Inverse Problem of integration: Differentiation

$$R: C_0([0,1]) \to C_1([0,1])$$

$$(Rf)(x) := \int_0^x f(t)\,dt.$$

$\|\cdot\|_\infty$ in both.

$$\|Rf\|_\infty = \sup_x \left|\int_0^x f(t)\,dt\right|$$

$$\leq \sup_x \int_0^x |f(t)|\,dt$$

$$\leq (1)\,\|f\|_\infty \quad \Rightarrow R \text{ is continuous}$$

$$\tilde{f} = f + u$$

$$\|R\tilde{f} - Rf\| = \|Rf + Ru - Rf\|$$

$$= \|Ru\| \leq \|u\|_\infty \Rightarrow \text{Error is limited}$$

given $Rf = g$, compute f.

$$R^{-1}: C_1([0,1]) \to C_0([0,1])$$

$$(R^{-1}g)(x) = g'(x)$$

$$\tilde{g} = g + u$$

$\|\text{noise}\|_\infty \leq \delta$

Let $g \in C_1$. Let $\tilde{g} = g + (\delta \sin(n\pi x))$

$$\tilde{g}'(x) = g'(x) + \delta \cdot n\pi \sin(n\pi x)$$

$$\|R^{-1}\tilde{g} - R^{-1}g\|_\infty = \|\delta \cdot n \cdot \pi \cdot \sin(n\pi x)\|_\infty$$

$$= \delta \cdot n \cdot \pi$$

$$\|\text{error}\| = \delta \cdot (n \cdot \pi)$$

$$\|R^{-1}u\|_\infty = \|u'\|_\infty \qquad u(x) = \sin(n\pi x)$$

$$= n \cdot \pi \qquad \|u\|_\infty = 1$$

no C: $\|R^{-1}u\| \leq C \cdot \|u\|_\infty$

$$\frac{\|R^{-1}u\|_\infty}{\|u\|_\infty} \xrightarrow{n\to\infty} \infty$$

no upper bound, R^{-1} discontinuous.

Finite Differences

Inverse Problem: measurement operator $R: X \to Y$

$Rf = g$, compute f given g. Image f, Data g.

R^{-1} exists? continuous? well-posed / ill-posed

Basic examples of the course

1) Image Denoising: $R = I \Rightarrow R^{-1} = I$



Speckle-Noise

R: „space of all acceptable images" → Photos

Inverse Problem? „image space" → „photo space"

→ Restrict the origin space to „sensible" images.
⇒ No inverse.

→ Numerous images mapped to the same distorted image. No inverse.

I: invertible, continuous.

Norm in Y: measurement error

Norm in X: Distance between images?

2) Radon-Transform, Computerized Tomography, Paramada Estimation for the Transport Equation, Geometric (integral) equations



test disk through my body
not scattered, but attenuated

$I(x)$ intensity for $x \in L$.

$I(x_0) = I_0$ at the beginning.

$I(y_0) = I_1$ after leaving the body,

obviously, $I_0 > I_1$.

$I(t+h) = I(t) - I(t)\mu(t)h$

$\Rightarrow \frac{I(t+h) - I(t)}{h} = -\mu(t) \cdot I(t)$

$\hookrightarrow I'(t)$

$\Rightarrow I(t) = e^{-\int_0^t \mu(s)\,ds} \cdot I_0$, μ = attenuation coefficient

$\Rightarrow I_1 = e^{-\int_L \mu(s)\,ds} \cdot I_0 \Rightarrow \int_L \mu(s)\,ds = -\log \frac{I_1}{I_0}$

$\Rightarrow$ The CT measures line integrals of the function μ

$\Rightarrow$ Inverse problem: Given (all) line integrals of a function, can you compute the function?

Mathematical formulation:

Let $\mathcal{S} = \{ f \in C^\infty(\mathbb{R}^n) \mid \|x^\alpha D^\beta f\|_\infty < \infty \ \forall \alpha, \beta \}$.

(Schwartz space of rapidly decaying functions),

let $C = S^{n-1} \times \mathbb{R}$, S^{n-1} is the unit ball in $\mathbb{R}^n$, let $(\theta, s) \in C$.

$E(\theta, s) = \{ x : x \cdot \theta = s \}$ is an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$, with normal θ and distance $|s|$ from the origin.

$n=2$: line, $n=3$: plane.

parametrizes all hyperplanes. Assume we can measure all integrals.



Let $f \in \mathcal{S}$. $(Rf)(\theta, s) := \int_{x \cdot \theta = s} f(x)\,d\sigma(x)$

Rf is a line integral for $n=2$ and a plane integral for $n=3$.

$R: \mathcal{S} \to \mathcal{S}_2$, $\mathcal{S}_2 = \{ g : C \to \mathbb{R}, g \in C^\infty$, rapidly decaying in $s \}$.

Direct problem: Given $f(=\mu)$, compute $g = Rf$.

Inverse problem: Given g, compute f : $Rf = g$.

Line integrals in $\mathbb{R}^n$.

X-Ray transform, direction θ, initial point $y \in \theta^\perp$:

$(Sf)(\theta, y) = \int_{\mathbb{R}} f(y + t\theta)\,dt$



Given all line integrals of a function, Compute the true attenuation function.

§2 Basics

2.1 Definition: Let X, Y normed vector spaces.

Let $A : X \to Y$. The problem

Find a unknown $x \in X$ such that $Ax = y$ $(y \in Y)$

is properly / well posed, iff

1) $\forall y \in Y \; \exists x : Ax = y$ (solvability)
2) $\exists! x : Ax = y$ (uniqueness)
3) $A^{-1} : Y \to X$ is continuous w.r.t. the norms in X, Y

If one of the conditions is violated, the problem is ill-posed.

Remarks: a) In the sections, A will be linear (or affine)

$\Rightarrow$ Usually, X, Y infinite-dimensional function spaces

Remarks: Operator norm of A:

$$\|A\|_{X,Y} := \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|Ax\|$$

If $\|A\|_{X,Y} < \infty$, then A is continuous.

$\forall x \in X : \|Ax\|_Y \leq \|A\|_{X,Y} \|x\|_X$

Assume $x_n \to x$. $\|Ax_n - Ax\|$

$= \|A(x_n - x)\|$

$\leq \|A\| \, \|x_n - x\|$

$\to 0$

$\Rightarrow Ax_n \to Ax \Rightarrow A$ is continuous.

If $\|A\|$ is unbounded:

$\forall n \in \mathbb{N} : \exists x_n : \|x_n\| = 1, \; \|Ax_n\| > n.$

$y_n := \frac{1}{n} x_n : \Rightarrow \|y_n\| = \frac{1}{n} \to 0$

$\Rightarrow y_n \to 0$

$\|Ay_n\| = \frac{1}{n} \|Ax_n\| > 1$

$\Rightarrow Ay_n \not\to 0$

$\Rightarrow A$ is discontinuous.

Compute Ax, $\tilde{x} : \|\tilde{x} - x\| \leq \varepsilon$.

$\|A\tilde{x} - Ax\| \leq \|A\| \cdot \varepsilon$

ε error-amplification factor.

2.1 Remark: Continuity depends on the used norms.

Example: a) Derivative:

$A : C^0 \to C^0$

$Af \to f'$

is discontinuous w.r.t. $\|\cdot\|_\infty$

continuous $\|\cdot\|_{C^1}$ in $C_1 : \|f\|_{C^1} \leq \|f\|_\infty + \|f'\|_\infty$

$\to$ discontinuous wrt $\|\cdot\|_{L^2}$

$\to$ continuous wrt $\|\cdot\|_{H^1} : \|f\|_{H^1} \leq \|f\|_{L^2} + \|f'\|_{L^2}$

2) A as in the definition of the inverse problem, A^{-1} exists, discontinuous wrt $\|\cdot\|_Y, \|\cdot\|_Y$.

Define $\|y\|_{Y'} := \|A^{-1}y\|_X + \|y\|_Y$

$\|A\|' = \sup_{y \neq 0} \frac{\|A^{-1}y\|_X}{\|A^{-1}y\|_X + \|y\|_Y} = \frac{\|A^{-1}y\|_X}{\|y\|_Y + \|A^{-1}y\|_X} < 1$

$\Rightarrow A^{-1}$ continuous.

Trick does not help in solving the problem.

If X is infinite-dimensional, then there exists a linear operator

$A : X \to Y$

which is discontinuous.

X ∞-dim: $\{x_\lambda\}_{\lambda \in \Lambda}$ is only (clearly) linearly independent, $\|x_\lambda\| = 1$.

$Ax_\lambda = \lambda \cdot x, \; y \in Y, \|y\| = 1.$

$A\left(\sum \lambda_i x_i\right) := \sum \lambda_i Ax_i$

$\frac{\|Ax_\lambda\|}{\|x_\lambda\|} = \lambda \to \infty \Rightarrow \|A\|$ unbounded

$\Rightarrow A$ discontinuous.

X finite dim: All linear operators are continuous.

2.3 Definition: Let X, Y function spaces, inverse problem $Ax = y$, A^{-1} exists, discontinuous.

Order of ill-posedness m:

If A^{-1} is continuous, when adding the norms of the solution up to degree m, and discontinuous for any $m - \eta$, the inverse problem is ill-posed of order m.

Ad: $g \quad \|g\|_{W^m_2} = \|g\|_{L^2} + \|g^{(m)}\|_{L^2} \to$ order m.

$\|g\|_{W^m_2} = \|g\|_{L^2} + \|g'\|_{L^2} + \ldots + \|g^{(m)}\|_{L^2} \to$ also 2.

Examples: 1) Integration: order 1

2) Diffusion equation: order as exponentially ill-posed

3) Radon Transform: order $\frac{n}{2}$

(2.4)

Let $\mathcal{X} \subset \mathbb{R}^m$, open,

$L^p(\mathcal{X}) : \quad f : \mathcal{X} \to \mathbb{R}, \quad \|f\|_p^p = \int_{\mathcal{X}} |f(x)|^p dx < \infty$

$p=2 \rightarrow L^2(\mathcal{X})$ Hilbert space

$(f,g) = \int_{\mathcal{X}} f(x)\, g(x)\, dx$

$\|f\|_2^2 = (f,f) = \int_{\mathcal{X}} f(x)^2 dx.$

$|(f,g)| \leq \|f\| \cdot \|g\|$ Cauchy-Schwarz

$|(f,g)| \leq \|f\|_p \cdot \|g\|_q \quad \frac{1}{p} + \frac{1}{q} = 1$

Def 2.4 : $\mathcal{Z} \subset \mathbb{R}^n$, $\mathcal{X} \subset \mathbb{R}^m$ measurable, open,

$k : \mathcal{Z} \times \mathcal{X} \to \mathbb{R}$ measurable.

K linear integral operator with kernel k;

$u : \mathcal{X} \to \mathbb{R}$, $x \in \mathcal{Z}$

$(Ku)(x) = \int_{\mathcal{X}} k(x,y)\, u(y)\, dy$ = $(k(x,\cdot), u)_{L^2}$

Example convolution:

$\mathcal{Z} = \mathcal{X} = \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$

$k(x,y) = g(x-y)$

$(Ku)(x) = \int_{\mathcal{X}} g(x-y)\, u(y)\, dy$

Theorem 2.5:

Let $\mathcal{Z}$, $\mathcal{X}$ bounded.

$k \in L_2(\mathcal{Z} \times \mathcal{X})$

Then, $K : L_2(\mathcal{X}) \to L_2(\mathcal{Z})$,

is continuous,

$\|Ku\|_2^2 = \int_{\mathcal{Z}} \int_{\mathcal{X}} |k(x,y)\, u(y)|\, dy\, dx|^2$

$\leq \underbrace{\int_{\mathcal{Z}} \int_{\mathcal{X}} k^2(x,y)\, dy\, dx}_{= \|k\|_{L^2(\mathcal{Z}\times\mathcal{X})}} \cdot \underbrace{\int_{\mathcal{Z}} \int_{\mathcal{X}} u(y)^2\, dy\, dx}_{\|u\|_2^2 \cdot |\mathcal{Z}|}$

$\leq \|k\| \cdot \|u\|_2^2 \cdot |\mathcal{Z}|$

$\|Ku\|_2^2 = \int_{\mathcal{Z}} \left(\int_{\mathcal{X}} k(x,y)\, u(y)\, dy \right)^2 dx$

$\leq \int_{\mathcal{Z}} \int_{\mathcal{X}} k(x,y)^2\, dy \int_{\mathcal{X}} u(y)^2 dy\, dx$

$\leq \|u\|_2^2 \cdot \|k\|_2^2$

$\leq C \|u\|_2^2 \Rightarrow$ continuous wrt L^2.

Operators in Hilbert spaces

X, Y Hilbert spaces

$K : X \to Y$, linear, continuous.

$\exists$ unique $K^* : Y \to X$, linear, continuous

$\forall x \in X, \forall y \in Y : (Kx, y)_Y = (x, K^* y)_X$

$X = L^2(\mathcal{X})$, $Y = L^2(\mathcal{Z})$, $k \in L^2(\mathcal{Z} \times \mathcal{X})$:

$Ku \in X : (Ku)(x) = \int_{\mathcal{X}} k(x,y)\, u(y)\, dy.$

$v \in Y : \underbrace{(Ku, v)_{L^2(\mathcal{Z})}}_{(k^*v)(y)} = \int_{\mathcal{Z}} Ku(x)\, v(x)\, dx$

$= \int_{\mathcal{Z}} \int_{\mathcal{X}} k(x,y)\, u(y)\, v(x)\, dy\, dx$

$= \int_{\mathcal{X}} \underbrace{\int_{\mathcal{Z}} k(x,y)\, v(x)\, dx}_{(K^*v)(y),\ K^* : L^2(\mathcal{Z}) \to L^2(\mathcal{X})}\, u(y)\, dy$

$= \int_{\mathcal{X}} (k^*v)(y)\, u(y)\, dy$

$= (u, k^*v)_{L^2(\mathcal{X})}$

Hilbert space X

Subset M (closed, linear subspace of X)

$\overline{M}$ = closure $= \{ x \in M \}$

$x \in X : \exists\, y \in \overline{M},\ z \in M^\perp : x = y + z.$

$M^\perp = \{ z : (y,z) = 0 \ \forall y \in M \}$

$X = \overline{M} \oplus M^\perp, \quad M^{\perp\perp} = \overline{M}$

$x \in X, \quad y := \arg\min_{w \in \overline{M}} \|x - w\|_2^2$

$\alpha := \inf_{w \in M} \|x - w\|$

Sequence $(w_n) : \|x - w_n\| \to \alpha$

Parallelogram equation: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$2(\|x - w_n\|^2 + \|x - w_m\|^2) \to \alpha$

$\to 4\alpha^2$

$= \underbrace{\|x - w_n + y - w_m\|^2}_{4\,\|x - \frac{w_n + w_m}{2}\|^2} + \|w_m - w_n\|^2$

$\in M$, $\to \alpha^2$

$\geq 4\alpha^2 + \|w_m - w_n\|$

$\to 0$

(w_n) is a Cauchy sequence,

converges towards y, $\|x - y\| = \alpha$

y_1, y_2 initializers, $w_n = y_1$, $w_{zero} = y_2 \Rightarrow$ converges

$\Rightarrow y_1 = y_2$.

$x = y + \underbrace{(x-y)}_{z}$

$y \in \overline{M}$, $(x-y) \in M^\perp$.

$\phi(\lambda) = \|x - y + \lambda v\|_2^2 \quad v \in M, \quad \Rightarrow \phi$ has Minimum at $\lambda = 0$.

$= \|x-y\|_2^2 + 2\lambda\,(x-y, v) + \lambda^2 \|v\|_2^2$

$\Rightarrow \phi'(0) = 0 \Rightarrow (x-y, v) = 0$

$\Rightarrow (x-y) \in M^\perp$.

$\Rightarrow X = \overline{M} \oplus M^\perp$

$x = \underbrace{(P_x)}_{\in} \oplus \underbrace{(x - P_x)}_{\perp}$

orthogonal Projection.

Examples for integral operators

Def 2.6 Let $f, g \in L^1(\mathbb{R}^n)$.
Define the convolution of f and g:
$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy$, $x \in \mathbb{R}^n$.
$(f * g) : \mathbb{R}^n \to \mathbb{R}$.

Lemma 2.7: $f * g$ is well-defined.
$f * g \in L^1(\mathbb{R}^n)$.

$$\|f * g\|_1 = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y) g(y)\, dy \right| dx$$
$$\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dy\, dx$$
$$= \int_{\mathbb{R}^n} \underbrace{\int_{\mathbb{R}^n} |f(x-y)|\, dx}_{\|f\|_1} |g(y)|\, dy$$
$$= \|f\|_1 \cdot \|g\|_1$$

Fix $g \in L^1$, convolution maps from $L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$
$\|f * g\|_1 \leq \|g\| \cdot \|f\| \to$ Convolution continuous wrt L^1.

Def 2.8: Let $f \in \mathcal{S}$. Hilbert transform:
$$(\mathcal{H}(f))(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy$$
$$:= \lim_{\varepsilon \to 0} \int_{|y-x| \geq \varepsilon} \frac{f(y)}{x-y}\, dy$$

Existence, Invertible. Exercise.
Inverse is continuous wrt $\|\cdot\|_2$.

Remark: $\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\, dx$
$\check{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\xi} f(\xi)\, d\xi$
$\hat{\check{f}} = f \quad (f \in L^2)$
$\|f\|_2 = \|\hat{f}\|_2 = \|\check{f}\|_2$ (Plancherel)
FT is continuous.
inverse FT is continuous.

Remark: $k : X \to Y$, Banach, linear.
$\|k\|_{X,Y} < 1$.
$A := I - K \Rightarrow A$ is continuous.
solve $Af = g$, $f := \sum_{k=0}^{\infty} k^k g$

$$\|\sum_{k=0}^{N} k^k g\| \leq \sum_{k=0}^{N} \|k\|^k \cdot \|g\| \Rightarrow \text{partial sums are Cauchy sequence}$$
$$\leq \frac{1}{1-\|k\|} \Rightarrow \text{converges}$$
$\Rightarrow f$ well-defined

$$Af = (I-k) \lim_{N \to \infty} \sum_{k=0}^{N} k^k g$$
$$= \lim_{N \to \infty} (I-k) \sum_{k=0}^{N} k^k g = g$$
$$= g - k^{N+1} g \to 0$$

A is invertible. $\|A^{-1} g\| = \|f\| \leq \frac{1}{1-\|k\|} \|g\|$
$\Rightarrow A^{-1}$ is continuous
Solve $Af = g$ is well-posed.

Characterization of least squares. Minimum norm solutions

Theorem 2.13 The following is equivalent:

1) u is least squares solution of $Ku = g$.
2) $K^*Ku = K^*g$
3) $Ku = Pg$, P orthogonal projection onto $\mathrm{Range}(K)$.

(1) $\Rightarrow$ (2): Let u LSQ.

$v = K^*(Ku - g)$.

$f(\lambda) = \|K(u+\lambda v) - g\|_2^2$ Minimum at $\lambda = 0$.

$= \|Ku - g\|^2 + 2\lambda(Ku - g, Kv) + \lambda^2 \|Kv\|^2$

$\Rightarrow f'(0) = 0 \Rightarrow (Ku - g, Kv) = 0$

$= (K^*(Ku - g), v)$

$= \|v\|^2 \Rightarrow v = 0 \Rightarrow K^*Ku = K^*g$.

