic large time behavior, we set $\varepsilon = 0.01$ under the scaling (3.54). The society has a certain characteristic percentage of strong opinion leaders in the long-run average, following [5] in our numerical tests we suppose that five per cent of the whole population may typically be opinion leaders. To gain an overview, the leaders’ profiles have been increased tenfold in all figures. We assume here that the regularization term $v = 1$. For a uniform random variable with scaled variance $\zeta^2 = \bar{\zeta}^2 = \hat{\zeta}^2 = 0.01$, the diffusion function has been chosen similar to that in the previous section (3.68). For these choices the preservation of the bounds can be tested in the numerical simulations easily. In the following examples, we show some test cases with a single population of leaders which we discussed in the previous sections. Later on, the case of multiple populations of leaders with different strategies depending on various time intervals is presented. This corresponds to the realistic idea of competition between populations of leaders. According to the theoretical analysis, we set the interaction functions $P(\cdot, \cdot) \equiv 1$ and $R(\cdot, \cdot) \equiv 1$. The other scaled computational parameters for the different test cases are given in

Figure 4.5: Steady profile $g$: $p = q = 0.5$ with $\omega_+ = -0.3, \omega_+ = 0.8$ (left), $\omega_+ = -0.8, \omega_+ = 0.6$ (right)
4.2.1 Example 1. Leaders’ strategy and its effect

Firstly a single population of leaders, where the opinion of the leaders influence the opinion of ordinary people and maintain their own, is represented by the system of Boltzmann equations that, reads

\[
\begin{align*}
\frac{d}{dt} \int \varphi(\omega)f_F(\omega,t) d\omega &= (Q_F(f_F, f_F), \varphi) + (Q_{FL}(f_F, f_L), \varphi) \\
\frac{d}{dt} \int \varphi(\tilde{\omega})f_L(\tilde{\omega},t) d\tilde{\omega} &= (Q_L(f_L, f_L), \varphi).
\end{align*}
\tag{4.1}
\]

In this example it is shown that a non-monotone behavior of \( m_L(T) \) is obtained, if we choose a suitable control problem. In Figure 4.6 we present the evolution of the kinetic densities \( f_F(\omega,t) \) and \( f_L(\tilde{\omega},t) \) over the time interval \([0, 1]\). Note that the functions \( P(\cdot, \cdot), R(\cdot, \cdot), S(\cdot, \cdot) \) are constant. The initial distribution \( f_F \sim U([-1, -0.5]) \), where \( U(\cdot) \) is the uniform distribution respectively \( f_L \sim N(\omega_d, 0.05) \) with \( N, \) is considered as the normal distribution. Moreover we defined

\[
\dot{c}_{FL} = c_{FL}/\rho, \quad \dot{c}_L = c_{L}/\rho.
\tag{4.2}
\]

The presence of the non-monotone behavior leads us to a conclusion: the leaders use interplay between populistic and radical strategy to drive the followers toward their desired state which is typical for populistic radical parties in an election event. Comparatively, we consider a bound confidence model for the leader-follower interaction with the promise function \( S(\cdot) \)

\[
S(\omega, \tilde{\omega}) = \chi(|\omega - \tilde{\omega}| \leq \Delta),
\tag{4.3}
\]

with \( 0 \leq \Delta \leq 2. \) We consider \( \Delta = 0.5 \) and the same initial data of the previous example.
Figure 4.6: Example 1a: Kinetic densities evolution over time interval [0, 1] for a single population of leaders. (From [6].)

Figure 4.7: Examples 1b: Kinetic densities at different times for a single population of leaders with bound confidence interaction. (From [6].)

The figures 4.7 and 4.8 represent an interesting result, which describe a realistic fact, that a small group of followers is attracted by the leaders at the beginning and finally most of the followers are driven towards the desired state.
4.2.2 Example 2. The case of multiple populations of leaders

In this case, more populations of leaders drive the followers with different strategies. The evolution of the kinetic density of the system is described by a Boltzmann approach. Let $M > 0$ be the number of families of leaders, and every leader is described by a density $f_{L_p}, p = 1, \ldots, M$ such that

$$\int_I f_{L_p}(\tilde{\omega})d\tilde{\omega} = \rho_p. \quad (4.4)$$

Furthermore we assume that only one population of followers exists, with the density $f_F$ (identical to the previous test), and each of them interacts with both the other individuals from the same population and with every leader from every $p$th population. The corresponding Boltzmann equations with a suitable testfunction $\varphi$ are given

$$\begin{aligned}
\frac{d}{dt} \int_I \varphi(\omega)f_F(\omega, t)d\omega &= (Q_F(f_F, f_F), \varphi) + \sum_{p=1}^{M}(Q_{FL}(f_{L_p}, f_F), \varphi) \\
\frac{d}{dt} \int_I \varphi(\tilde{\omega})f_{L_p}(\tilde{\omega}, t)d\tilde{\omega} &= (Q_L(f_{L_p}, f_{L_p}), \varphi), \quad p = 1, \ldots, M.
\end{aligned} \quad (4.5)$$