(2) $\Rightarrow$ (3): $K^*Ku = K^*g$. $\forall v \in X$:

$\Rightarrow (K^*Ku - K^*g, v) = 0$

$\Rightarrow (Ku - g, Kv) = 0$

$Ku - g \in \mathrm{Range}(K)^\perp$, $Kv \in \mathrm{Range}(K)$

$\Rightarrow Ku = P_{\overline{R(K)}}\, g$ possibly unsolvable!

Requires: $g \in R(K) \oplus \underbrace{R(K)^\perp}_{\ker(K^*)}$

(3) $\Rightarrow$ (1): $Ku = Pg$

$\forall v \in X$:

$\|Kv - g\|^2 = \|\underbrace{Ku - g}_{= Pg - g \in R(K)^\perp} + \underbrace{K(v-u)}_{\in R(K)}\|^2$

$= \|Ku - g\|^2 + \|K(v-u)\|^2$

$\geq \|Ku - g\|^2$

$\Rightarrow u$ is least squares solution.

Theorem 2.13 The following is equivalent:

1) u is least squares solution of $Ku = g$.
2) $K^*Ku = K^*g$
3) $Ku = Pg$, P orthogonal projection onto $\overline{\text{Range}(K)}$.

Corollary 2.14

1) $Ku = g$ has LSQ
   iff $g \in R(K) \oplus ke(K^*) = \overline{R(K)} \oplus R(K)^\perp$
2) $\overline{R(K)} \oplus R(K)^\perp = \overline{R(K)} \oplus N(K) = Y$
   $\Rightarrow$ $\{g : Ku = g \text{ has LSQ}\}$ dense in Y.
3) u, v LSQ $\Rightarrow K^*Ku = g = K^*Kv$
   $\Rightarrow (K^*K(u-v), u-v) = 0$
   $\Rightarrow (K(u-v), K(u-v)) \Rightarrow K(u-v) = 0$
   $\Rightarrow u - v \in N(K)$
4) The set of all least squares solutions is convex, closed,
   If it is nonempty, the MNS is unique.
5) If $Ku = g$ has LSQ,
   u MNS $\Leftrightarrow$ 1) u is LSQ
   2) $u \in \overline{R(K^*)}$

Let v MNS.

$v = u + \tilde{u} \to \tilde{u} \in \overline{R(K^*)}^\perp = N(K)$

$P_{\overline{R(K^*)}}$

$K^*g = K^*Kv = K^*Ku + \underbrace{K^*K\tilde{u}}_{=0} \Rightarrow u$ is LSQ.

$\|v\|^2 = \|u\|^2 + \|\tilde{u}\|^2 \Rightarrow \|\tilde{u}\| = 0$

$v = u \Rightarrow v \in R(K^*)$

Least squares and minimum norm

$K: X \to Y$ (Banach, Hilbert)

linear, continuous.

K possibly: not surjective

not injective.

Solve $Ku = f$, $f \in Y$, $u \in X$.

Def 2.9: u is a Best-approximation

iff $\|Ku - f\|_Y \leq \|Kv - f\|_Y \quad \forall v \in X$.

X, Y Hilbert: u is least squares solution.

Def 2.10: $u \in X$ is minimal norm solution

(Moore-Penrose-solution)

iff $\|u\| \leq \|v\|$ $\forall$ Best-approximations v.

Notation, $\ker(K) = N(K) = \text{Nullspace of } K$

$:= \{u \in X : Ku = 0\}$.

$\text{Range}(K) = R(K) = \text{Im}(K)$

$:= \{v \in Y : \exists u \in X : Ku = v\}$.

Lemma 2.11: $\ker(K)$ is closed.

$\text{Range}(K)$ is not necessarily closed.

$(u_n) \subset \ker(K)$, $u_n \to u$

$\Rightarrow \underbrace{Ku_n}_{=0} \to Ku \Rightarrow u \in \ker(K)$

$X: L^2, \quad k \in L^2, \quad k \in \mathcal{J}(\mathbb{R})$

$Ku(x) = \int_{\mathbb{R}} k(x,y)\, u(y)\, dy$

$\Rightarrow Ku \in C^\infty \Rightarrow$ not closed w.r.t. C^1, C^2-norm.

X, Y Hilbert

Lemma 2.12: $R(K)^\perp = \ker(K^*)$

$\overline{R(K^*)} = \ker(K)^\perp$

1) $y \in \ker(K^*) \Rightarrow (K^* y, x) = 0 \quad \forall x \in X$

$\Rightarrow (y, Kx) = 0$

$\Rightarrow y \in R(K)^\perp$

$y \in R(K)^\perp \Rightarrow (y, KK^* y) = 0$

$(K^* y, K^* y)$

$\|K^* y\|^2$

$\Rightarrow y \in \ker(K^*)$

2) $\overline{R(K^*)}^\perp = \ker(K)$

$R(K^*) = \ker(K)^\perp$

Compact operators

Example: Laplace equation, Sheet 2, exercise 4.



$-\Delta u = 0$ on $B_1(0) \subset \mathbb{R}^2$.

$u = f$ on $\|x\| = 1$, $f \in L^2([-\pi,\pi])$.

$K : L_2([-\pi,\pi]) \to L_2([-\pi,\pi])$, $R_0 < 1$.

$(Kf)(\varphi) = u(R_0, \varphi)$ in polar coordinates.

$$f(\varphi) = \sum_k a_k e^{ik\varphi}, \quad a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-ik\varphi} d\varphi.$$

$$u(r,\varphi) = \sum_k a_k r^{|k|} e^{ik\varphi}$$

$$= \frac{1}{2\pi} \sum_k \int_{-\pi}^{\pi} f(\varphi) e^{-ik\varphi} d\varphi \, r^{|k|} e^{ik\varphi}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_k r^{|k|} e^{-ik(\varphi-\varphi)} f(\varphi) d\varphi$$

$$\kappa(\varphi,\varphi) = G_r(\varphi - \varphi)$$

$\Rightarrow$ convolution,

$$= \sum_{k=0}^{\infty} r^k e^{-ik(\varphi-\varphi)} + \sum_{k=0}^{\infty} r^k e^{ik(\varphi-\varphi)} - 1$$

$$= \frac{1}{1 - r e^{-i(\varphi-\varphi)}} + \frac{1}{1 - r e^{i(\varphi-\varphi)}} - 1$$

$$= \frac{1 - r^2}{1 + r^2 - 2r\cos(\varphi-\varphi)} =: G_r(\varphi - \varphi) \quad (r = R_0)$$

$$Kf(\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_r(\varphi - \varphi) f(\varphi) d\varphi$$

$L^2([-\pi,\pi])$

$\Rightarrow$ K compact $\Rightarrow$ K continuous.

$K^{-1}g$ discontinuous.

Satz: $K \subseteq X$
$\exists$ strong min V bez. g^1 $\forall g \in G$
$\forall$ + neg. existing $\forall$ $\forall$ bez.
$\forall$ $\forall$ $\forall$ $

$K: X \to Y$, K cp, X, Y Hilbert.

2.22 Bessel's inequality

Let (u_k) ONS in X, $u \in X$.

$$\|u\|_2^2 \geq \sum_{i=1}^{\infty} (v_i, u)^2$$

$$\left\| \sum_{i=1}^{n} \alpha_i u_i \right\|_2^2 = \left(\sum_{i=1}^{n} \alpha_i u_i, \sum_{j=1}^{n} \alpha_j u_j \right)$$

$$= \sum_{i=1}^{n} \alpha_i^2$$

$$0 \leq \left\| u - \sum_{i=1}^{n} (u, u_i) u_i \right\|_2^2$$

$$= \|u\|_2^2 - 2 \sum_{i=1}^{n} (u, u_i)(u_i, u_i) - \sum_{i=1}^{n} (u, u_i)^2$$

$$= \|u\|_2^2 - \sum_{i=1}^{n} (u, u_i)^2$$

$$\Rightarrow \sum_{i=1}^{n} (u, u_i)^2 \leq \|u\|_2^2$$

$$\Rightarrow \sum_{i=1}^{\infty} (u, u_i)^2 \leq \|u\|_2^2$$

Definition 2.23

Let (u_k) ONS in X.

(u_k) complete $\overset{\text{def}}{\iff} X = \overline{\operatorname{span} \langle u_k \rangle}$

$U_n = \operatorname{span} \langle u_1, \dots, u_n \rangle \quad g_n \in U_n, \ g_n \to u \quad \forall u \in X.$

Let $u \in X$, $w \in U_n \Rightarrow w = \sum_{k=1}^{n} c_k u_k$

$\alpha_k = (u, u_k)$.

$$\|u - w\|_2^2 = \|u\|^2 - 2 \sum_{k=1}^{n} c_k \underbrace{(u, u_k)}_{\alpha_k} + \sum_{k=1}^{n} c_k^2$$

$$= \|u\|^2 + \sum_{k=1}^{n} (c_k - \alpha_k)^2 - \sum_{k=1}^{n} \alpha_k^2$$

minimal: $c_k = \alpha_k$

$v_n := \sum_{k=1}^{n} \alpha_k u_k$ is Bestapprox. of u in U_n.

(u_k) complete $\Rightarrow \exists g \in U_n, \|u - g\| \leq \varepsilon$.

$\|u - v_n\| \leq \|u - g\| \leq \varepsilon$.

$v_n \to u \Rightarrow \|v_n\|_2^2 \to \|u\|_2^2$

$$\sum_{k=1}^{n} \alpha_k^2 \to \sum_{k=1}^{\infty} \alpha_k^2$$

$$\Rightarrow \sum_{k=1}^{\infty} \alpha_k^2 = \|u\|_2^2$$

2.24 Parseval's identity:

(u_k) is complete $\Leftrightarrow \|u\|^2 = \sum_{k=1}^{\infty} (u, v_k)^2$

"$\Rightarrow$": ✓

"$\Leftarrow$": $\left\| u - \sum_{i=1}^{n} (u, v_i) v_i \right\|_2^2 = \|u\|_2^2 - \sum_{i=1}^{n} (v_i, v)^2 \to 0$

$$u = \sum_{i=1}^{\infty} (u, u_i) u_i$$

Eigenvalues / Eigenvectors

2.25 Let $k: X \to X$ compact, X Banach.
Then $\dim \ker (I-k) < \infty$.
Proof: Assume $\dim \ker (I-k) = \infty$.
$\Rightarrow$ in Gram-Schmidt construct (u_n) in $\ker(I-k)$:
$\|u_n\| = 1$, $\|u_n - u_k\|^2 \geq 2$, $k \neq n$
$\|ku_n - u_n\| = 0$
$\Rightarrow (ku_n)$ has no cpt subsequences.
$\Rightarrow k$ is compact.

2.26 Corollary: If $\lambda \neq 0$, an Eigenvalue of k, then the Eigenspace has finite dimension.
$\ker(\lambda I - k) = \ker(\frac{1}{\lambda} I - \frac{1}{\lambda^2} k)$

2.27 Definition: $k: X \to X$ is self-adjoint if $k = k^*$.

2.28 Lemma: a) Eigenvalues of sa. operators are real.
b) Let $ku_1 = \lambda_1 u_1$, $ku_2 = \lambda_2 u_2$, $\lambda_1 \neq \lambda_2$
$\Rightarrow (u_1, u_2) = 0$. ($k$ s.a.)

Proof: a) Let $ku = \lambda u$, $u \neq 0$.
$\lambda (u,u) = (\lambda u, u) = (ku, u) = (u, ku) =$
$= (u, \lambda u) = \bar{\lambda} (u,u)$
$\Rightarrow \lambda = \bar{\lambda}$.
b) $\lambda_1 (u_1, u_2) = (ku_1, u_2)$
$= (u_1, ku_2)$
$= (u_1, \lambda_2 u_2) = \lambda_2 (u_1, u_2)$
$\Rightarrow (u_1, u_2) = 0$.

2.29 Theorem: Let k cp., s.a. operator, $k: X \to X$.
Then $\lambda = \|k\|$ or $\lambda = -\|k\|$ is an Eigenvalue of k.

$\|k\| \geq |\mu|$ for all Eigenvalues μ of k.
Proof: $\|k\| = \sup_{\substack{u \in X \\ u \neq 0}} \frac{\|ku\|}{\|u\|} = \sup_{\|u\|=1} \|ku\|$
$ku = \mu u$, $v \Rightarrow \frac{\|ku\|}{\|u\|} = |\mu|$
$\leq \|k\|$

$(*_n)$: $\|u_n\| = 1$, $\|ku_n\| \to \|k\|$.
$0 < \|k^2 u_n - \|k\|^2 u_n\|^2$
$= \|k^2 u_n\|^2 - 2\|k\|^2 \|ku_n\|^2 + \|k\|^4 \|u_n\|^2$
(ku_n, ku_n)
$= \|k^2 u_n\|^2 - \|k\|^4 \|ku_n\|^2$
$\leq \|k\|^2 \|ku_n\|^2 - \|k\|^4 \|u_n\|^2$
$\to \|k\|^3 \quad \to \|k\|^4$
$\to 0$
k is compact $\Rightarrow k^2$ compact
u_n bounded $\Rightarrow k^2 u_n \to \|k\|^2 g$
$\|k^2 k^2 \cdot \frac{1}{g} \cdot \|k\|^2 u_n\| \to 0$
$u_n \to g$.
$k^2 g = \lim k^2 u_n = \|k\|^2 \cdot g$
$\Rightarrow k^2 \frac{1}{g} \cdot g = \|k\|^2 \cdot g$
$(k - \|k\| \cdot I)\underbrace{(k + \|k\| \cdot I)\, g}_{\neq 0} = 0$
$\Rightarrow 0$ is Ew of k for $-\|k\|$
$\Rightarrow$ Ew of k for $\|k\|$

Examples:
$k: L^2([0,1]) \to (\cdot, [0,1])$
$(ku)(x) = \int_0^1 e(x,y) u(y)\, dy \quad k \in C^1$.
$k(x,y) = k(y,x) \Rightarrow k$ s.a. $\left(\int = \int_0^1\right)$
$(ku, v) = \int_0^1 \int_0^1 e(x,y) u(y)\, dy\, v(x)\, dx$
$= \int_0^1 \int_0^1 e(y,x)\, v(x)\, dx\, u(y)\, dy$
$= (u, (kv))$
$u(x,y) = \alpha(x) \cdot \alpha(y)$
$\|ku\|_2^2 = \int_0^1 (ku)(x)^2\, dx$
$= \int_0^1 \left[\int_0^1 \alpha(x) \alpha(y) u(y)\, dy\right]^2 dx$
$\overset{CS}{\leq} \int_0^1\alpha(x)^2 \int_0^1 \alpha(y)^2 dy \int_0^1 u(y)^2\, dy$
$= \|\alpha\|^4 \cdot \|u\|_2^2$
$\Rightarrow \|k\| \leq \|\alpha\|^2$
$(ku)(x) = \int_0^1 \alpha(x) \alpha(y) u(y)\, dy$
$= \int_0^1 \alpha(y)^2 dy \cdot \alpha(x)$
$\Rightarrow \alpha$ is Eigenvalue of k w.E. $\|\alpha\|^2$
$\Rightarrow \|k\| = \|\alpha\|^2$, $\|\alpha\|^2$ is Eigenvalue of k.

(1) $k : X \to X$ cpt, $\{u_i\}$ ONS

$(u_i \mid ONS$

$\|u\|^2 \geq \sum_i (u_i, u)^2$

$\|\sum \alpha_i u_i\|^2 = (\sum \alpha_i u_i, \sum \alpha_j u_j) = \sum \alpha_i^2$

$0 \leq \|u - \sum_{i=1}^{n} (u, u_i) u_i\|^2 = \|u\|^2 - 2 \sum_{i=1}^{n} (u, u_i)^2 + \sum (u, u_i)^2$

$\Rightarrow \|u\|^2 \geq \sum (u_i, u)^2$

Let $\{u_i\}$ ONS, $u \in \overline{\text{span}\{u_i\}}$

$\Rightarrow u = \sum_{i=1}^{\infty} \langle u, u_i \rangle u_i$

$\|u - \sum_{i=1}^{n} \alpha_i u_i\| \to 0$

$\Rightarrow \|u - \sum_{i=1}^{n} \alpha_i u_i\|$ is Cauchy seq wlog

$\Rightarrow \sum_{i=1}^{\infty} \alpha_i^2$ is Cauchy seq wlog

$u = \sum_{i=1}^{\infty} \alpha_i u_i \Rightarrow (u, u_i) = \alpha_i$

$\|u\|^2 = \sum \alpha_i^2 = \sum (u, u_i)^2$

2.23 Def: $\{u_n\}$ ONS in X. $\{u_n\}$ complete

iff $X = \overline{\text{span}\{u_n\}}$ — Lebedev

Let $u \in X$, $U_n = \text{span}\{u_1, \dots, u_n\}$, use $U_n = \sum_{k=1}^{n} \alpha_k u_k$, $\alpha_k = (u, u_k)$.

$\|u - u_n\|^2 = \|u\|^2 - 2\sum_{k=1}^{n} \alpha_k (u, u_k) + \sum_{k=1}^{n} \alpha_k^2$

$= \|u\|^2 + \sum_{k=1}^{n} (\alpha_k - \alpha_k)^2 - \sum \alpha_k^2$

minimal for $c_k = \alpha_k$.

$V_n = \sum_{k=1}^{n} \alpha_k u_k$ is Best approx of u in U_n.

$\{u_n\}$ complete $\Rightarrow \forall u \in U_n$ $\forall u - u \| < \varepsilon$

$\Rightarrow \forall u \in X \Rightarrow \|u - u_n\| < \varepsilon$

$= \sum_{k=n}^{\infty} \alpha_k^2$

2.24 $\{u_n\}$ is complete $\Leftrightarrow \|u\|^2 = \sum_{k=1}^{\infty} (u, u_k)^2$.

Parseval identity

$\Leftarrow$: $\|u - \sum_{i=1}^{n} (u, u_i) u_i\|^2 = \|u\|^2 - \sum_{i=1}^{n} (u, u_i)^2$

$\to 0$

2.25 Let $k : X \to X$ compact. Then

$\dim \ker (\lambda - k) < \infty$.

Proof: Assume $\dim \ker (\lambda - k) = \infty$.

As in Riesz' Lemma: $\|u_n\| = 1$, $\|u_n - u_m\| \geq \frac{\lambda}{2}$

u_n bound, $\Rightarrow k u_{n_j} \to u$ has no cgt subsequence.

2.26 Cor: If $\lambda \neq 0$ is an Eigenvalue of k then the Eigenspace has finite dimension.

$\ker (\lambda I - k) = \ker (I - \frac{1}{\lambda} k)$

Def 2.27 $k : X \to X$ is self-adjoint iff $k = k^*$.

Lemma 2.28 Eigenvalues of s.a. operators are real

Let $k u_n = \lambda_n u_n$, $k u_j = \lambda_j u_j$, $\lambda_n \neq \lambda_j$

$\Rightarrow (u_n, u_j) = 0$

a) $\mu (u, u) = (\mu u, u) = (k u, u) = (u, k^* u) = (u, \mu u) = \bar{\mu} (u, u)$

b) $\lambda_j (u_n, u_j) = (k u_j, u_j) = (k u_j, u_n)$

$= (u_j, k u_n)$

$= (u_j, \lambda_n u_j) = \lambda_j (u_n, u_j)$

$\Rightarrow (u_n, u_j) = 0$.

Theorem 2.29: Let k cpt, self-adjoint.

Then $\|k\|$ or $-\|k\|$ is Eigenvalue of k.

$\|k\| > |\mu|$ for all Eigenvalues μ of k.

Proof: $\|k\| = \sup_{\substack{u \in X \\ \|u\|=1}} \|k u\| = \sup_{\substack{u \in X \\ \|u\|=1}} \|k u\|^2$

$k = k^* \Rightarrow \|k\|^2 = \sup_{\|u\|=1} \|k^2 u\|$

$\leq \|k\|^2$

$\forall u_n \exists u_n, \|k u_n\| \to \|k\|$.

$0 \leq \|k^2 u_n - \|k u_n\|^2 u_n\|^2$

$= \|k^2 u_n\|^2 - 2\|k u_n\|^2 (k^2 u_n, u_n) + (\|k u_n\|^2)^2$

$\leq \|k\|^2 \|k u_n\|^2 - \|k u_n\|^4 \to 0$

$\forall \varepsilon > 0 \Rightarrow k^2 \varepsilon > \exists \tilde{\varepsilon} u_n \to \|k\|^2 \cdot \varepsilon$

$(k - \|k\| I)(k + \|k\|) \varepsilon = 0$

$\underbrace{\qquad}_{\to 0} \quad \underbrace{\qquad}_{\to 2\|k\|}$

$\Rightarrow$ $\|k\|$ is Eigenvalue, or $-\|k\|$ Eigenvalue for $\|k\|$

$k : X \to X$, s.a., cp. $k \neq 0$

$k u_1 = \lambda_1 \cdot u_1$, $\|u_1\| = \|k\|$, $u_1 \neq 0$.

$k_2 u = k u - \lambda_1 (u, u_1) \cdot u_1$

k_2 s.a., cp

$k_2 u_1 = k u_1 - \lambda_1 (u_1, u_1) \cdot u_1 = 0$

$u_1 \in \ker(k_2)$.

If $\|k_2\| \neq 0$; $k_2 u_2 = \lambda_2 u_2$, $|\lambda_2| = \|k_2\| \neq 0$, $\|u_2\| = 1$.

$(u_1, u_2) = 0$.

$\lambda_2 u_2 = k_2 u_2 = k u_2 - \lambda_1 (u_1, u_2) \cdot u_1 = k u_2$

$\Rightarrow u_2$ is Eigenvector of k wrt λ_2.

$|\lambda_1| \geq |\lambda_2|$

$k_n u = k u - \sum_{k=1}^{n-1} \lambda_k (u, u_k) \cdot u_k$

If $k_n \neq 0$: k_n has EV $\lambda_{max} \neq 0$

$\quad \Rightarrow$ Eigenvector u_{max}, $\|u_{max}\| = 1$.

$\vdots$

System of Eigenvectors (u_k) and Eigenvalues λ_{k_1}, $\lambda_k \neq 0$, $|\lambda_{k_1}| \geq |\lambda_{k_2}| \geq \dots$ of k.

$t_n \neq 0$. $t_n \to 0$

λ_m

If λ_m is bounded from below > 0.

$|\lambda_n| \to c \neq 0$

If $\frac{1}{\lambda_n} u_n \perp \frac{1}{|\lambda_n|} u_n$, $\|u_n\| \leq C$.

$k\left(\frac{1}{\lambda_n} u_n\right) = u_n$ must have a convergent subsequence.

u_n ONS, $\|u_n - u_k\| \geq \frac{1}{C^2}$, $l \neq k$.

$\Rightarrow u_n$ has no cgt. seqs.

$\lightning$

$|\lambda_n| \to 0$.

$k : X \to X$ cpt s.a. (λ_i, u_i):

u_i ONS of Eigenvectors, $k u_i = \lambda_i u_i$,

$(\lambda_i) \in |\lambda_i|$, $\quad$ diabo $(k - \lambda_i \mathbb{1}) \neq$

$\quad \Rightarrow \lambda_i$ occurs $\leq$ dim.

$H = \left\{ u \in X : u = \sum_k \alpha_k u_k \right\}$

$v \in H^\perp \Rightarrow (v, u_k) = 0$

$\Rightarrow (k v, u_k) = (v, k u_k) = \lambda_k \cdot (v, u_k) = 0$

$\Rightarrow k v \in H^\perp \Rightarrow k : H^\perp \to H^\perp$

If $k \neq 0$: If $\ker \lambda \neq$ Eigenvalue $\lambda \neq \pm \|k\| \neq 0$ with corresponding Eigenvectors.

$(\lambda, u) \quad u \in H^\perp \Rightarrow u$ cannot be represented as linear combination of u_k

$\lightning$ u_k complete set of Eigenvectors.

$K : H^\perp \to H^\perp$, $k = 0 \Rightarrow \ker(k) = H^\perp$.

$X = H \oplus H^\perp \qquad X = \overline{H} \oplus H^\perp$

w_n Cauchy sequences in H

$w_n = \sum_k \alpha_k^{(n)} u_k$, $\|w_n\| \leq C$ $\forall n$.

$\|w_n - w_m\|^2 = \sum_k (\alpha_k^{(n)} - \alpha_k^{(m)})^2 \to 0$

For fixed k, $\alpha_k^{(n)} \to \alpha_k$

$\|w_n\|^2 = \sum_k (\alpha_k^{(n)})^2 \leq C^2$

$\Rightarrow \sum_k (\alpha_k \bigcirc)^2 \leq C^2$

$\quad$ converges

$\Rightarrow u = \sum_k \alpha_k u_k$

$w_n \to u \in H \Rightarrow \overline{H} = H$.

$X = H \oplus H^\perp$

Theorem 3.30: Let $K : X \to X$ cpt, s.a., X Hilbert.

(u_k) solve all orthonormal eigenvectors for Eigenvalues $\lambda_k \neq 0$.

Let $u \in X$. Then

$u = \underset{\in H}{u_1} \oplus \underset{\in H^\perp}{u_2} \Rightarrow k u_2 = 0$

$u_2 = \sum_k (u, u_k) \cdot u_k$

$\Rightarrow u = \sum_k (u, u_k) \cdot u_k + u_2$, $k u_2 = 0$.

07

Literature: Hochstadt, Integral operators

Corrections in the PDFs

Scriptum available in Learnweb

Theorem 2.30: Let $K: X \to X$ cp, s.a., X Hilbert.

(u_k) set of all orthonormal eigenvectors for Eigenvalues $\lambda_k \neq 0$.

$$u \in U \Rightarrow u = \sum_k (u, v_k) \cdot v_k + u^\perp, \quad K u^\perp = 0, (u^\perp, v_k) = 0.$$

$$(\Leftrightarrow A = U D U^+, A \text{ symmetric}, D \text{ diagonal}, U^+ U = U U^+ = I : \dim X < \infty).$$

$u \in U$, $\tilde{u} = u + n$, $\|n\|_2 \leq \varepsilon$.

$$\|Ku - K\tilde{u}\| \leq \|K\| \cdot \varepsilon$$

$$n = \sum_k (n, u_k) \cdot u_k + n^\perp \quad \text{n, not } \varepsilon \text{ !}$$

$$\|Ku - K\tilde{u}\|_2 = \|K\left(\sum_k (n, v_k) \cdot u_k + n^\perp\right)\|_2$$

$$= \left\| \sum_k \lambda_k (n, v_k) \cdot v_k \right\|$$

$$\leq \sum_k |\lambda_k| |(n, v_k)|$$

$$|\lambda_1| = \|K\|$$

$$|\lambda_k| \leq |\lambda_1|$$

SVD:

K compact $\Rightarrow K u = \sum_k \sigma_k (u, u_k) \cdot v_k$

ONS $(u_k), (v_k)$, $\sigma_k > 0$

1) $\sum_k (u, u_k)^2 \le \|u\|^2$ $\quad$ $(u, u_k) \cdot u_k$

$|(u, u_k)| \le \sigma_k |(u, u_k)|$

$\sigma_k \to 0$

2) Analytically computing the SVD is difficult in general.

3) Discretising the problem

$\Rightarrow$ Matrix $\Rightarrow$ SVD of the matrix

"converges" to the SVD of K.

4) Don't use it.

5) Most of the SVDs of operators in imaging share a common property:

Example:

$K : L^2([0,\pi]) \to L^2([0,\pi])$

$(Ku)(x) = \int_0^\pi k(x,t)\, u(t)\, dt$

$= \int_0^\pi k(x,t) u(t)\, dt \qquad k(x,t) = \begin{cases} 1, & t \le x \\ 0, & t > x \end{cases}$

$(K^* v)(t) = \int_0^\pi k(x,t)\, v(x)\, dx$

$= \int_t^\pi v(x)\, dx.$

$(K^* K) u = \sigma^2 \cdot u$

$\Rightarrow \sigma^2 u(t) = \int_t^\pi \int_0^x u(s)\, ds\, dx$

$\Rightarrow u(\pi) = 0, u'(0) = 0.$

$\Rightarrow \sigma^2 u''(t) = -u(t).$

$u''(t) - \frac{1}{\sigma^2} u(t) = 0, \; u(\pi) = 0, u'(0) = 0.$

$\cos\left(\left(\frac{n}{2} + n\right)\pi\right) \cdot t = u_n(t), \; \sigma_n^2 = \left(\frac{1}{\frac{n}{2} + n}\right)^2$

$\sigma_n = \frac{2}{n+2\pi}$

$v_n(t) = \sin\left(\frac{t}{\sigma_n}\right) \cdot \sqrt{2}$

$u_n(t) = \cos\left(\frac{t}{\sigma_n}\right) \cdot \sqrt{2}$

$\sigma_n = \frac{2}{n}\frac{1}{(2n+1)} \to 0, \quad \sigma\left(\frac{1}{n}\right)$

Singular functions are low frequency for n small, or n large.

$Ku = g$

$K^* K u = K^* g$

$u \in \text{Range}(K^*)$

$Ku = \sum_i \sigma_i (u, u_i) v_i$

$K^* v = \sum_k \sigma_k (v, v_k) \cdot u_k \in$

$K^* K u = K^* \sum_i \sigma_i (u, v_i) v_i$

$= \sum_i \sigma_i^2 (u, u_i) \cdot u_i$

$w \in \text{SR}(K) \Rightarrow w = \sum_i \mu_i u_i + u^\perp$

$K^* K w = K^* g = \sum_k \sigma_k (g, v_k) \cdot u_k$

$\sum_i \sigma_i^2 \mu_i u_i$

$\Rightarrow \sigma_k^2 \mu_k = \sigma_k \cdot (g, v_k)$

$\Rightarrow \mu_k = \frac{1}{\sigma_k} (g, v_k)$

$w = \sum_k \frac{1}{\sigma_k} (g, v_k) \cdot u_k + u^\perp$

w MNS: $w = \sum_k \frac{1}{\sigma_k} (g, v_k) \cdot u_k$

$\sum_k (g, v_k)^2 \le \|g\|^2$

$\sum_k (g, v_k) \cdot u_k$ exists

infinite sing. value $\Rightarrow$ ill-posed

$\sigma_k \to 0 \Rightarrow \frac{1}{\sigma_k} \to \infty$

2.41: Definition

Let $K: X \to X$ cp. s.a.

K positive (semi-) definite

if/f $(x, Kx) > (\geq) 0 \quad \forall x \in X, x \neq 0$.

2.42: Remark

1) K positive semidefinite

$\Leftrightarrow$ All Eigenvalues $\lambda_k \geq 0$.

2) $K: X \to Y \Rightarrow K^*K: X \to X$, pos. semidef.

cp. $K K^*: Y \to Y$, "

$(K^*Kx, x) = (Kx, Kx) = (\lambda_k k^* K x)$

$= \|Kx\|^2 \geq 0$

Theorem 2.43:

Let $K: X \to Y$, linear + compact, X, Y Hilbert.

$K^*K, K K^*$ are positive semidefinite

Let $(u_k), \lambda_k$ a complete system of Eigenvalues and Eigenvectors of K^*K as in 2.40.

$\lambda_k = \sigma_k^2, \sigma_k > 0$.

Let $v_k = \frac{K u_k}{\|K u_k\|}$. Also v_k is an ONS.

u_k are Eigenvectors of KK^* for the Eigenvalues $\lambda_k = \sigma_k^2$.

$\forall u \in X: Ku = \sum_k \sigma_k (u, u_k) \cdot v_k$.

$\forall v \in Y: K^* v = \sum_k \sigma_k (v, v_k) \cdot u_k$.

σ_k: Singular values of K.

u_k: (right) singular vectors of K

v_k: (left) " " " "

Proof: $\ker(K^*K) = \ker(K)$:

$x \in \ker(K^*K) \Rightarrow (x, K^*Kx) = (Kx, Kx)$

$= \|Kx\|^2 = 0$

$\Rightarrow x \in \ker(K)$.

$\sigma_k^2 u_k = K^*K u_k$, $\sigma_k \neq 0$, $u_k \perp 0$.

$\ker K$: $(u_k, v_k) = \frac{1}{\|Ku_k\|} \cdot \underbrace{(Ku_k, Ku_k)}_{\|Ku_k\|^2 \|u_k\|^2} = 0$.

(K^*Ku_k, u_k)

$\sigma_k^2 (u_k, u_k) = 0$

$KK^*(v_k) = \frac{1}{\|Ku_k\|} KK^*Ku_k$

$= \sigma_k^2 \cdot \frac{1}{\|Ku_k\|} \cdot Ku_k = \sigma_k^2 \cdot v_k$

$\Rightarrow v_k$ is an Ev of KK^* with Ev σ_k^2

Let $\{u_1, \dots, u_m\}$ ONS of K^*K for the Eigenvalue $\lambda \neq 0$, $u_1, \dots, u_m$ are Eigenvectors of KK^*

$\Rightarrow \dim \ker(K^*K - \lambda I) \geq \dim \ker(KK^* - \lambda I)$

$\Rightarrow$ The dimensions of the Eigenspaces of KK^* and K^*K for an Eigenvalue λ are the same.

$\|Ku_k\|^2 = (Ku_k, Ku_k)$

$= \sigma_k^2 \Rightarrow \sigma_k = \|Ku_k\|$

$v_k = \frac{1}{\|Ku_k\|} Ku_k = \frac{1}{\sigma_k} Ku_k$.

$K^*Ku_k = \sigma_k^2 u_k$

$\sigma_k K^* v_k = \sigma_k^2 u_k$

$K^* v_k = \sigma_k u_k$, $Ku_k = \sigma_k \cdot v_k$

$u \in X: u = \sum_k (u, u_k) \cdot u_k + u^\perp$

$Ku = \sum_k (u, u_k) \cdot Ku_k$

$= \sum_k \sigma_k (u, u_k) \cdot v_k$

$v \in Y: v = \sum_k (v, v_k) \cdot v_k + v^\perp$, $v^\perp \in \ker(KK^*)$

$K^* v = \sum_k (v, v_k) \cdot K^* v_k$

$= \sum_k \sigma_k (v, v_k) \cdot u_k$

$k_\alpha^\dagger g := \sum_k g_\alpha(\sigma_k)\,(g, v_k)\cdot u_k$, $g_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$.

$K^\dagger g = \sum_k \frac{1}{\sigma_k}\,(g, v_k)\cdot u_k$

$g_\alpha(\sigma) = \frac{\sigma}{\alpha^2+\sigma^2}$: Tikhonov

$g_\alpha(\sigma) = \begin{cases} 0 : \sigma \le \alpha \\ 1/\sigma : \sigma > \alpha \end{cases}$: Truncated SVD

1) $g_\alpha(\sigma) \xrightarrow[\alpha\to 0]{} \frac{1}{\sigma}$ (pointwise)

2) $\sup_\sigma g_\alpha(\sigma) \le C_\alpha$ (guarantees existence)

3) $\sup_{\alpha,\sigma} \sigma \cdot g_\alpha(\sigma) \le C$

Tikhonov: $\alpha^2+\sigma^2-2\sigma^2 = 0 \Rightarrow \sigma = \alpha \Rightarrow C_\alpha = \frac{1}{2\alpha}$

$C = 1$

Truncated SVD: $C_\alpha = \frac{1}{\alpha}$

$C = 1$.

Lemma 3.5: $\|K_\alpha^\dagger\| \le C_\alpha$, $K_\alpha^\dagger g \to K^\dagger g$ $\forall g \in \mathcal{D}(K^\dagger)$.

Let $g \in X$. $\|K_\alpha^\dagger g\|^2 = \|\sum_k g_\alpha(\sigma_k)(g, v_k)\cdot u_k\|^2$

$= \sum_k g_\alpha(\sigma_k)^2\,(g, v_k)^2$

$\le C_\alpha^2 \|g\|^2$ (Bessels inequality)

Fix $g \in \mathcal{D}(K^\dagger) \Rightarrow \sum_k \frac{1}{\sigma_k^2}(g, v_k)^2 < \infty$.

(Picard-criterion)

$\|K_\alpha^\dagger g - K^\dagger g\|_2^2 = \sum_k \left(\frac{1}{\sigma_k} - g_\alpha(\sigma_k)\right)^2 (g, v_k)^2$

$= \sum_k \frac{1}{\sigma_k^2}\left(1 - \underbrace{\sigma_k g_\alpha(\sigma_k)}_{\le C}\right)^2 \langle g, v_k\rangle^2$

$\le C^2$ independent of α

$\alpha \to 0$: $\|K_\alpha^\dagger g - K^\dagger g\|_2^2 \to \sum_k \frac{1}{\sigma_k^2}\left(1 - \sigma_k \lim_{\alpha\to 0} g_\alpha(\sigma_k)\right)^2 \langle g, v_k\rangle$

$\to \frac{1}{\sigma_k}$

$= 0$.

Cor. 3.5: Choose $\alpha(\delta, g^\delta) \xrightarrow[\delta\to 0]{} 0$, $\delta \cdot \|K_{\alpha(\delta, g^\delta)}^\dagger\| \to 0$.

If $\|K_\alpha^\dagger\| = g(\alpha)$ strictly decreasing

$\alpha(\delta) = g^{-1}\left(\frac{1}{\sqrt{\delta}}\right) \Rightarrow \alpha(\delta) \xrightarrow[\delta\to 0]{} 0$, $\delta \cdot \|K_{\alpha(\delta)}^\dagger\| = \sqrt{\delta} \to 0$.

Then $\alpha, K_\alpha^\dagger$ is a regularization of $K^\dagger f = g$.

Proof: Let $g \in \mathcal{D}(K^\dagger)$ fixed, $\|g - g^\delta\| \le \delta$.

$\|K_{\alpha(\delta)} g^\delta - K^\dagger g\| \le \underbrace{\|K_{\alpha(\delta)} g^\delta - K_{\alpha(\delta)} g\|}_{\le \delta \cdot \|K_{\alpha(\delta)}\|} + \underbrace{\|K_{\alpha(\delta)} g - K^\dagger g\|}_{\to 0}$

$\to 0$ (for $\delta \to 0$)

$\to 0$ $\quad o(\sqrt{\delta})$

Corollary 2.34:

Let $k: X \to Y$ cpt, X, Y Hilbert.
Let $(\sigma_\nu, u_\nu, v_\nu)$ singular system.