In essence, the leaders are supposed to target at minimizing the control problem [3.8], for that reason every leaders’ population follows two factor to combine their strategy: the target opinions $\omega_{d_p}$, and the leaders’ behavior towards a radical ($\psi_p \approx 1$) or
populistic strategy \((\mu_p \approx 1)\). Similar to the previous example we define

\[
\hat{c}_{FL_p} = c_{FL_p}/\rho_p, \quad \hat{c}_{LP_p} = c_{LP_p}/\rho_p, \quad p = 1, \ldots, M.
\] (4.6)

Accidently, a link between our arguments and a Hotelling’s type model \([9]\) is furnished in this numerical test. The other model describes, how two shop owners, when they sell the same product with the same price and in the same street, must locate their shops to reach the maximal number of customers, uniformly distributed along the street. In other words, how to maximize their profits depending on their shop location in the same street. Strangely, the model shows that, without changing the prices, this can be achieved if their shops get closer. In \([9]\) the electoral dynamics are applied in this context and compare the result to the reason why the programs of political parties’ are often perceived as similar. In Figures 4.9 and 4.10 the two populations of leaders are described by the densities \(f_{FL_1} \sim U([-1, 1])\) and \(f_{FL_2} \sim U([-1, 1])\), each of them exercising different control over a population of followers. The initial distribution of \(f_{LP_p}\) is normally distributed with a mean value in the desired state \(\omega_{d_p}, p = 1, 2\) and \(\sigma = 0.05\). These Figures show that the model leads to a centrist population of followers. Especially marked, the followers’ opinion is located with a range between leaders’ mean opinions.

![Figure 4.9: Example 2: Kinetic densities at different times for two populations of leaders. (From [6].)](image)
4.2.3 Example 3. Two populations of leaders with time-dependent strategies

In the last example, a multi-population model for opinion formation with time-dependent coefficients is discussed. The idea of modifiable and flexible strategy for every family of leaders $p = 1, \ldots, M$ is involved with this approach. The coefficients $\psi$ and $\mu$ we introduced in (3.4) evolve in time and are defined for every $t \in [0, T]$ as

$$
\psi_p(t) = \frac{1}{2} \int_{\omega_{dp} - \delta}^{\omega_{dp} + \delta} f_F(\omega) d\omega + \frac{1}{2} \int_{m_{dp} - \overline{\delta}}^{m_{dp} + \overline{\delta}} f_F(\omega) d\omega
$$

$$
\mu_p(t) = 1 - \psi_p(t)
$$

(4.7)

where both $\delta, \overline{\delta} \in [0, 1]$ are constant and as usual $m_{L_p}$ is the average opinion of the $p$-th leader. The competition between the populations of leaders, where the leaders vary their strategy in populistic or radical attitude in order to succeed, is presented by this choice of coefficients. In particular, it depends on the local perception of the followers, whether the strategy has an edge over the other or not.
In Figure 4.11 and Figure 4.12 two populations of leaders and a single population of follower are considered. Initially the leaders are normally distributed with mean values $\omega_{d_1}$ and $\omega_{d_2}$ and parameters $\delta = \hat{\delta} = 0.5$, respectively the population of followers is represented by a skewed distribution $f_F \sim \Gamma(2, \frac{1}{4})$ over the interval $[-1, 1]$, where $\Gamma(\cdot, \cdot)$ is the Gamma distribution. As we see in Figure 4.11 since $\hat{c}_{FL_1} = 0.1$ and $\hat{c}_{FL_2} = 1$, the frequencies of interactions are supposed to be unbalanced. Usually, in this case we predict that the followers will be attracted to a position by one leaders group. However, as it is shown in Figure 4.12 on account of communication strategies pursued by the minority leaders group, it tends to be different positions. In a bipolar electoral context, a typical example of this behavior would be the coalition of the different political parties or groups for a particular purpose, usually for a limited time.
4.3 Comparison