The LSQ of $Ku = g$ are

$$\sum_\nu \frac{1}{\sigma_\nu}(g, v_\nu)\cdot u_\nu + u^\perp \qquad u^\perp \in N(K).$$

The MNS of $Ku = g$ is

$$\sum_\nu \frac{1}{\sigma_\nu}(g, v_\nu)\cdot u_\nu.$$

Proof: Compare coefficients in $K^*Ku = K^*g$
$u \in \mathrm{Range}(K^*)$.

$$g \in R(K) \oplus \underbrace{R(K)^\perp}_{=N(K^*)} \qquad KK^*v = 0$$

$$g = k_N + v$$

$$(g, u_\nu) = (Kx + v, v_\nu) = (x, K^*v_\nu) + (v, v_\nu) = (x, \sigma_\nu u_\nu)$$

$$\sum_\nu (x, u_\nu)\cdot u_\nu \text{ converges.}$$

Definition 2.35 Degree of ill-posedness

An inverse problem is
mildly ill-posed of order α:
iff $\exists\, C: \sigma_n \le C \cdot \frac{1}{n^\alpha} \quad \forall n$.
$(\sigma_n = O(\frac{1}{n^\alpha}))$

severely ill-posed: if no α, C with this property exist.

Definition 2.36 Generalized Inverse, Moore-Penrose-Inverse

Let $K: X \to Y$, Hilbert,

$K^+: \mathrm{Range}(K) \oplus \mathrm{Ker}(K^*) \to X$,

$K^+g := u^+$, u^+ MNS of $Ku = g$

is called Generalized Inverse.

---

3. Regularization

Compute $f^+: K^+g = \sum_\nu \frac{1}{\sigma_\nu}(g, v_\nu)\cdot u_\nu$, $g \in R(K) \oplus R(K)^\perp$.

$g^\delta \in Y: \|g - g^\delta\| \le \delta$.

$K_\alpha^+ g^\delta := \sum_\nu \frac{1}{\sigma_\nu}(g^\delta, v_\nu)\cdot u_\nu$ well-defined on X continuous.

Idea for $\alpha = \alpha(\delta, g^\delta)$:

$\alpha = \alpha(\delta, g^\delta)$: $\|K_\alpha^+\| \le \frac{1}{\alpha}$

$$\|f_\delta^+ - f^+\| \le \underbrace{\|K_\alpha^+ g^\delta - K_\alpha^+ g\|}_{\le \|K_\alpha^+\|\cdot\delta} + \underbrace{\|K_\alpha^+ g - K^+ g\|}_{\text{Worst operator}}$$

Worst operator:
$$\sum_\nu \frac{1}{\sigma_\nu}(g, v_\nu)\cdot u_\nu$$
$|\sigma_\nu| \le \alpha$

Definition 3.1: Let $K: X \to Y$. A family of operators $K_\alpha^+, \alpha \in \mathbb{R}^+$,
$K_\alpha^+: Y \to X$, is a regularization of K^+, iff
there is a parameter choice $\alpha(\delta, g^\delta)$,
such that $\forall\, g \in R(K) \oplus R(K)^\perp$:

$$\lim_{\delta \to 0} \sup_{g^\delta \in Y,\ \|g - g^\delta\| \le \delta} \|K_{\alpha(\delta, g^\delta)}^+ g^\delta - K^+ g\| = 0$$

$$|\alpha(\delta, g^\delta)| \to 0 \quad (\text{necessary: } \alpha \to 0)$$

$$\forall\, u: K^+u = K^+.$$

$\alpha = \alpha(\delta)$: a-priori choice.

Remark: Many kinds of regularization exist.

Corollary 3.2: Let (g_n) such that

$$\|g_n - g\| \le \delta_n, \quad \delta_n \to 0.$$

$K_{\alpha(\delta_n, g_n)}^+ g_n \xrightarrow{n\to\infty} K^+ g$ $\Leftrightarrow$ K_α^+ is regularization.

Remark: If $\alpha = \alpha(g^\delta) \to K^+$ is continuous.

Tikhonov-Regularization

(Tikhonov 1963, Phillips 1962)

Definition 3.3: f^α Tikhonov-Regularization of $Kf = g$

iff $f^\alpha = \arg\min_f \|Kf - g\|^2 + \alpha^2 \|f\|^2$ $(\alpha = \alpha(\delta, g_\delta)), \alpha > 0$.

$\overline{K}_\alpha : X \to Y \times X$, $\overline{K}_\alpha f := \begin{pmatrix} Kf \\ \alpha I \end{pmatrix} = \begin{pmatrix} K \\ \alpha I \end{pmatrix} f$ & $\overline{g} := \begin{pmatrix} g \\ 0 \end{pmatrix}$.

$\|\overline{K}_\alpha f - \overline{g}\|_2^2 = \|Kf - g\|^2 + \alpha^2 \|f\|^2$

$\Rightarrow f^\alpha$ is least-squares solution of $\overline{K}_\alpha f^\alpha = \overline{g}$

$\overline{K}_\alpha^* \overline{K}_\alpha f^\alpha = \overline{K}_\alpha^* y \Rightarrow (K^*K + \alpha^2 I) f^\alpha = K^* y$

(regularized normal equation)

$A := K^*K + \alpha^2 I$. (Remark: A is not compact!)

1) A is s.a., continuous if K is continuous.
2) A is injective: $(Au, u) = (Ku, Ku)_Y + \alpha^2 (u, u)_X \geq \alpha^2 \|u\|^2 > 0$ $(u \neq 0)$
3) A is positive definite.
4) A is invertible: $X = \overline{\text{Range}(A)} \oplus \underbrace{\text{Ker}(A)}_{=\{0\}}$ Show: Range (A) is closed.

$\|u\|^2 \leq \frac{1}{\alpha^2} (Au, u) \leq \frac{1}{\alpha^2} \|Au\| \cdot \|u\|$ C-S

$\Rightarrow \|u\| \leq \frac{1}{\alpha^2} \|Au\|$

$Au_n \to v$

$\Rightarrow \|u_n - u_m\| \leq \frac{1}{\alpha^2} \|A(u_n - u_m)\|$ Cauchy Sequence

$\Rightarrow u_n \to u \Rightarrow \begin{matrix} Au_n \to Au \\ Au_n \to v \end{matrix} \Rightarrow v \in \text{Range}(A) \Rightarrow \text{Range}(A)$ closed.

5) A^{-1} is continuous:

$\|A^{-1} v\| \leq \frac{1}{\alpha^2} \|v\| \Rightarrow \|A^{-1}\| \leq \frac{1}{\alpha^2}$

Lemma 3.4: Let $K : X \to Y$ continuous, linear, X, Y Hilbert.

Then Tikhonov-Regularization is well-defined and continuous.

$f_\alpha^\# = (K^*K + \alpha^2 I)^{-1} K^* g$.

Proof: $f_\alpha^\# = A^{-1} K^* g$, $\|A^{-1} K^*\| \leq \|A^{-1}\| \cdot \|K^*\| < \infty$.

What if K is compact? Is this a regularization?

(σ_k, u_k, v_k) singular system, $f_\alpha^\# = \sum_k a_k u_k + u^\perp$

$(K^*K + \alpha^2 I) f_\alpha^\# = \sum_k \sigma_k^2 (u_k, f_\alpha^\#) \cdot u_k + f_\alpha^\# \cdot \alpha^2$

$= \sum_k a_k (\sigma_k^2 + \alpha^2) u_k + u^\perp \cdot \alpha^2$ } Compare coefficients

$K^* g = \sum_k \sigma_k (g, v_k) \cdot u_k$

$\Rightarrow u^\perp = 0$, $a_k = \frac{\sigma_k}{\alpha^2 + \sigma_k^2} (g, v_k)$

$\Rightarrow f_\alpha^\# = \sum_k \frac{\sigma_k}{\alpha^2 + \sigma_k^2} (g, v_k) \cdot u_k$

$f^\# = K^\# g = \sum_k \frac{1}{\sigma_k} (g, v_k) \cdot u_k$ } pointwise convergence of coefficients (for $\alpha \to 0$)

Orthonormal basis of eigenvectors

Conversely: Recursive Gram-Schmidt orthonormalization

1) $K = P^{-1} K^* P$, $P^{-1} = P^*$

$K^* = (P^{-1} K^* P)^* = P^* K^* (P^{-1})^* = P^{-1} K P$

$\Rightarrow$ same spectrum $\sigma(K)$

2) Projection operator

$P_\lambda = \frac{1}{2\pi i} \int_{|z-\lambda|=\varepsilon} (zI - K)^{-1} dz$

$P_\lambda^2 = \frac{1}{(2\pi i)^2} \int_{|z-\lambda|=\varepsilon} \int_{|w-\lambda|=\varepsilon} (zI-K)^{-1} (wI-K)^{-1} dz\, dw$

$= \frac{1}{(2\pi i)^2} \int_{|z-\lambda|=\varepsilon} \int_{|w-\lambda|=\varepsilon} \frac{(zI-K)^{-1} - (wI-K)^{-1}}{w-z} dz\, dw$

3) For closed $M \subseteq \mathbb{C}^n$ eigenspace, let the index be $\operatorname{ind}(M) := \dim \ker P_\lambda$

Assume $K = \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix}$ is upper triangular,

$K_{11} = \frac{1}{2\pi i} \int_{|z-\lambda|=\varepsilon} (zI - K_{11})^{-1} dz$

$P_\lambda^2 = \frac{1}{(2\pi i)^2} \int_{|z-\lambda|=\varepsilon} \int_{|w-\lambda|=\varepsilon} \frac{(zI-K)^{-1} - (wI-K)^{-1}}{w-z} dz\, dw$

$= \frac{1}{(2\pi i)^2} \int_{|z-\lambda|=\varepsilon} \int_{|w-\lambda|=\varepsilon} \frac{(zI-K_{11})^{-1} - (wI-K_{11})^{-1}}{w-z} dz\, dw$

$\quad + \frac{1}{(2\pi i)^2} \int_{|z-\lambda|=\varepsilon} \int_{|w-\lambda|=\varepsilon} \frac{(zI-K)^{-1} - (zI-K_{11})^{-1}}{w-z} dz\, dw$

$\quad - \frac{1}{(2\pi i)^2} \int_{|z-\lambda|=\varepsilon} \int_{|w-\lambda|=\varepsilon} \frac{(wI-K)^{-1} - (wI-K_{11})^{-1}}{w-z} dz\, dw$

$= P_\lambda^2 + \frac{1}{2\pi i} \int_{|z-\lambda|=\varepsilon} (zI-K)^{-1} \begin{pmatrix} 0 & K_{12} \\ 0 & 0 \end{pmatrix} (zI-K_{11})^{-1} dz$

$\quad - \frac{1}{2\pi i} \int_{|w-\lambda|=\varepsilon} (wI-K)^{-1} \begin{pmatrix} 0 & K_{12} \\ 0 & 0 \end{pmatrix} (wI-K_{11})^{-1} dw$

$= P_\lambda^2 + \frac{1}{2\pi i} \int_{|z-\lambda|=\varepsilon} \begin{pmatrix} (zI-K_{11})^{-1} & (zI-K)^{-1} K_{12} (zI-K_{22})^{-1} \\ 0 & (zI-K_{22})^{-1} \end{pmatrix} \begin{pmatrix} 0 & K_{12} \\ 0 & 0 \end{pmatrix} dz$

$\quad - \frac{1}{2\pi i} \int_{|w-\lambda|=\varepsilon} \begin{pmatrix} (wI-K)^{-1} & (wI-K)^{-1} K_{12} (wI-K_{22})^{-1} \\ 0 & (wI-K_{22})^{-1} \end{pmatrix} \begin{pmatrix} 0 & K_{12} \\ 0 & 0 \end{pmatrix} dw$

$= P_\lambda^2 + \frac{1}{2\pi i} \int_{|z-\lambda|=\varepsilon} \begin{pmatrix} 0 & (zI-K)^{-1} K_{12} (zI-K_{22})^{-1} \\ 0 & (zI-K_{22})^{-1} \end{pmatrix} dz$

$\quad - \frac{1}{2\pi i} \int_{|w-\lambda|=\varepsilon} \begin{pmatrix} 0 & (wI-K)^{-1} K_{12} (wI-K_{22})^{-1} \\ 0 & (wI-K_{22})^{-1} \end{pmatrix} dw$

$= P_\lambda^2 + \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}$ (Residue integrability)

Now $\operatorname{ind} P_\lambda^2 = \operatorname{ind} P_\lambda$ if $Q = 0$, i.e. $K_{12} = 0$ or $\lambda \notin \sigma(K_{22})$

Choose ε s.t. $\sigma(K_{22})$ avoids circle $|z-\lambda| = \varepsilon$

$\Rightarrow$ invariant.

Examples:

1) Nilpotent $K = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$, $K^n = 0$

$(zI-K)^{-1} = \frac{1}{z^n} \begin{pmatrix} z^{n-1} & z^{n-2} & \cdots & 1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & z^{n-2} \\ & & & z^{n-1} \end{pmatrix}$, $\begin{cases} P_\lambda = I & \lambda = 0 \\ P_\lambda = 0 & \lambda \neq 0 \end{cases}$ $\Rightarrow \operatorname{ind} = n$ for $\lambda = 0$

$\Rightarrow \det = \lambda^n = \det(\lambda I - K)$ characteristic polynomial

$\Rightarrow \begin{cases} \text{Cayley-Hamilton} & p(K) = 0 \\ \text{minimal polynomial} & \text{smallest } p \text{ with } p(K) = 0 \end{cases}$

$\Rightarrow$ Jordan normal form

2) Diagonal $K = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$

$(zI-K)^{-1} = \operatorname{diag}\left(\frac{1}{z-\lambda_1}, \dots, \frac{1}{z-\lambda_n}\right)$

$P_\lambda = \begin{cases} I & \lambda \in \{\lambda_1, \dots, \lambda_n\} \\ 0 & \lambda \notin \{\lambda_1, \dots, \lambda_n\} \end{cases}$, $\operatorname{ind} = \#\{j : \lambda_j = \lambda\}$

$\Rightarrow \det = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) = \det(\lambda I - K)$

$\Rightarrow$ Cayley-Hamilton

3) Companion matrix $K = \begin{pmatrix} 0 & & & -a_0 \\ 1 & \ddots & & -a_1 \\ & \ddots & 0 & \vdots \\ & & 1 & -a_{n-1} \end{pmatrix}$

$\det(zI-K) = z^n + a_{n-1} z^{n-1} + \cdots + a_0 = p(z)$

$(zI-K)^{-1} = \frac{1}{p(z)} \begin{pmatrix} z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_1 & \cdots & 1 \\ z^{n-2} + a_{n-1} z^{n-3} + \cdots + a_2 & \cdots & z \\ \vdots & & \vdots \\ 1 & \cdots & z^{n-1} \end{pmatrix}$

$\Rightarrow P_\lambda = I$ for $\lambda \in \sigma(K)$, $P_\lambda = 0$ else

$\Rightarrow \det = p(\lambda) = \det(\lambda I - K)$

$\Rightarrow$ Cayley-Hamilton

Jordan Problem: Given $K \in \mathbb{C}^{n \times n}$, $\sigma(K) = \{\lambda_1, \dots, \lambda_m\}$, find S s.t. $S^{-1} K S$ is diagonal.

Definition: $\begin{cases} \text{eigenspace} & E_\lambda := \ker(\lambda I - K) \\ \text{generalized eigenspace} & G_\lambda := \bigcup_{k=1}^{\infty} \ker(\lambda I - K)^k \end{cases}$

Ideal: $\mathbb{C}^n = \bigoplus_{j=1}^m E_{\lambda_j}$ (sometimes $\mathbb{C}^n = \bigoplus_{j=1}^m G_{\lambda_j}$)

Theorem 1.7: $\begin{cases} \text{(i) } \lambda^k \in \sigma(K^k) \\ \text{(ii) } \sigma(K^k) \subseteq \{\lambda^k : \lambda \in \sigma(K)\} \end{cases}$ $\Leftrightarrow$ $\begin{cases} \text{(i) } \lambda^k \in \sigma(K^k) \\ \text{(ii) } \sigma(K^k) \subseteq \{\lambda^k : \lambda \in \sigma(K)\} \end{cases}$

Thm $\dim G_\lambda = \operatorname{ind}(\lambda)$ where $\operatorname{ind}(\lambda) = \begin{cases} \text{multiplicity of } \lambda \text{ as root of } \det(zI-K) \\ \text{algebraic multiplicity of } \lambda \end{cases}$

Proof: $(u^*, v^*) \frac{1}{2\pi i} \int_{|z-\lambda|=\varepsilon} (zI-K)^{-1} dz \begin{pmatrix} x \\ y \end{pmatrix}$

$= \frac{1}{2\pi i} \int_{|z-\lambda|=\varepsilon} \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} (u^* K^j x + u^* K^j y + v^* K^j x + v^* K^j y) dz$

$= u^* K^0 x + v^* K^0 y = u^* x + v^* y$

$\Rightarrow P_\lambda^2 = P_\lambda$, $\operatorname{ind} P_\lambda = \operatorname{tr} P_\lambda = \dim \operatorname{im} P_\lambda$

$\Rightarrow \begin{cases} \text{(i) } \dim \operatorname{im} P_\lambda \geq 1 & \text{if } \lambda \in \sigma(K) \\ \text{(ii) } \dim \operatorname{im} P_\lambda = 0 & \text{if } \lambda \notin \sigma(K) \end{cases}$

Assume $K = \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix}$ is upper triangular, $\sigma(K_{11}) \cap \sigma(K_{22}) = \emptyset$

$\Rightarrow P_\lambda = \begin{pmatrix} P_\lambda & Q \\ 0 & 0 \end{pmatrix}$ where $P_\lambda = \frac{1}{2\pi i} \int_{|z-\lambda|=\varepsilon} (zI-K_{11})^{-1} dz$

$\Rightarrow \begin{cases} P_\lambda = P_\lambda^2 + Q \\ P_\lambda = P_\lambda^2 \end{cases} \Rightarrow Q = 0$

$\Rightarrow \begin{cases} \dim \operatorname{im} P_\lambda = \dim \operatorname{im} P_\lambda & \text{if } \lambda \in \sigma(K_{11}) \\ \dim \operatorname{im} P_\lambda = 0 & \text{if } \lambda \notin \sigma(K_{11}) \end{cases}

Regularization by Finite-Dimensional Approximation

1) Discretization:

$$(Ku)(t) = \int_0^\pi k(t,s)u(s)ds$$

$$Ku(x_l) \approx h \cdot \sum_{j=0}^{n} k(x_l, x_j)\, u(x_j) \cdot \textcircled{1}$$

$x_0 - x_n$ equidistant

i.e. $[0,\pi]$.

$$\widetilde{K} = \left(k(x_l, x_j)\right)$$

$$\begin{pmatrix} (Ku)(x_0) \\ \vdots \\ (Ku)(x_n) \end{pmatrix} \approx \widetilde{K} h \textcircled{1} \begin{pmatrix} u(x_0) \\ \vdots \\ u(x_n) \end{pmatrix}$$

Trapez: $\widetilde{K} \cdot h \begin{pmatrix} a_0 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \begin{pmatrix} u(x_0) \\ \vdots \\ u(x_n) \end{pmatrix} \Rightarrow \widetilde{K} D_1$ is the finite-dim. operator that approximates K

solve $\widetilde{K} \cdot D_1 \cdot u^{Trap} = g$

Simpson: $\widetilde{K} \cdot h \begin{pmatrix} a_0 & & & & 0 \\ & a_1 & & & \\ & & a_2 & & \\ & & & \ddots & \\ 0 & & & & a_{2n} \end{pmatrix} \begin{pmatrix} u(x_0) \\ \vdots \\ u(x_n) \end{pmatrix} \Rightarrow \widetilde{K} D_2$ "

solve $\widetilde{K} \cdot D_2 \cdot u^{Simp} = g$

$\Rightarrow$ Assume $\widetilde{K}$ invertible.

$$D_2 \cdot u^{Trap} = D_2 \cdot u^{Simps}$$

$$\sim I \sim u^{Trap} = D_1 \cdot u^{Simps}$$

Truncated SVD:

$$u_\alpha^\delta = K_\alpha^+ g = \sum_{\sigma_k \geq \alpha} \frac{1}{\sigma_k} (g, v_k) \cdot u_k$$

$\Rightarrow$ Look for solutions in $\mathrm{span}\langle u_1 \dots u_n \rangle$

$Ku \approx g$

$\Rightarrow (v_k, Ku^\delta) \approx (v_k, g) \Leftrightarrow \varphi_k(Ku^\delta - g) \approx 0$

$\varphi_k(v) = (v_k, v)$ Linear functionals

2) Moment-Methods, Galerkin-Methods:

As above, u_k, v_k not necessarily singular system.

"Regularization by projection"

Theorem: $\overline{\mathrm{span}\, u_k} = X$

$\Rightarrow$ Discretization is Regularization.

$\overline{\mathrm{span}\, v_k} \supseteq \overline{R(K^*)}$

Regularization by SVD

$K_\alpha^+ g := \sum_k g_\alpha(\sigma_k)\,(g, v_k)\cdot u_k$, $g_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$

$g_\alpha(\sigma) \xrightarrow[\alpha\to 0]{} \frac{1}{\sigma}$, $\sup_\sigma \sigma\, g_\alpha(\sigma) \le C_\alpha$, $\sup_\alpha g_\alpha(\sigma) \le C$

Tikhonov: $g_\alpha(\sigma) = \frac{\sigma}{\alpha^2+\sigma^2}$, $C_\alpha = \frac{1}{2\alpha}$ — compute w.o. SVD

T-truncated SVD: $g_\alpha(\sigma) = \begin{cases} 0, & \sigma \le \alpha \\ \frac{1}{\sigma}, & \sigma > \alpha \end{cases}$ — compute with SVD

3.5: $\|K_\alpha^+\| \le C_\alpha$, $K_\alpha^+ g \xrightarrow[\alpha\to 0]{} K^+ g$ pointwise

3.6: $\alpha(\delta, g_\delta) \xrightarrow{\delta\to 0} 0$, $\delta \cdot \|K_{\alpha(\delta,g_\delta)}^+\| \to 0$

$\Rightarrow (\alpha_\delta, K_{\alpha_\delta})$ is regularization.

$\alpha := \alpha(\delta, g_\delta)$: $\|K_\alpha^+ g_\delta - K^+ g\| \le \|K_\alpha^+ (g - g_\delta)\| + \|K_\alpha^+ g - K^+ g\|$

$\le \delta \cdot \|K_\alpha^+\| \to 0$

Lavrentiev: $K_\alpha^+ g := \sum_k \frac{1}{\sigma_k + \alpha}\,(g, v_k)\cdot u_k$

$\Rightarrow K_\alpha^+ g := \sum_k \frac{1}{\sigma_k+\alpha}\,(g, v_k)\cdot u_k$

$\Rightarrow g_\alpha(\sigma) = \frac{1}{\sigma+\alpha}$, $C_\alpha = \frac{1}{\alpha}$, $C = 1$, $g_\alpha(\sigma) \xrightarrow[\alpha\to 0]{} \frac{1}{\sigma}$.

Let K s.a., pos. semidefinite, $\overline{\mathrm{Range}(K)} = X$

$K K^+ = K^+ K = K^+$, $u_k = v_k$

$Ku = \sum_k \sigma_k (u, u_k)\cdot u_k$

$g = \sum_k (g, u_k) u_k \Rightarrow \sum_k \sigma_k (K\alpha^+ g, u_k)\cdot u_k + \alpha \sum_k (K_\alpha^+ g, u_k)\cdot u_k$

$(K+\alpha I) K_\alpha^+ g = \sum_k (g, u_k)\cdot u_k = g$ $= \sum_k (\sigma_k + \alpha)\,(K_\alpha^+ g, u_i)\, u_i$

$\Rightarrow K_\alpha^+$ can be computed without SVD.

Iterative Regularization.

Idea 1: Solve the normal equation with a known iterative algorithm.

Landweber (conjugate gradient ...) $K^*Ku = K^* g$

$u^{(n+1)} = u^{(n)} - \omega K^*(K u^{(n)} - g)$, $u^{(0)} = 0$

if it converges to $u : K^*Ku = K^* g$

Let $u^{(n)} = \sum_k \beta_k^{(n)} (g, v_k)\cdot u_k$

$\Rightarrow u^{(n+1)} = \sum_k \left((1-\omega\sigma_k^2)\beta_k^{(n)} + \omega\sigma\right)(g, v_k)\cdot u_k$

$\Rightarrow \beta_k^{(n+1)} = (1-\omega\sigma_k^2)\beta_k^{(n)} + \omega\cdot\sigma_k$

$\Rightarrow \beta_k^{(n+1)} = \sum_{\ell=0}^{n} (1-\omega\sigma_k^2)^\ell \cdot \omega\cdot\sigma_k$

$= \frac{1-q_k^{n+1}}{\omega\sigma_k^2}\cdot\omega\cdot\sigma_k = (1-q_k^{n+1})\cdot\frac{1}{\sigma_k}$, q_k

$\sigma_k \in (0, \sigma_1]$. Let $\omega < \frac{2}{\sigma_1^2} \Rightarrow |q_k| < 1$.

$n = \lfloor \frac{\alpha}{2} \rfloor$ $\quad \alpha > 0 : n > \infty$.

$g_\alpha(\sigma) = \frac{(1-(1-\omega\cdot\sigma^2)^{n+1})}{\sigma}$

$K_\alpha^+ := \sum_{k=0}^{\infty} g_\alpha(\sigma_k)\,(g, v_k)\cdot u_k$ (nth iteration of Landweber)

$g_\alpha(\sigma) \to \frac{1}{\sigma}$ ✓

$\sigma \cdot g_\alpha(\sigma) \le 1 + |1-\omega\cdot\sigma^2|^{n+1} \le 2$

$g_\alpha(\sigma) \le \sum_{\ell=0}^{n} |1-\omega\sigma_k^2|^\ell \cdot \omega\cdot\sigma_k$

$\le (n+1)\cdot\omega\cdot\sigma_1$ ✓

$\Rightarrow C_\alpha = \left(\frac{\alpha}{2}+1\right)\omega\cdot\|K\|$

$\Rightarrow$ Let $\alpha = \alpha(\delta, u_\delta)$, $\alpha \xrightarrow[\delta\to 0]{} 0$, $\delta\, C_\alpha \xrightarrow[\alpha\to 0]{} 0$

Idea 2: Iterated Regularization.

$Ku = g$. Have approx. u_n

$u_{n+1} = u_n + d_n$

$K(u_n + d_n) = g \Rightarrow K d_n = (g - K u_n)$

$\Rightarrow$ Inverse Problem for d_n $\quad 0$

Determine regularized solution d_n

$u_{n+1} = u_n + d_n$. (successive)

$\to$ In the same manner.

(9) Fourier transforms

4.1 Definition Let $f \in C^\infty(\mathbb{R}^n)$.

$$\hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx$$

4.2 Lemma: Let $f \in C^\infty(\mathbb{R}^n)$.

a) If $f(x) \in L^1(\mathbb{R}^n)$
$$\Rightarrow \|\hat{f}\|_\infty \leq (2\pi)^{-n/2} \|f\|_1$$

b) $\lim_{\xi \to \xi_0} \hat{f}(\xi) = (2\pi)^{-n/2} \lim_{\xi \to \xi_0} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx$
(uniform convergence, Lebesgue's theorem)
$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi_0} dx$$
$$= \hat{f}(\xi_0)$$
we will see this implies $f \in C^0$!

c) $f: \mathbb{R} \to \mathbb{R}$, $\hat{f}(\xi) = \frac{d}{i\pi\xi} \int_{\mathbb{R}} e^{-ix\xi} f'(x) dx$
$= -i\xi \hat{f}(\xi)$ $\frac{d}{i\pi\xi} \int_{\mathbb{R}} -ix e^{-ix\xi} f(x) dx$
$= i \hat{f}'(\xi)$

$g(x) := x f(x)$.
$\hat{g}(\xi) = \frac{d}{i\pi\xi} \int_{\mathbb{R}} e^{-ix\xi} \cdot x \cdot f(x) dx$
$= i \frac{d}{i\pi\xi} \int_{\mathbb{R}} (-ix) e^{-ix\xi} f(x) dx$
$= i \hat{f}'(\xi)$

d) Let $\alpha \in \mathbb{N}^n \to \mathbb{R}$, or $\in \mathbb{Z}^n$.
$(D^\alpha f) := \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdot \left(\frac{1}{i}\frac{\partial}{\partial x_2}\right)^{\alpha_2} f$.
$x \in \mathbb{R}^n \Rightarrow x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
$|\alpha| = \alpha_1 + \cdots + \alpha_n$
$\Rightarrow \widehat{(D^\alpha f)}(\xi) = (-i\xi)^\alpha \hat{f}(\xi)$
$\widehat{(x^\alpha f)}(\xi) = i^{|\alpha|} D^\alpha \hat{f}(\xi)$

e) $f_a(x) := f(x+a)$
$\hat{f}_a(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x+a) e^{-ix\xi} dx$
$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\xi + ia\xi} dx$
$= e^{ia\xi} \hat{f}(\xi)$

f) $f_\lambda(x) := f(\lambda x)$
$\hat{f}_\lambda(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\lambda x) e^{-ix\xi} dx$
$= \frac{1}{|\lambda|^n} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\xi/\lambda} dx$
$= \frac{1}{|\lambda|^n} \hat{f}\left(\frac{\xi}{\lambda}\right)$

g) $f(x) := (\lambda + a x)^{-1}$
$\Rightarrow \lambda f + a x f = 1$
$\Rightarrow i\lambda \hat{f}(\xi) - a \hat{f}'(\xi) = 0$
$\Rightarrow \int \text{solves ODE for } \hat{f}$
$\hat{f}(0) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-x^2/2} = 1$
$\Rightarrow \hat{f}(\xi) = e^{-a\xi^2/2}$

$\Rightarrow \hat{f}$ are solutions to the same initial value problem
$\Rightarrow$ Picard-Lindelöf $\Rightarrow \hat{f} = \tilde{f}$

h) Example: $f: \mathbb{R}^n \to \mathbb{R}$, $f := \chi_{[-a,a]^n}$
$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{[-a,a]^n} e^{-ix\cdot\xi} dx$
$= \frac{1}{(2\pi)^{n/2}} \left[\frac{e^{-ix\cdot\xi}}{-i\xi} x^{-i\xi}\right]_{-a}^{a}$ (*)
$= \frac{1}{(2\pi)^{n/2}} \left(\frac{-1}{i\xi}\right) (\sin(\xi) - \sin(-\xi))$
$= \sqrt{\frac{2}{\pi}} \sin(\xi)$, $\sin(\xi) := \begin{cases} \frac{\sin\xi}{\xi}, & \xi \neq 0 \\ 1, & \xi = 0 \end{cases}$

$f: \mathbb{R}^n \to \mathbb{R}$, $f := \chi_{[-a,a]^n}$
$\hat{f}(\xi) = \left(\frac{2}{\pi}\right)^{n/2} \sin(\xi_n) \cdots \sin(\xi_n)$.
$\sin$ decays like $\frac{1}{x}$ so $\hat{f}$ is not in L^1!
$\Rightarrow$ Need a function from L^1 to L^1.

Theorem V.3: f can be extended continuously from equation from L^1 to L^2.
(Parseval, Analysis 3)

Theorem 4.4: Let $f: \mathbb{R}^n \to \mathbb{R}$, $f \in C^1$ (or $\hat{f} \in C^1$).
If $\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{ix\cdot\xi} dx$.
Then
$\check{\hat{f}}(x) = f(x)$.
$\Rightarrow$ is the inverse Fourier transform.

Regularization: Practical explanation

$Ku = g$, $\|g - g_\delta\| \le \delta$, $g \in \mathcal{D}(K^\dagger)$, (σ_k, u_k, v_k) SVD.

$$u^\dagger = K^\dagger g = \sum_k \frac{1}{\sigma_k} (g, v_k) u_k.$$

$$u_\alpha^\dagger = K_\alpha^\dagger g_\delta = \sum_k g_\alpha(\sigma_k)(g_\delta, v_k) \cdot u_k.$$

Assume $g = g_\delta$ (but we don't know), $\delta > 0$. $\Rightarrow \alpha > 0$,

$$\|u_\alpha^\dagger - u^\dagger\|^2 = \sum_k \underbrace{\left(\sigma_k g_\alpha(\sigma_k) - 1\right)^2}_{\to 0 \text{ for } \alpha \to 0} \frac{(g, v_k)^2}{\sigma_k^2}$$

$$\left.\begin{array}{l} g_\alpha(\sigma) \to \frac{1}{\sigma} \\ \sigma\, g_\alpha(\sigma) \to 0 \end{array}\right\} \begin{array}{l} \alpha \to 0 \\ \sigma \to 0 \end{array}$$

α small for fixed $\sigma \sim 0$

σ small for fixed $\alpha \sim 1$

(easily checked for all methods)

Assuming u_k highly oscillating for large k: We take away these parts, effectively smoothing. (TSVD)

How bad is that?

Assume $u_k, v_k = e^{ikx}$. Assume $g : [-\pi, \pi] \to \mathbb{R}$ n times differentiable.

$$g(x) = \sum_k a_k e^{ikx} \cdot g^{(n)} = \sum_k k^n a_k e^{ikx} \cdot i^n$$

$$\Rightarrow \sum_k (k^n a_k)^2 < \infty$$

$\Rightarrow a_k$ decay like $\frac{1}{k^n}$.

In the following: $K = I$ (not compact!).

$X = Y = L_2([-\pi, \pi])$.

$u_k = v_k = \frac{1}{2\pi} e^{ikx}$, $\sigma_k = 1$ (not decaying!).

$\left(u = \sum_k \sigma_k (u_i, u_k) \cdot v_k = u^\dagger\right)$.

Tikhonov: $u_\alpha^\dagger = \arg\min_u \|Ku - g\|^2 + \alpha^2 \|u\|^2$

$$\left(K^* K + \alpha^2 I\right) u_\alpha^\dagger = K^* g \Rightarrow u_\alpha^\dagger = \frac{1}{1+\alpha^2} g.$$

not useful

Tikhonov with $\|u^{(1)}\|_2^2$:

$$u_\alpha^\dagger = \arg\min_u \|Ku - g\|^2 + \alpha^2 \|u^{(1)}\|^2$$

$$g = \sum_k g_k u_k, \; u = \sum_k a_k u_k, \; u' = \sum_k k \cdot a_k \cdot u_k$$

$$= \arg\min_u \left[(a_k - g_k)^2 + \alpha^2 a_k^2 k^{2n}\right]$$

$$a_k^2 (1 + \alpha^2 k^{2n}) - 2 a_k g_k$$

$$\Rightarrow \min \text{ at } \frac{g_k}{1 + \alpha^2 k^{2n}} \quad (\text{"Damping"})$$

General case: $g_\alpha(\sigma_k) = \frac{\sigma_k}{\sigma_k^2 + \alpha^2 k^{2n}}$

Conclusion:
- Choosing an appropriate norm is important,
- All regularizations with $K_\alpha^\dagger$ linear share the same properties.

Idea: Use non-linear $K_\alpha^\dagger$ $\Rightarrow$ **Numerical experiment**

$$\min \|Ku - g\|^2 + \|\nabla u\|_1 \cdot \alpha$$

Convolutions: $\Omega = \mathbb{R}^n$, $\mathcal{D} = \mathcal{D}(\Omega)$.

$* : \mathcal{D}' \times \mathcal{D} \to C^\infty(\mathbb{R}^n)$.

$(T_f * g) \equiv f * g = T_{f*g}$

$(T_f * g)(\varphi) = T_{f*g}(\varphi)$

$$= \int_{\mathbb{R}^n} (f*g)(x)\,\varphi(x)\,dx$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)\,g(x-y)\,dy\,\varphi(x)\,dx$$

$g(x-y) =: g_x(y)$ (cp. support)

$$= T_f(g_x) = T_{T_f(g_x)}(\varphi) \equiv T_f(g_x)$$

$\equiv T_f(g_x)$ (function in x).

$(T*g)(x) := T(g_x)$. $T(g_x)$ may have no compact support

① $(\delta * g)(x) = \delta(g(x-\cdot))$ T_f for $f \equiv 1$.

$= g(x)$

② Let $g_z(x) = g(z+x)$.

$(f * g_z)(y) = \int_{\mathbb{R}} f(x)\,g(z+y-x)\,dx$

$= (f*g)(z+y) = (f*g)_z$

$\Rightarrow$ translation-invariant!

"Shifting the convolution means shifting the image"

All translation-invariant operators are convolutions, find g from $g = \delta * g$.

Fourier transform:

$(\hat{T}_f) \equiv \hat{f} \equiv T_{\hat{f}}$

$(\hat{T}_f)(\varphi) = T_{\hat{f}}(\varphi) = \int \hat{f}(x)\,\varphi(x)\,dx$

$= \int f(x)\,\hat{\varphi}(x)\,dx$

$= T_f(\hat{\varphi})$.

$\hat{T}(\varphi) := T(\hat{\varphi})$.

$\varphi, \hat{\varphi}$ compact support $\Rightarrow \varphi = 0$.

Distributions (Rudin) (Generalized functions)

Let $\Omega \subset \mathbb{R}^n$ open $\neq \emptyset$.

Let $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ topological space. **Test functions**

$\varphi_n \to \varphi$ (in $\mathcal{D}$)
1) $\exists$ a cpt $K \subset \Omega$ s.t. $\operatorname{supp} \varphi_n \subset K \; \forall n$
2) $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly convergent
very strong!

$\mathcal{D}(\Omega)$ is non-empty. $f(x) = \begin{cases} e^{\frac{-1}{1-|x|^2}}, & |x| < 1 \\ 0, & \text{else} \end{cases}$



$\mathcal{D}(\Omega)' = \{ T : \mathcal{D}(\Omega) \to \mathbb{R} \mid T \text{ continuous, linear} \}$
$(\varphi_n \to \varphi \Rightarrow T(\varphi_n) \to T(\varphi))$ **Distributions**

topological space, weak convergence:
$T_n \to T$ if $T_n(\varphi) \to T(\varphi) \; \forall \varphi \in \mathcal{D}(\Omega)$
(pointwise convergence).

Examples:

1) Let $L^1_{loc}(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \int_K |f| < \infty \; \forall K \subset \Omega, K \text{ cpt} \}$
(constant function 1, $\Omega = \mathbb{R}^n$), all called f.

$T_f(\varphi) := \int_\Omega f(x)\varphi(x)\,dx$ (well-defined), linear.
$\operatorname{supp} \varphi \subset K$, $|\varphi(x)| \le C$.