In order to compare the two models reasonably, the control term $u$ from the second model is set to zero. It leads to Dürring’s model [5] and the steady state of the followers’ population is a solution of

$$
\left(\frac{\omega a_F - m_F}{\tau_F} + \frac{\omega a_L - m_L}{2\tau_{FL}}\right) g_{F,\infty}(\omega)
+ \left(\frac{\lambda_F a_F}{2\tau_F} + \frac{\lambda_L a_L}{4\tau_{FL}}\right) \frac{d}{d\omega} (D^2(\omega) g_{F,\infty}(\omega)) = 0.
$$

(4.8)

For simplification, we consider the case $P \equiv 1$. $\tau_F, \tau_{FL}$ are suitable relaxation times which allow control of the interaction frequencies of leaders and followers, similar to (3.34) and (3.35). Again, the masses of the opinion leaders and followers are denoted by $a_L = \int g_{L,\infty}$ and $a_F = \int g_{F,\infty}$ and their average opinions (first order moments) by $m_L$ and $m_F$. The parameters $\lambda$ are defined: $\lambda_F = \sigma^2/\gamma_F$ and $\lambda_{FL} = \hat{\sigma}^2/\gamma_{FL}$. Note that the $\gamma_F$ and $\gamma_{FL}$ are the constant compromise parameters. If we solve the differential equation (4.8), we have

$$
g_{F,\infty} = \frac{c_1}{(1-w^2)^{2\alpha}} \exp \left\{ -k \int_0^w \left[ \frac{va_F - m_F}{\tau_F(1-v^2)^{2\alpha}} + \frac{va_L - m_L}{2\tau_{FL}(1-v^2)^{2\alpha}} \right] dv \right\},
$$

(4.9)

with

$$
k = \frac{4\tau_F \tau_{FL}}{2\lambda_F a_F \tau_F + \lambda_{FL} a_L \tau_F}.
$$

The integration of (4.8) leads to $a_L m_F - m_L a_F = 0$. Now the equation (4.8) is given by

$$
g_{F,\infty} = \frac{c_1}{(1-w^2)^{2\alpha}} \exp \left\{ -k \left( \frac{a_F}{\tau_F} + \frac{a_L}{2\tau_{FL}} \right) \int_0^w \frac{v}{(1-v^2)^{2\alpha}} dv \right\}
\cdot \exp \left\{ km_L \left( \frac{1}{2\tau_{FL}} + \frac{a_F}{\tau_F a_L} \right) \int_0^w \frac{1}{(1-v^2)^{2\alpha}} dv \right\}.
$$

Note that since $|m_L| < a_L$, it concludes $c_1$ only can be determined if $\alpha > \frac{1}{2}$. To compare the two models in a meaningful way, two populations of leaders with the opinions $w_- = -0.6$ and $w_+ = 0.8$ are presented. We decide the mass of each population of
leaders is 50% of the total mass of the population of leaders. By the same token for
the choice formation, we have the probabilities that one can choose \( w_- \) and \( w_+ \) are
\( p = q = 0.5 \). The following paramters are chosen for the comparison.

\[
\alpha = 1, \quad a_F = 1, \quad a_L = 0.05, \quad m_L = a_L(0.5w_- + 0.5w_+) = 0.005, \tag{4.10}
\]

\[
m_F = \frac{m_L a_F}{a_L} = 0.01, \quad \tau_{FL} = 1, \quad \lambda_F = \lambda_{FL} = 1.
\]

Again, the steady state of the choice formation is a solution of

\[
(\omega - m)g + \frac{\lambda}{2} \frac{d}{d\omega} ((1 - w^2)^{2\alpha} g) = 0 \tag{4.11}
\]

Since \( \alpha = 1, \lambda = 1 \) and \( m = pw_- + qw_+ = 0.1 \). The steady state \( g_{c,\infty} \) of the choice
formation is given by

\[
g_{c,\infty} = 1.2525 \exp \left\{ -\frac{1}{2} \left[ \frac{0.1125}{1 + w} + \frac{0.1375}{1 - w} + 0.5125 \log (1 - w) + 0.4875 \log (1 + w) \right] \right\}. \tag{4.12}
\]

Obviously, the two differential equations (4.8) and (4.11) are similar, if we set \( \tau_F \to \infty \) in (4.8). We obtain

\[
\left( \frac{\omega a_L - m_L}{2\tau_{FL}} \right) g_{F,\infty}(\omega) + \left( \frac{\lambda_L a_L}{4\tau_{FL}} \right) \frac{d}{d\omega} (D^2(\omega)g_{F,\infty}(\omega)) = 0. \tag{4.13}
\]

With a certain choice of parameters of the second model, we assume we would get a
similar steady state compare to the first model. In the following Figure, we show the
different steady states depending on various \( \tau_F \).