$\varphi_n \to \varphi \Rightarrow \operatorname{supp} \varphi_n \subset K$

$T_f(\varphi_n) = \int_\Omega f(x)\varphi_n(x)\,dx$

$= \int_K f(x)\varphi_n(x)\,dx$ (uniform)

$= \int_K f(x)\varphi(x)\,dx = T_f(\varphi)$

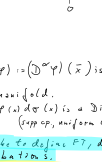
$\Rightarrow T_f$ is continuous

$\Rightarrow T_f \in \mathcal{D}(\Omega)'$.

We identify $f \equiv T_f$.

In that sense, $L^1_{loc}(\Omega) \subset \mathcal{D}(\Omega)'$.

2) $T(\varphi) = \varphi(0)$ ($T = \delta$, Dirac-Dirichlet-Lutzen)
linear. $\varphi_n \to \varphi \Rightarrow T(\varphi_n) = \varphi_n(0) \to \varphi(0) = T\varphi$, $T \in \mathcal{D}(\Omega)'$.
Can δ be represented as a function $\delta = T_\delta$? No, $\delta \notin L^1_{loc}(\Omega) \subsetneq \mathcal{D}(\Omega)'$



$T_{f_n}(\varphi) = \int_0^{\frac{1}{n}} n\varphi(x)\,dx$
$\to \varphi(0) = \delta(\varphi)$
$\delta(x) = \begin{cases} \infty, 0 \\ 0, \text{else} \end{cases}$
$\int \delta = 1$

3) $T_{\partial^\alpha f}(\varphi) := (\partial^\alpha \varphi)(\bar{x})$ is Distribution.

4) Manifold.
$T(\varphi) = \int_M \varphi(x)\,d\sigma(x)$ is a Distribution.
(topology, uniform conv.)

We would like to define T', derivative for distributions.

$(T_\delta) = (f)' = (f')' = T_{f'}$

$\Rightarrow (T_f)'(\varphi) := T_{f'}(\varphi) = \int f'(x)\varphi(x)\,dx$
$= -\int f(x)\varphi'(x)\,dx$
$= -T_f(\varphi')$

Definition: $T'(\varphi) := -T(\varphi')$

This is always defined!

All distributions are differentiable.

f weakly differentiable $\Rightarrow T_f' = T_{f'}$.
What if f is not weakly differentiable?

$f(x) = \operatorname{sgn}(x)$, $\Omega = \mathbb{R}$.
$(T_f)'(\varphi) = -T_f(\varphi')$
$= -\int_{\mathbb{R}} f(x)\varphi'(x)\,dx$
$= \int_{-\infty}^{0} \varphi'(x)\,dx - \int_0^{\infty} \varphi'(x)\,dx$
$= 2\varphi(0) = 2\delta(\varphi) \neq T_\delta(\varphi)$

$\Rightarrow$ we extend the definition of differentiability beyond weak sense.

More about Fourier Transform

Fourier III

Parseval I:

Theorem: Let $f, g \in \mathcal{L}^2(\mathbb{R}^n)$. Then $\int f \bar{g} = \int \hat{f} \bar{\hat{g}}$

$\Rightarrow \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$, $\hat{f}, \hat{g} \in \mathcal{L}^2(\mathbb{R}^n)$

$\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ $\quad \|g\|_{L^2} = \|\hat{g}\|_{L^2}$

$\Rightarrow$

$$\int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx\, \overline{g(\xi)}\,d\xi$$

(Fubini) $= \int_{\mathbb{R}^n} f(x) \overline{\int_{\mathbb{R}^n} g(\xi) e^{-ix\xi}\,d\xi}\,dx$

$$= \int_{\mathbb{R}^n} f(x)\overline{\hat{g}(x)}\,dx$$

Parseval II:

Theorem: Let $f, g \in \mathcal{L}^2$

$\Rightarrow \langle f, g\rangle_{L^2} = \langle \hat{f}, \hat{g}\rangle_{L^2}$

$\langle f, \hat{g}\rangle_{L^2} = \langle \hat{f}, g\rangle_{L^2}$

$\|f\|_{L^2} = \|\hat{f}\|_{L^2}$

Proof: 1) $\hat{\bar{g}}(\xi) = \int_{\mathbb{R}^n} \overline{g(x)} e^{-ix\xi}\,d\xi$

$$= \overline{\int_{\mathbb{R}^n} g(x) e^{ix\xi}\,d\xi}$$

$$= \overline{\check{g}(\xi)}$$

2) $\check{g} = \tilde{g}$

3) $g = \check{\hat{g}}$.

Convolution:

Definition: $f, g \in \mathcal{L}^2 \Rightarrow (f*g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\,dy$.

$f*g \in \mathcal{L}^\infty$, $\|f*g\|_\infty \le \|f\|_{L^2}\|g\|_{L^2}$

Remark: $f*g = g*f$.

Remark: $F = \operatorname{supp} f$, $G = \operatorname{supp} g$

$\Rightarrow \operatorname{supp}(f*g) \subset F+G$.

Convolution Theorem: $f, g \in \mathcal{L}^2$.

$\widehat{(f*g)}(\xi) = (2\pi)^{n/2}\hat{f}(\xi)\hat{g}(\xi)$.

$$\widehat{(f*g)}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (f*g)(x) e^{-ix\xi}\,dx$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)\, g(x-y) \quad e^{-i\xi(y+x-y)}\,dy\,dx$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y) e^{-iy\xi} \underbrace{\int_{\mathbb{R}^n} g(x-y) e^{-i\xi(x-y)}\,dx}_{(2\pi)^{n/2}\hat{g}(\xi)}\,dy$$

$$= (2\pi)^{n/2} \hat{f}(\xi)\hat{g}(\xi)$$

Example: Let $f(x) = \chi_{[-a,a]}(x)$.

$$(f*f)(x) = \int_{\mathbb{R}} f(y) f(x-y)\,dy$$

$$= \int_{-a}^{a} f(x-y)\,dy$$

- $\operatorname{supp} \subset [-2,2]$

$x-y \ge -a \Rightarrow y \le x+a$

$x-y \le a \Rightarrow y \ge x-a$

$\min(a, x+a)$

$\max(-a, x-a)$ $\quad dy = \begin{cases} 0, & |x| > 2a \\ a-x, & x \ge 0 \\ x+a, & x \le 0 \end{cases} = 2a - |x| \cdot \chi_{[-2a,2a]}$

$$\widehat{(f*f)}(\xi) = \hat{f}(\xi)\cdot\hat{f}(\xi)\cdot\sqrt{2\pi}$$

$$= \frac{2}{\pi}\,\operatorname{sinc}(\xi)^2\sqrt{2\pi}$$

Remark: $f, g \in \mathcal{L}^2 \Rightarrow \hat{f}, \hat{g} \in \mathcal{L}^2 \Rightarrow \hat{f}\cdot\hat{g} \in \mathcal{L}^1$

$\Rightarrow f*g = \check{(\hat{f}\cdot\hat{g})}$

is well-defined.

Remark: $(\chi_{[a,b]})\check{}\,\mathcal{D}' \to \mathcal{L}^1$.

$|\check{\mathcal{D}}(\xi)| \cdot |\hat{g}(\xi)| \le C\cdot|\hat{g}(\xi)|$

$\Rightarrow |\hat{g}(\xi)| \le C\cdot\frac{1}{|\xi^n|} \le C\cdot\frac{1}{|\xi^n|+1}$

Schwartz-Space, tempered distributions

Let $\mathcal{S} = \{ f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta \in \mathbb{N}_0^n : \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty \}$

$\Rightarrow$ All functions in $\mathcal{S}$ decay faster than any polynomial, together with their derivatives.

$\mathcal{S}'$:= tempered distributions.

Everything we proved for $\mathcal{D}'$ is valid for $\mathcal{S}'$.
($\mathcal{D}$ is dense in $\mathcal{S}$).

- $\mathcal{D} \subsetneq \mathcal{S}$: $f(x) = e^{-\|x\|^2/2} \in \mathcal{S}$, $f \notin \mathcal{D}$.
  $g(x) = e^{\|x\|^2/2} \notin L^1_{loc}$
  $\Rightarrow T_g(f) = \int_{\mathbb{R}^n} 1 \, dx = \infty$.

Note: $f \in \mathcal{S} \Rightarrow \hat{f} \in \mathcal{S}$. $\Rightarrow \hat{T}(f) := T(\hat{f})$ ✓
$\check{T}(f) := T(\check{f})$
$\Rightarrow \widehat{T(f)} = \check{T}(f)$.

Examples: $\hat{\delta}(\varphi) = \delta(\hat{\varphi}) = \hat{\varphi}(0) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} 1 \cdot \varphi(x) \, dx$

$\hat{\delta} = (2\pi)^{-n/2} \cdot 1 = \check{\delta}$
$1 = (2\pi)^{n/2} \check{\delta} = \widehat{\delta}$

w.r. $\delta(x) = (2\pi)^{-n/2} \int_{\mathbb{R}} \hat{\delta}(\xi) e^{ix\xi} \, d\xi$

$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \, d\xi$.

$1 \sim \chi_{[-R,R]}$, $R \to \infty$.

$\sim \frac{1}{2\pi} \int_{-R}^{R} e^{ix\xi} \, d\xi = \frac{R}{\pi} \operatorname{sinc}(Rx) =: \delta_R(x)$

convergence in the distributional sense: $\delta_R \to \delta$

$\int_{\mathbb{R}} \delta_R(x) \varphi(x) \, dx = (\varphi * \check{\delta}_R)(0) \xrightarrow{R \to \infty} \varphi(0)$

$(\varphi * \delta_R)\hat{}(\xi) = \hat{\varphi} \cdot \hat{\delta}_R = \hat{\varphi} \cdot \chi_{[-R,R]}$ $\quad \sqrt{2\pi}$

$= \int_{-R}^{R} \hat{\varphi}(x) e^{-ix\xi} \, dx \to \hat{\varphi}(\xi)$

$\mathcal{F}\mathcal{F}^{-1} \varphi = \varphi$

2) $T(\varphi) = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} \, dx = \int_{\mathbb{R}} \frac{\varphi(x)}{x} \, dx$ (Cauchy Prin. Value)

Existence: $\int_{\varepsilon \leq |x| \leq \eta} \frac{\varphi(x)}{x} \, dx = \int_{\varepsilon \leq |x| \leq \eta} \frac{\varphi(x) - \varphi(0)}{x} \, dx$

$\leq \int_{\varepsilon \leq |x| \leq \eta} \|\varphi'\|_\infty \, dx$

$\leq 2 \cdot \|\varphi'\|_\infty$.

$\int_{\mathbb{R}} \chi(x) := \begin{cases} 0, & |x| \leq \varepsilon \\ \frac{1}{x}, & \varepsilon \leq |x| \leq \frac{1}{\varepsilon} \\ 0, & |x| \geq \frac{1}{\varepsilon} \end{cases}$ $\quad \varphi \in L^2 \Rightarrow \hat{\varphi} \in L^2$ $\quad T(\varphi) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \chi(x) \varphi(x) \, dx$

$\hat{\chi}_\varepsilon(\xi) = \frac{1}{\sqrt{2\pi}} \int_{|x| \geq \varepsilon} \frac{1}{x} e^{-ix\xi} \, dx$ (under/over flow, limit of L^2-functions $\xrightarrow{\mathcal{F}}$)

$= \frac{-i}{\sqrt{2\pi}} \int_{|x| \geq \varepsilon} \frac{\sin x\xi}{x} \, dx \quad x \to \frac{x}{\xi}$

$\to \frac{-i}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\sin x}{x} \, dx \cdot \operatorname{sgn}(\xi)$

$\underbrace{\int_{\mathbb{R}} \operatorname{sinc}(x) \, dx}_{(\operatorname{sinc} * 1)(0)}$

$= -i \left(\frac{\pi}{2} \right)^{n/2} \operatorname{sgn}(\xi)$.

$\hat{T}(\varphi) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \hat{\chi}_\varepsilon(x) \hat{\varphi}(x) \, dx$

$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \hat{\chi}_\varepsilon(x) \check{\varphi}(x) \, dx$ (Parseval)

$= \int_{\mathbb{R}} \lim_{\varepsilon \to 0} \hat{\chi}_\varepsilon(x) \check{\varphi}(x) \, dx$

$= -i \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}} \operatorname{sgn}(x) \check{\varphi}(x) \, dx$.

$\frac{1}{x} = -i \sqrt{\frac{\pi}{2}} \cdot \operatorname{sgn}$

Hilbert-Transform:

$\forall f \in \mathcal{S}$: $(Hf)(x) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x - y} \, dy$

$= \frac{1}{\pi} \left(f * \frac{1}{x} \right)(x)$.

$\widehat{Hf}(\xi) = \frac{1}{\pi} (2\pi)^{n/2} \left(\frac{1}{x} \right)\hat{}(\xi) \cdot \hat{f}(\xi)$

$= -i \operatorname{sign}(\xi) \hat{f}(\xi)$

Example: rotationally symmetric functions

Definition: Let $f: \mathbb{C} \to \mathbb{C}$,

$f(z) := e^{\kappa(z-\frac{1}{z})} \cdot \frac{1}{z}$ — Bessel functions

$(\kappa \in \mathbb{R}$ fixed$)$.

f has an essential singularity (pole of order ∞) at 0, holomorphic away from 0

$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} J_n(\kappa) z^n$ (Laurent series)

$J_n(\kappa)$ is the Bessel function of the first kind.

We are interested in $\kappa > 0$, $n > 0$, n integer.

Lemma: $J_n(\kappa) = \frac{1}{2\pi i} \int_C e^{\frac{\kappa}{2}(z-\frac{1}{z})} \frac{1}{z^{n+1}} dz$ — Cauchy Contour Integral

$z = e^{i\varphi}, z^{-1} = e^{-i\varphi}, \varphi \in [0, 2\pi]$, $J_n(\kappa) \frac{1}{2}(z - \frac{1}{z}) = i \sin\varphi$,

$= \frac{1}{2\pi} \int_0^{2\pi} e^{i\kappa \sin\varphi} e^{-i(n+1)\varphi} \cdot ie^{i\varphi} d\varphi$

$= \frac{1}{2\pi} \int_0^{2\pi} e^{i\kappa \sin\varphi - in\varphi} d\varphi \qquad \psi \to \varphi - \frac{\pi}{2}$

$= \frac{1}{2\pi} \int_0^{2\pi} e^{i\kappa\cos\varphi - in\varphi} d\varphi$

Lemma: $J_n(\kappa)$ solves Bessel's equation

$x^2 f''(x) + x f'(x) + (x^2 - n^2) f(x) = 0$

Proof: Partial integration

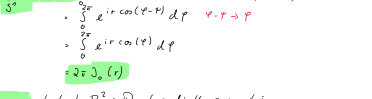
Lemma: $J_n(\kappa) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} \left(\frac{\kappa}{2}\right)^{2m+n}$ — Notation: $\frac{1}{m!}$ (factorial), $\frac{1}{(m+n)!}$ (factorial)

Proof: Solves Bessel's equation,

$J_n(0) = 0$, $J_n'(0) = 0 \Rightarrow = 0$.

$\Rightarrow$ Picard-Lindelöf

Lemma: Debye's relations:



$J_n(\kappa) \sim 0$ for $\kappa \ll n$.

no proof. Bulk, special functions?

Lemma: $\int_0^{2\pi} e^{ir\Theta(t)} \cdot \Theta'(t) dt \overset{s}{=} \int_0^{2\pi} e^{ir\Theta(t)} \cdot \Theta'(t) dt \qquad \Theta(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$

$= \int_0^{2\pi} e^{ir\cos(\varphi - t)} d\varphi \qquad \varphi - t \to \varphi$

$= \int_0^{2\pi} e^{ir\cos(\varphi)} d\varphi$

$= 2\pi J_0(r)$

Theorem: Let $f: \mathbb{R}^2 \to \mathbb{R}$, f radially symmetric ($f = f_0(\|x\|)$).

Then $\hat{f}(\xi) = \int_0^{\infty} r f_0(r) J_0(r\|\xi\|) dr$

$\hat{f}$ is radially symmetric.

Proof: $\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x) e^{i x \cdot \xi} dx$

$= \frac{1}{2\pi} \int_0^{\infty} \int_{S^1} f(r,\theta) e^{ir\cdot\theta\cdot\xi} r^1 d\theta\, dr$ — Polar coordinates

$= \frac{1}{2\pi} \int_0^{\infty} r \int_{S^1} e^{-ir\|\xi\|\cdot\theta} \frac{\xi}{\|\xi\|} d\theta\, dr$

$= \frac{1}{2\pi} \int_0^{\infty} r \cdot J_0(r) J_0(r\|\xi\|) dr$

Theorem: Let $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x) := \frac{1}{\|x\|}$

Then $\hat{f}(\xi) = \frac{1}{\|\xi\|}$

Proof: $\frac{1}{\|x\|} \in L^2_{loc}(\mathbb{R}^2) \Rightarrow \hat{f}$ exists in a distributional sense.

$(\notin L^1, \notin L^2)$

$\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{\|x\|} e^{-ix\xi} dx$ (in disk, cause)

$= \int_0^{\infty} r \cdot \frac{1}{r} \cdot J_0(r\|\xi\|) dr \qquad r\|\xi\| \to r$

$= \frac{1}{\|\xi\|} \cdot \underbrace{\int_0^{\infty} J_0(r) dr}_{=1} = \frac{1}{\|\xi\|}$

Note: Thm is a 1-d version.

$\frac{1}{\|x\|}$ is a fixpoint of the FT in 2D

(like $e^{-x^2/2}$ in 1D).

$e^{-x^2/2} \to$ inversion formula for FT

$\frac{1}{\|x\|} \to$ inversion formula for Radon

Fourier Slice Theorem
(Projection Theorem)

Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\theta \in S^{n-1}$, $\sigma \in \mathbb{R}$. Then

$$\hat{\mathcal{R}f}(\theta,\sigma) = (2\pi)^{(n-1)/2}\, \hat{f}(\sigma\cdot\theta).$$

↑ FT w.r.t 2nd argument

Main Result. In particular:

The Radon transform is invertible,

1) Take FT of data for fixed θ
2) Assemble $\hat{f}(\sigma\cdot\theta)$
3) Inverse FT.

Proof: $\hat{\mathcal{R}f}(\theta,\sigma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{R}f)(\theta,s)\, e^{-is\sigma}\, ds$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{x\cdot\theta=s} f(x)\, dx\, e^{-is\sigma}\, ds$$

(Fubini) $$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} f(x)\, e^{-ix(\sigma\cdot\theta)}\, dx$$

$$= (2\pi)^{(n-1)/2}\, \hat{f}(\sigma\cdot\theta).$$

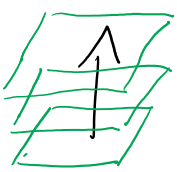
Röntgen Transform:

$$(\hat{Pf})(\theta,\xi) = \sqrt{2\pi}\, \hat{f}(\xi),\quad \theta \in S^{n-1}.$$

FT w.r.t. $\theta^{\perp}$

⇒ over determined, just choose $\theta \perp \xi$ $(n>2)$

Too many data: Stack planes in one direction



Part of the data is sufficient.

$$\hat{Pf}(\theta,\xi) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\substack{x\cdot\theta=s \\ \theta\cdot\xi=0}} e^{-ix\cdot\xi} (Pf)(\theta,x)\, dx$$

$$= \frac{1}{(2\pi)^{(n-1)/2}} \int_{x\cdot\theta=s} e^{-ix\cdot\xi} \int_{\mathbb{R}} f(x+t\cdot\theta)\, dt\, dx$$

$$= \frac{1}{(2\pi)^{(n-1)/2}} \int_{x\cdot\theta=s} \int_{\mathbb{R}} e^{-i\underbrace{(x+t\cdot\theta)}_{=:z}\cdot\xi} f(\underbrace{x+t\cdot\theta}_{z})\, dt\, dx$$

$$= \sqrt{2\pi}\, \hat{f}(\xi)$$

Radon-Transform

Let $f : \mathbb{R}^2 \to \mathbb{R}$.

Then $(\mathcal{R}f)(\theta, s) := \int_{\theta \perp} f(q)\, dq(s)$, $\theta \in S^{n-1}, \mathcal{R}_\theta$ of $(n-1)$-dim. Int measure.

We define this "integral of nearby points" integral exists $\forall \theta \in S^{n-1}$ for "nice" functions f (piecewise constant).

n=2: Line integral: $\int_{\theta \perp} f \, dq(s)$, $\theta = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$, $\theta^\perp(\gamma) = \begin{pmatrix} -\sin\gamma \\ \cos\gamma \end{pmatrix}$

n=3: Plane integral is not useful for tomography.

Definition: Let $C \to \int S^{n-1} \times \mathbb{R}^3$ (n=2: Cylinder in $\mathbb{R}^3$)

$\mathcal{R} : \mathcal{X}(\mathbb{R}^n) \to \mathcal{X}(C)$ is the Radon transform.

$(\mathcal{R}f)(\theta, s) = \int_{x \cdot \theta = s} f(x)\, dx$

$= \int_{x \cdot \theta = s} f(x)\, dx$

$= \int_{y \in \theta^\perp} f(s\theta + y)\, dy$ (map to θ, theory $\subset \theta$)

$= \int_{\mathbb{R}^n} f(x)\, \delta(x \cdot \theta - s)\, dx$ 1D-Delta

$\int_{y \in \theta^\perp} f(s\theta + y)\, dy = \int_{t \in \mathbb{R}} \int_{y \in \theta^\perp} f(s\theta + y)\, \delta(t)\, dt\, dy$

$= \int_{\mathbb{R}^n} f(x)\, \delta(x \cdot \theta - s)\, dx$ $x = y + t\cdot\theta$, $t \cdot s = \theta$

$= \int_{\mathbb{R}^n} f(x)\, \delta(x \cdot \theta - s)\, dx$ $s \cdot \theta \to x$

advantage: normal $(n-1)$-measure.

Definition: Let $f \in \mathcal{X}(\mathbb{R}^n)$.

$C \subseteq \mathbb{R}^{2n} : \{(\theta, s) \in S^{n-1} \times \mathbb{R}^n : x \cdot \theta = s\}$.

$P_\theta(\theta, s) := \int_{\mathbb{R}^n} f(x)\, \delta(x \cdot \theta - s)\, dx$ Radon-Transform

"Line in direction Θ through x", x is the Projection of 0 onto the line (=orthogonal)

2D: $(\mathcal{R}f)(\theta, s) = (\mathcal{R}f)(\theta', s \cdot \theta)$.

We will use R now. Theorems for P are similar (x 2).

$(\mathcal{R}f)(\theta, s) := \int_{x \cdot \theta = s} f(x)\, dx = \int_{\mathbb{R}^{n-1}} f(\ldots) \, dx = D(f(-\theta, -s))$

$\Rightarrow$ Only half of S^{n-1} needed.

$\Rightarrow \theta(\gamma)$, $\forall \gamma \in [0, \pi)$ ($[0, 180°)$).

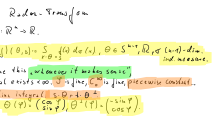
Assume: $\operatorname{supp} f \subseteq K(0)$

$\Rightarrow D(f(\theta, s)) = 0$ for $|s| > r$.

$(\mathcal{R}f)(\theta, s)$ Measurement for discrete values.

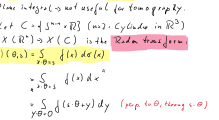
Specific angles $\Theta(\gamma_k)$ distances s_k

$\gamma_k = \frac{k\pi}{p}$, $k = 0 \ldots p-1$, $s_k \in L$, $k = \frac{s}{q}$, $L = -q \ldots q$.

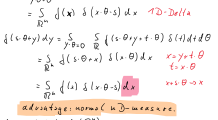


$\Theta(0)$ horizontal measurement in dir. $\theta^\perp$ (vertical)

Result: 1 measurement vector of $\mathcal{R}f_{\text{ghost}}$

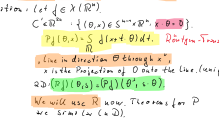


Result: $\in \mathbb{R}^{\text{ghost}}$ same as above.

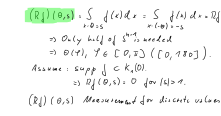


$\mathbb{R}^{(\text{ghost}) \cdot p}$ $\ni (m_1, m_2 \ldots m_{p \cdot q})$

Stack into Columns of matrix V



Sinogram $\sim$sine wave



ground $\mathbb{R}$ (Default = Matlab)

$\Rightarrow$ Program for generating data

Problem:



If the X-Ray-source is too close to the object, the data is ill-defined.

parallel scanning vs. fan beam-scanning.

Riesz-Potential and Filtered Backprojection

Let $\alpha < n$, $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \|\xi\|^{-\alpha} \hat{f}(\xi) \in L^1(\mathbb{R}^n)$

Let $\alpha < 1$, $g \in \mathcal{S}(C) \Rightarrow |\sigma|^{-\alpha} \hat{g}(\theta, \sigma) \in L^1(C)$.

Definition: $I^\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $\widehat{(I^\alpha f)}(\xi) = \|\xi\|^{-\alpha} \hat{f}(\xi)$ $\alpha < n$

$I^\alpha : \mathcal{S}(C) \to \mathcal{S}(C)$, $\widehat{(I^\alpha g)}(\theta, \sigma) = |\sigma|^{-\alpha} \hat{g}(\theta, \sigma)$ $\alpha < 1$

I^α is the Riesz-Potential.

(⇒ Performs the MPin Fourier space for ρ-filtered Lyonogram)

Theorem: Let $0 \le \alpha < n$. Then for $f \in \mathcal{S}(\mathbb{R}^n)$

$$f = \frac{1}{2} (2\pi)^{1-n} I^{-\alpha} R^* I^{\alpha - n + 1} R f$$

$\alpha = n-1$: ρ-filtered Lyonogram

$\alpha = 0$: Backprojection of filtered data

(FBP, filtered Backprojection, working horse of CT)

Proof: $f = I^{-\alpha} I^\alpha f$.

$$(I^\alpha f)(x) = (I^\alpha f)(x)$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|\xi\|^{-\alpha} \hat{f}(\xi) e^{ix\xi} d\xi$$

$$= (2\pi)^{-n/2} \int_{S^{n-1}} \int_{\mathbb{R}^+} \sigma^{n-1-\alpha} \hat{f}(\sigma \cdot \theta) e^{i\sigma x \cdot \theta} d\sigma \, d\theta$$

$$= (2\pi)^{-\frac{n}{2} - \frac{(n-1)}{2}} \int_{S^{n-1}} \int_{\mathbb{R}^+} |\sigma|^{n-1-\alpha} \widehat{Rf}(\theta, \sigma) e^{i\sigma x \cdot \theta} d\sigma \, d\theta$$

$\sigma \to -\sigma$: $|\sigma|^{n-1-\alpha}$

$\theta \to -\theta$: $\widehat{Rf}(-\theta, -\sigma)$ ⇒ $= \frac{1}{2} \left(\int_{\mathbb{R}^+} + \int_{\mathbb{R}^-} \right) = \frac{1}{2} \int_{\mathbb{R}}$

$e^{i\sigma x \cdot \theta}$

$$= \frac{1}{2} (2\pi)^{n - \frac{1}{2}} \int_{S^{n-1}} \int_{\mathbb{R}} \underbrace{\left[|\sigma|^{n-1-\alpha} \widehat{Rf}(\theta, \sigma) \right]}_{\widehat{(I^{-n+1+\alpha} Rf)}(\theta, \sigma)} e^{i\sigma x \cdot \theta} d\sigma \, d\theta$$

$$= \frac{1}{2} (2\pi)^{n-1} \left(R^* I^{-n+1+\alpha} Rf \right)(x) \quad \checkmark$$

Mathematical Backprojection

Let $\theta \in S^{n-1}$, $x \in \mathbb{R}^n$

Let $L = \{z : z \cdot \theta = x \cdot \theta\}$.

goes through x, perpendicular to θ: $g(\theta, x\cdot\theta)$

Now assume that $(Rf)(\theta, s) = g(\theta, s)$ is measured for all θ, s.

$$(R^* g)(x) = \int_{S^{n-1}} g(\theta, x\cdot\theta)\, d\theta$$

is called Back projection of g = discrete version.

Theorem: R^* is the L^2-adjoint of R.

Let $f \in L^2(\mathbb{R}^n)$, $g \in L^2(C)$.

$$\begin{aligned}
(Rf, g)_{L^2(C)} &= \int_{S^{n-1}} \int_{\mathbb{R}} (Rf)(\theta, s)\, g(\theta, s)\, ds\, d\theta \\
&= \int_{S^{n-1}} \int_{\mathbb{R}} \int_{x\cdot\theta = s} f(x)\, g(\theta, s)\, dx\, ds\, d\theta \\
&\overset{\text{(Fubini)}}{=} \int_{S^{n-1}} \int_{\mathbb{R}^n} f(x)\, g(\theta, x\cdot\theta)\, dx\, d\theta \\
&= \int_{\mathbb{R}^n} f(x) \int_{S^{n-1}} g(\theta, x\cdot\theta)\, d\theta\, dx \\
&= \int_{\mathbb{R}^n} f(x)\, (R^* g)(x)\, dx = (f, R^* g)_{L^2(\mathbb{R}^n)}
\end{aligned}$$

ρ-filtered Layogram for $n = 2$:

$$\begin{aligned}
(R^* R f)(x) &= \int_{S^1} Rf(\theta, x\cdot\theta)\, d\theta \\
&= \int_{S^1} \int_{y\cdot\theta = x\cdot\theta} f(y)\, dy\, d\theta \qquad y \to x+y \\
&= \int_{S^1} \int_{y\cdot\theta = 0} f(x+y)\, dy\, d\theta \qquad \text{(Polar coordinates)} \\
&= 2 \int_{\mathbb{R}^2} f(x+y) \cdot \frac{1}{\|y\|}\, dy \\
&= 2 \left(\frac{1}{\|x\|} * f \right) \Rightarrow \text{Convolution Theorem} \\
&\qquad \Rightarrow \text{Recon Alg.}
\end{aligned}$$

Theorem: $(R^* h) * f = R^*(h * Rf)$, $h \in \mathcal{S}(C)$, $f \in \mathcal{S}(\mathbb{R}^n)$.

convolution in image space — convolution in data space wrt 2nd var.

$\Rightarrow$ convolution in image space can be performed in data space.

$$\begin{aligned}
R^*(h * Rf)(x) &= \int_{S^{n-1}} (h * Rf)(\theta, x\cdot\theta)\, d\theta \\
&= \int_{S^{n-1}} \int_{\mathbb{R}} h(\theta, x\cdot\theta - s)\, Rf(s)\, ds\, d\theta \\
&= \int_{S^{n-1}} \int_{\mathbb{R}} \int_{y\cdot\theta = s} h(\theta, x\cdot\theta - s) f(y)\, dy\, ds\, d\theta \\
&= \int_{\mathbb{R}^n} \underbrace{\int_{S^{n-1}} h(\theta, (x-y)\cdot\theta)\, d\theta}_{(R^* h)(x-y)} f(y)\, dy \\
&= \left[(R^* h) * f \right](x)
\end{aligned}$$

Discussion of Fourier Slice

$\hat{g}(\theta,s) = \int f(x,y)\, dl$ (1D)

$\theta \in [0, \pi)$, $s \in [-r, r]$, $r = \frac{2}{\sqrt{2}}$

$x = r s_1 \cos\theta - r s_2 \sin\theta$, $y = r s_1 \sin\theta + r s_2 \cos\theta$

supp $\{f \in K_1(0)\} \subset C_0^\infty$

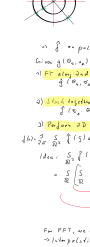
$\Rightarrow$ f is C^∞

$\left.\begin{matrix} \theta_1 \in [0, \pi_1] \\ s_1 \in [-r_1, r_1] \end{matrix}\right\}$ Parallel scanning

$\Rightarrow$ use

θ_1, s_1 fixed

$\Rightarrow$ can compute $\hat{f}(\xi_1, \xi_2)$



All lines: Can compute $\hat{g}$ everywhere.

Discretized: Can compute for some $\theta_1, \ldots, \theta_n$

$\Rightarrow$ $\hat{g}$ on polar grid

Given $\hat{g}(\theta_1, s_1)$:

1) FT along 2nd variable: $\hat{g}(\theta_1, \sigma_2)$ $\Rightarrow$ Put Polar Transform (not necessary)

2) Stack together $\hat{g}(\theta_1, \sigma_2)$

3) Perform 2D inverse FT

$\hat{f}(\sigma_1, \sigma_2) = \int_{\mathbb{R}^2} f(s_1, s_2) e^{-2\pi i (\sigma_1 s_1 + \sigma_2 s_2)} ds_1 ds_2$ How?

Idea: $\int_{\mathbb{R}^2} f(s_1, s_2) e^{-2\pi i (\sigma_1 s_1 + \sigma_2 s_2)} ds_1 ds_2$

$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(s_1, s_2) e^{-2\pi i \sigma_1 s_1} ds_1 \right] e^{-2\pi i \sigma_2 s_2} ds_2$

$\rightarrow$ 1D FT

$\rightarrow$ 1D FT

For FFT, we need data on a grid

$\rightarrow$ Interpolation is difficult.

$\hat{f}$ is highly oscillatory.

$\Rightarrow$ Integration of oscillated data requires

$\Rightarrow$ Solution: Non-equispaced Fourier Transform

Developed at Stanford.

All is ok, implementation Fourier slice is difficult (Human visual system is not sensitive to low frequency)

$\Rightarrow$ Radon transform is leading

Special case:

Only one direction ($\theta = 0$ horizontal, line in direction $\frac{\pi}{2}$ (vertical).



$\hat{f}(\sigma_1, \sigma_2) = \hat{g}(\sigma_1)$ $\hat{f}(\xi_1, \xi_2) = \overline{\hat{f}(-\xi_1, -\xi_2)}$

$\hat{g}(\sigma_1, \sigma_2) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(s_1, s_2) e^{-2\pi i (\sigma_1 s_1 + \sigma_2 s_2)} ds_1 ds_2$

$s_2 = 0$

$= \int_{\mathbb{R}} f(s_1) e^{-2\pi i \sigma_1 s_1} ds_1$

$= \hat{g}(\sigma_1)$

1) constant along $(\sigma_1, -)$ assumed (vertical lines) value

2) changes along σ_1 are real

Simple explanation:

To find the values at some points, simply take the measured values on the line that crosses the point.



Extension: When we have multiple lines, the idea is still the same

$\rightarrow$ Add the values you will find the right line.

Back projection

Solve an problem

Easily implemented.

Why does this not work?

Geometric:



Algebraic:

Assuming by adding the value will be correct

$\Rightarrow$ adding is correct. 0

$\Rightarrow$ higher the more often it

Further goes with $\mathrm{BP} \neq \mathrm{FSF}$.

$\Rightarrow$ Multiply with FSF in the Fourier Domain!

$\Rightarrow$ $\mathrm{BP} = \mathcal{F}^{-1} \circ \mathrm{FSF} \circ \mathcal{F}$

$\Rightarrow$ $\mathrm{BP} = \mathcal{F}^{-1} \circ \mathrm{FSF} \circ \mathcal{F} \circ \mathcal{F}^{-1} \circ \mathcal{F}$

$\Rightarrow$ $\mathrm{BP} = \mathcal{F}^{-1} \circ \mathrm{FSF} \circ \mathcal{F}$

$\Rightarrow$ Compute BP

$\Rightarrow$ Compute $\widehat{\mathrm{BP}}$ (FFT)

$\Rightarrow$ Compute $\widehat{\mathrm{BP}} \cdot \mathrm{FSF} = \hat{f}$ $\quad f(s) = \mathrm{FSF}$

$\Rightarrow$ Compute $\mathcal{F}^{-1} \circ \hat{f}$

Up to constants ∇

Remark: Properties of the Fourier Transform come into Properties of the Radon Transform.

Example: $\mathrm{R}(\overline{\Delta^2 f})(\theta, s) = \overline{\Delta^2}\, \mathrm{R}(f)(\theta, s)$ $\quad \Delta^2$ not real

$\mathrm{R}(\overline{\Delta^2 f})(\theta, s) = \overline{\Delta^2} \int_{\mathbb{R}^2} f(x,y)\, 12\pi^2 e^{-2\pi i (\sigma_1 x + \sigma_2 y)} \, dx\, dy$

$= \overline{\Delta^2} \int_{\mathbb{R}^2} \mathcal{F}(f)(\sigma_1, \sigma_2)\, e^{2\pi i (\sigma_1 x + \sigma_2 y)} \, d\sigma_1 \, d\sigma_2$

$= \overline{\Delta^2}\, \mathcal{F}^{-1}(\hat{f})(\theta, s)$

filtered Back projection for d-plane Radon transform

$\hat{f}(\xi) = \frac{2}{(2\pi)^{n-1}} \mathcal{I}^{1-n} \mathcal{R}^* \hat{g}$

Inversion: $(\mathcal{R}^* g)(x) = \int_{S^{n-1}} g(\theta, x\cdot\theta)\, d\theta$

$\left(\mathcal{I}^{1-n} f\right)^\wedge(\xi) = |\xi|^{1-n} \hat{f}(\xi)$ ← not continuous at 0

$\mathcal{R} = $ filtering + convolution

FBP: $f = \frac{2}{(2\pi)^{n-1}} \mathcal{R}^* \mathcal{I}^{1-n} g$

$\left(\mathcal{I}^{1-n} g\right)^\wedge(\theta,\sigma) = |\sigma|^{1-n} \hat{g}(\theta,\sigma)$

Discussion:

1) $n=3$ Reminder: $\mathcal{D}^d \hat{f}(\xi) = i^{|\alpha|} \xi^\alpha \hat{f}(\xi)$ (Lemma 4.2)

(n odd)

Inversion $\mathcal{I}^{-2} \hat{f}(\xi) = |\xi|^2 \hat{f}(\xi)$

$= (\xi_1^2 + \xi_2^2 + \xi_3^2)\, \hat{f}(\xi)$

$= -\widehat{\Delta f}(\xi) \Rightarrow \mathcal{I}^{-2} f = -\Delta f$

$f = \frac{-1}{8\pi^2} \Delta_x \left(\int_{S^2} g(\theta, x\cdot\theta)\, d\theta \right)$

FBP:

$\mathcal{I}^{-2} g(\theta,s) = |\sigma|^2 \hat{g}(\theta,\sigma)$

$= -\hat{g}''(\theta,\sigma) \Rightarrow \mathcal{I}^{-2} g = -g''$ wrt 2nd var

(Radon, 1917)

$f(x) = \frac{-1}{8\pi^2} \int_{S^2} g''(\theta, x\cdot\theta)\, d\theta$

works only for n odd.