![Figure 4.13: Steady profile g: \( a_L = 0.05, m_L = 0.005 \) (left), \( a_L = 0.1, m_L = 0.01 \) (right)](image)

As we have seen, the steady states \( g_{\infty} \) surprisingly well approximated the steady state
of choice formation \( g_{c, \infty} \) independent of the values of \( \tau_F \). Consequently, we continued studying the other parameters of the steady state in the second model. On the right-hand side of the Figure 4.13 we chose the mass of the total population of leader \( a_L = 0.1 \). However, the steady states in spite of a bigger value of \( \tau_F \) do not diversify compare to the left Figure. In the next Figure we present the extreme cases.

![Figure 4.14: Steady profile \( g \): \( a_L = 0.5, m_L = 0.05 \) (left), \( a_L = 0.7, m_L = 0.07 \) (right)](image)

In Figure 4.14 we set \( a_L = 0.5 \) (left) and \( a_L = 0.7 \) (right). Apparently, a bigger value of \( \tau_F \), the two options are well separated and locate in the two extreme opinions. Now let us compare the equation (4.11) with (4.13), we note that \( m_L = ma_L \) and \( m \) is the expectation value of the Maxwellian. For the simplification we compare the drift and the diffusion terms in (4.11) and (4.13) with each other. We have

\[
\frac{a_L(w - m)}{2\tau_{FL}} = (w - m), \quad \frac{\lambda_{FL} a_L}{4\tau_{FL}} = \frac{\lambda}{2}
\]

(4.14)

From the first equation we get \( 2\tau_{FL} = a_L \) and it follows \( \lambda_{FL} = \lambda \). Although it is solvable, it leads to an unrealistic result. Since the values of \( a_L \leq 1 \), \( \tau_{FL} \geq 1 \) and the values of \( \lambda = \sigma^2/\gamma \) should be kept small, because a too-pronounced self-thinking (a relatively big diffusion coefficient \( \sigma^2 \)) destroys the different opinions.

In essence, as the Figures 4.13 and 4.14 have shown, with a certain choice of parameters, it is possible to simulate choice formation with the second model.
5 Conclusion

In summary, we have presented two kinetic models fitted to the study of opinion dynamics while overcoming certain difficulties. The individuals in the society are simulated by the particles in the kinetic theory. In order to describe the two models in a suitable environment and model the possible spontaneous change of opinion of each individual, we pointed out the main characteristic of the opinion formation is the binary interaction based on the compromise and the self-thinking process. These two processes sufficiently illustrate a large variety of human behaviors in a society. Following the same line of thought, we were able to characterize the formation of choice under the influence of a Maxwellian background in the first model. In this case each individual is pushed and influenced to form an opinion from prescribed events. Later, contrasting the first model, we focused on the leadership hierarchy in our society. To present the idea of interaction between the leaders and the followers, a Boltzmann type control was introduced. Through a suitable cost function, we were able to represent the leaders’ strategy, which depends on the desire of the leaders and the mean opinion of the followers, in trying to control followers’ opinions. However, the control problem is not easy to be solved in general. Thanks to the instantaneous binary control approximation, the control strategy of the leaders was well involved in the microscopic leaders’ interactions of the corresponding Boltzmann equation. Then macroscopic quantities of essential importance for the evolution, such as moments, were studied in both models. We found out, that it is difficult to solve these analytically and give us explicitly the steady state of opinion from which one can give an account of information on the behavior of opinion. In order to overcome this complexity, the asymptotic limit was introduced and the Fokker-Planck type for the distribution of opinion among individuals was obtained. It simplified the asymptotic analysis and under specific conditions, it gave us explicitly the stationary description of opinion. Finally, not only we have been able to successfully formulate the two models, depending on different parameters, they accurately depict real life societal behavior under the set conditions, as exemplified in three separate cases in the second model. Although the two models successfully have been derived mathematically and compu-
tionally, the opinion dynamic among individuals is still a difficult task. Indeed the human brain is much more complex than a set of variables, such as everyone has a different background and character. But how can it be further developed? What will be the next step /challenge to define the opinion states of a population?
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