Notice that the Radon transform is local in the sense: To reconstruct in a region (like Leah), we only need data of hyperplanes going through the Leah.

Differentiation is local

2) $n=2$: (interesting case) (MATHEMATICIAN formula)

$\mathcal{I}^{-1} g(\theta,s) = |\sigma| \hat{g}(\theta,\sigma)$

$= -i\,\mathrm{sign}(\sigma) \cdot \sigma \hat{g}(\theta,\sigma)$

$= -i\,\mathrm{sign}(\sigma)\, \widehat{g'}(\theta,\sigma)$ (wrt 2nd var)

(Hilbert Transform! (4.77))

$= \left(H g'(\theta,\cdot)\right)^\wedge(\sigma)$

$\Rightarrow f(x) = \frac{1}{4\pi} \int_{S^1} \mathcal{H}\, g'(\theta, x\cdot\theta)\, d\theta$

Not local!

To reconstruct, we need all data going through the support of f.

In the case of the Leah: 96% of the radioactive food in the patient is wasted.

3) Cormack (Hounsfield, Nobel prize 1979)

Inventing devices Gordon

$f(x) = \sum_{k=0}^{\infty} a_k(r)\, e^{ik\theta}$ $r = \theta(\theta)$ expansion for f fixed

$g(\theta,s) = \sum_{k=0}^{\infty} g_k(s)\, e^{ik\theta}$ expansion for s fixed

a_k from g_k directly, involving Toeplitz-type/Hankel poly matrices.

→ exercises

4) Radon for Transforms:

Equivalent formulas exist.

5) Ill-posedness:

Injective ✓

Surjective :—

g is the range: $p_n(\theta) = \int_{S^{n-1}} g(\theta,s)\, ds$

$\in \mathcal{D}$ homogeneous.

⇒ characterize the range (Helgason)

Invertible on the range:

- Inversion involves the derivative (max order $n-1$)
- Singular function $\mathrm{sgn}(\sigma)$ oscillating like $e^{i\sigma\beta}$ (Helgason).

β-filtered Lipschitz $\mathrm{sgn}(\sigma)\, e^{i\sigma\beta}\, |\sigma|^\beta$

$\Rightarrow \varepsilon_k \in O\left(\frac{1}{k^\beta}\right)$ max order n.

However, integrating makes things smoother, degree $\frac{n}{2}$.

⇒ implementation.

So Good - Sobolev Stuff

Let $f \in H^s(\mathbb{R}^n)$ (weakly differentiable in L^2)

$$\|f\|_{H^s}^2 := \sum_{|\alpha|\le s} \|\partial^\alpha f\|_{L^2}^2 \qquad \left(= \|f\|_{L^2}^2 + \dots + \left\|\frac{\partial^s f}{\partial x_n^s}\right\|_{L^2}^2\right)$$

$$= \sum_{|\alpha|\le s} \|\widehat{\partial^\alpha f}\|_{L^2}^2 \qquad \text{(Parseval)}$$

$$= \sum_{|\alpha|\le s} \|\xi^\alpha \widehat{f}\|_{L^2}^2 \qquad \text{(computation rule)}$$

$$= \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + \xi_1^2 + \dots + \xi_n^2)^s \, d\xi$$

$$= \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi$$

H^2: $(1+|\xi|^2)^2$ (equivalent)

naturally H^{∞}: $\|f\|_{H^\infty} < \infty$,
$\|f\|_{H^\infty}^2 := \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^k \, d\xi$.

α fractional (∇: monotonous wrt. α), $\|\cdot\|_{H^\alpha}$, $\|\cdot\|_{H^\beta}$ not equivalent for $\alpha \neq \beta$.

$H^\infty(\mathbb{C})$: $\{ g : \|g\|_{H^\infty(\mathbb{C})} < \infty \}$ Example in $\mathbb{R}^1$: $\frac{1}{1+x^2}$ is in H^∞ but not in H^ω!

$\|g\|_{H^\infty(\mathbb{C})}^2 = \sum_{k=0}^{\infty} \int_{\mathbb{R}^2} (1+|\xi|^2)^k \int_{\partial B(0,\varepsilon)} |g(z)|^2 \, dz \, d\xi$

or integrate on equivalent to normal norm.

In the following: Restrict to compactly supported C^∞ functions, w.l.g. $\operatorname{supp} f \subset K_r(0)$.

Theorem: $\partial_{\mathbb{R}^n}^s \subset \left\{ \|f\|_{L^2} \le \varepsilon^s \|f\|_{H^{(n+1)s}(\mathbb{R}^n)} + C_\varepsilon \|f\|_{L^2} \right\}$, $f \in L^2(\Omega)$ with cp. support (f continuous)

① $\Rightarrow$ If f is just in C^s, $\|f\|$ is in $H^{(n+1)s}$ ($\Rightarrow$ f is smoother (k is integer/real)

② $g = \partial f$, $f = \mathcal{F}^{-1} g = \mathcal{F}^{-1} \frac{1}{\partial} g = \mathcal{F}^{-1} \frac{1}{\partial} \mathcal{F} g \in C_c^s \|g\|_{H^{(n+1)s}}$

$\Rightarrow \mathbb{R}^{n+1}$ is continuous wrt. $H^{(n+1)s} \subset L^2$.
∂ is not continuous wrt H^∞, L^2! ($\|\partial f\|_{H^\infty}, \|f\|_{H^\infty}$ not equivalent)

$\Rightarrow$ The inversion of the Radon Transform is ill-posed of the order $\frac{n+1}{2} = \frac{3}{2}, n=2$.

ill-posed: does not exist of first derivative is ∂

Inversion of the Radon Transform is a severely mildly ill-posed problem.

That's why we have excellent images.

Proof: $\|f\|_{H^{(n+1)s}(\mathbb{C})}$

$$= \int_{S^{n-1}} \int_{\mathbb{R}^n} (1+r\xi)^{(n+1)s} \, r^{n-1} \|\hat{f}\|_{L^2(B,\varepsilon)}^2 \, dr \, d\xi$$

Fourier Slice: $= (2\pi)^{n-1} \int_{S^{n-1}} \int_{\mathbb{R}^n} (1+r\xi)^{(n+1)s} \, r^{n-1} |\hat{f}(\xi \cdot \theta)|^2 \, dr \, d\theta$

$\theta = e_1, \xi = \xi$: $= (2\pi)^{n-1} \int_{S^{n-1}} \int_{\mathbb{R}^n} (1+r\xi)^{(n+1)s} \, r^{n-1} |\hat{f}(\xi \cdot \theta)|^2 \, dr \, d\theta$

$\xi := \xi \cdot \theta$: $= 2\,(2\pi)^{n-1} \int_{\mathbb{R}^n} \frac{(1+|\xi|)^{\frac{n+1}{2}s}}{|\xi|^{\frac{n-1}{2}}} \, |\hat{f}(\xi)|^2 \, d\xi$ (*)

$\geq 2\,(2\pi)^{n-1} \int_{\mathbb{R}^n} \frac{(1+|\xi|^2)^{\frac{n+1}{2}s}}{|\xi|^{\frac{n-1}{2}}} \, |\hat{f}(\xi)|^2 \, d\xi$

$\geq$ Using radius factor from substitution $r = (1+|\xi|^2)^{\frac{s}{2}}$ root

$\geq 2\,(2\pi)^{n-1} \int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{n+1}{2}s} \, |\hat{f}(\xi)|^2 \, d\xi$

Parseval: $= 2\,(2\pi)^{n-1} \|g\|_{H^{(n+1)s}}$ $\Rightarrow \leq$ LHS of equation.

(*) $\|f\|_{H^{(n+1)s/2}} \sim 2\,(2\pi)^{\frac{n-1}{2}} \int_{\mathbb{R}^n} (|\xi|^2)^{\frac{n-1}{4}} (1+|\xi|^2)^{\frac{n+1}{4}s} \, |\hat{f}(\xi)|^2 \, d\xi$

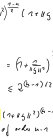
① $\|\xi\| \le 1$: $\left(\frac{1+|\xi|^2}{|\xi|^2}\right)^{\frac{n-1}{2}} = \left(1 + \frac{1}{|\xi|^2}\right)^{\frac{n-1}{2}} \leq 2^{(n-1)/2}$

$\Rightarrow \int_{\|\xi\|\le 1} \dots \, d\xi \leq 2^{(n-1)/2} \|f\|_{L^2}$. Parseval $\Rightarrow \|f\|_{H^{\frac{n+1}{2}s}} \leq \|f\|_{L^2}$

② $\|\xi\| \ge \eta$: $\int_{\|\xi\|\ge\eta} \frac{(1+|\xi|^2)^{(n+1)s}}{|\xi|^{n-1}} \, d\xi \leq C$ since $\eta > 0$

③ $\sup_{\|\xi\|\le\eta} |\hat{f}(\xi)| \le (2\pi)^{-n/2} \int_{K_r(0)} |f(x)| e^{-ix\xi} \, dx \leq \int_{K_r(0)} |f(x)| \, dx \leq (2\pi)^{-n/2} \int_{K_r(0)} 1 \, dx$

$\leq \frac{r^n \pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \|f\|_{L^2} (2\pi)^{-n/2}$



Merry Christmas!
Have a great Christmas break, stay healthy, see you on Jan 11.

→ Sampling: How exact must we measure to be sure not to miss a peak of size ε?

→ 3D Tomography: Improving scanning time + quality

Band-limited functions

⇒ Sound: Only frequencies in a certain range can be heard

⇒ Assume $\operatorname{supp} \hat{f} \subset K \ (= [-\Omega, \Omega]^n)$.

Definition: f is band-limited with band limit Ω iff $\hat{f}(\xi) = 0$ for $\|\xi\|_\infty > \Omega$.

$g \in Y$. Band-limited version of g:

Can only reconstruct $g(\xi)$ in $K = [-\Omega, \Omega]^n$.

$$f(x) = \left(\underbrace{\hat{g} \cdot \chi_{[-\Omega,\Omega]^n}}_{\text{sinc}} \right)^{\vee}(x)$$

$$= \left(g * \underbrace{\check{\chi}_{[-\Omega,\Omega]^n}}_{\text{sinc}} \right)(x) \cdot (2\pi)^{-n/2}$$

1D: $\check{\chi}_{[-\Omega,\Omega]}(\xi) = \Omega \sqrt{\frac{2}{\pi}}\, \operatorname{sinc}(\Omega x)$

Parseval: f is the best approximation to g in the space Y_Ω = space of band-limited functions in Y.

Remark: If f is compactly supported, $\hat{f}$ is not band-limited (except $f = 0$).

Proof: Paley-Wiener
Bloody 1933, Journals of the London Mathematical Society

If $|f(x)| \le A e^{-x^2/2}$
$|\hat{f}(\xi)| \le B \cdot e^{-\xi^2/2}$
$\Rightarrow f(x) = C \cdot e^{-x^2/2}$

⇒ When talking about band-limited, compactly supported functions we are talking about the empty set.

Remedy:

$f \in Y$ is essentially Ω-band-limited iff $\hat{f}(\xi)$ is negligible (small) for $|\xi| > \Omega$.

Read: f does not change if we replace it with its Ω-band-limited version.

Example 1: $g(x) = \chi_{[-1,1]}(x)$.

$$f(x) = \frac{\Omega}{\pi} \left(g * \operatorname{sinc}(\Omega \cdot) \right)(x)$$

$$= \frac{1}{\pi} \left(Si(\Omega(x+1)) - Si(\Omega(x-1)) \right)$$

$Si(x) = \int_0^x \operatorname{sinc}(y)\, dy$ special function.

Example 2: $g(x) = \delta(x)$, $\hat{\delta} = 1$

$\Rightarrow \hat{f}(\xi) = \chi_{[-\Omega,\Omega]}$

$\Rightarrow f(x) = \Omega \sqrt{\frac{2}{\pi}}\, \operatorname{sinc}(\Omega \cdot x)$

Interpretation: An ideal loudspeaker would play sounds like this function.

Chapter 6: Sampling Theory (Numerical analysis oriented)

Discussion of the exercise: $f \in S$.

$$\ell \in \mathbb{Z}^n: \quad \hat{f}(2\pi\ell) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-2\pi i \ell x} f(x)\,dx \qquad \left[\hat{f}(\ell)\right]$$

$$\sim \frac{1}{(2\pi)^{n/2}} \sum_{k \in \mathbb{Z}^n} e^{-2\pi i \ell k} f(k)$$

$$= \frac{1}{(2\pi)^{n/2}} \sum_{k \in \mathbb{Z}^n} f(k) \qquad (\ell = 0)$$

suspect: $\sum_{\ell \in \mathbb{Z}^n} \hat{f}(2\pi\ell) \sim \frac{1}{(2\pi)^{n/2}} \sum_{k \in \mathbb{Z}^n} f(k)$

Poisson's formula:

Poisson's summation formula

$$\sum_{\ell \in \mathbb{Z}^n} \hat{f}(2\pi\ell) = \frac{1}{(2\pi)^{n/2}} \sum_{k \in \mathbb{Z}^n} f(k)$$

$$g(x) = f(h \cdot x) \cdot e^{-ix\eta}; \quad h, \eta \text{ fixed} \qquad (4.2)$$

$$\frac{1}{h^n} \sum_{\ell \in \mathbb{Z}^n} \hat{f}\left(\frac{2\pi\ell}{h} + \eta\right) = \frac{1}{(2\pi)^{n/2}} \sum_{k \in \mathbb{Z}^n} f(k \cdot h)\, e^{-ik\eta}$$

$$g(\xi) := \sum_{\ell \in \mathbb{Z}^n} \hat{f}(\xi + 2\pi\ell) \qquad (\text{LHS for } \eta = 0)$$

2π-periodic

$$g(\xi) = \sum_{k \in \mathbb{Z}^n} \hat{g}_k e^{-i\xi k}, \qquad \text{Fourier expansion}$$

$$\hat{g}_k = (2\pi)^{-n} \int_{[-\pi,\pi]^n} g(\xi)\, e^{ik\xi}\,d\xi$$

$$= (2\pi)^{-n} \sum_{\ell \in \mathbb{Z}^n} \int_{[-\pi,\pi]^n} \hat{f}(\xi + 2\pi\ell)\, e^{i(\xi + 2\pi\ell)k}\,d\xi$$

$$= (2\pi)^{-n/2} \underbrace{(2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{ik\xi}\,d\xi}_{f(k)}$$

$$g(0) = \sum_{\ell \in \mathbb{Z}^n} \hat{f}(2\pi\ell)$$

$$\| \quad (2\pi)^{-n/2} \sum_{k \in \mathbb{Z}^n} f(k)$$

Basis for our analysis of algorithms.

1) $\Rightarrow$ $\hat{f}(\xi)$ can be measured for $\|\xi\| \leq \Omega$

$\Rightarrow$ Derivatives of order $\leq \frac{\pi}{\Omega}$ can be identified in image

2) „Full width half maximum"



$\hat{=}$ image of δ-Distribution

3) Equivalent formulation for 2π-periodic function f:

f Ω-band limited $\Leftrightarrow$

Fourier coefficients $\hat{f}_k = 0$, $|k| \geq \Omega$.

$\frac{1}{h}\sum_{k\in\mathbb{Z}} f\left(\frac{2\pi k}{h}+\vartheta\right) = \frac{\sqrt{2\pi}}{(2\pi)^{3/2}} \sum_{k\in\mathbb{Z}} \hat{f}(k,\ell)\, e^{ik\ell\vartheta}$ Poisson

Assume that $\hat{f}(k,\ell)$ has been measured (sample (k,ℓ)). We can compute only the sum of Evaluations of the P.F. But:

$\hat{f}(\xi) = (2\pi)^{-n/2}\, h^n \sum_{k\in\mathbb{Z}^n} f(k,\ell) e^{ik\ell\xi} \cdot \sum_{\ell\in\mathbb{Z}} \int e^{\frac{2\pi i \ell}{h}\xi}$

Assume that $\hat{f}$ is Ω- bandlimited (Ω even)

If $|\xi| > \Omega \Rightarrow \hat{f}(\xi) = 0$.

Let $|\xi| \leq \Omega$. If $|\xi + \frac{2\pi\ell}{h}| > \Omega$ for all ℓ,

then we can compute $\hat{f}(\xi)$ from $f(k,\ell)$!



$\gamma = 0, \frac{2\pi}{h}, -\frac{2\pi}{h}$

$\Rightarrow \xi + \gamma \in \Omega$

$|\xi| \leq \Omega \Rightarrow \xi + \gamma > \Omega \Rightarrow -\Omega + \frac{2\pi}{h} > \Omega \Rightarrow h \leq \frac{\pi}{\Omega}$

$|\xi| \leq \Omega \Rightarrow \xi + \gamma < -\Omega \Rightarrow \xi < -\Omega - \frac{2\pi}{h} \Rightarrow h \leq \frac{\pi}{\Omega}$

$\Rightarrow$ if $\hat{f}$ is Ω-band limited, $h \leq \frac{\pi}{\Omega}$,

then $\hat{f}$ is uniquely determined by the values of f on the grid $K\cdot h, K\in\mathbb{Z}^n$

$\Rightarrow$ Shannon, Nyquist, Kotelnikov, Whittaker, Wiener

Often formulation: The most oscillating sinusoidal light appears in an Ω-band limited function is $\sin(\Omega\cdot x)$.

We need to sample twice per period.

Note: This is in $\mathbb{R}^n$

Sampling with double frequency $\hat{=}$.

Theorem: Let $f \in \mathcal{S}$ Ω-band limited.

a) $\|\xi\| > \Omega \Rightarrow \hat{f}(\xi) = 0$

b) $\|\xi\| < \Omega$, $h < \frac{\pi}{\Omega} \Rightarrow$

$\hat{f}(\xi) = (2\pi)^{-n/2} \sum_{k\in\mathbb{Z}^n} f(k,\ell) e^{ik\ell\xi}$

$\Rightarrow$ $\hat{f}(\xi)$ determined

$\Rightarrow$ $f(x)$ determined by IFT

(The trapezoidal rule is exact)

c) $h < \frac{2\pi}{\Omega} \Rightarrow \hat{f}(0) = h(2\pi)^{-n/2} \sum_{k\in\mathbb{Z}^n} f(k,\ell) \approx \int f(x)\,dx \; (2\pi)^{-n/2}$

d) The sampling theorem holds for distributions (e.g. f is 2π-periodic), and then for $u \in \mathcal{S}'$ with $\mathrm{supp}\, u \in \mathcal{E}'$.

e) Let $h \leq \frac{\pi}{\Omega}$. Then

$\tilde{f}(x) = \sum_{k\in\mathbb{Z}^n} f(k\cdot h)\, \mathrm{sinc}\frac{\pi}{h}(x - k\cdot h)$

$= \sum_{k\in\mathbb{Z}^n} f(k,\ell)\, \sin\left(\frac{\pi}{h} - k\pi\right)$

Could choose Ω ♡

Proof

We have, according to our remarks,

$\hat{f}(\xi) = (2\pi)^{-n/2} \sum_{k\in\mathbb{Z}^n} f(k,\ell) e^{ik\ell\xi} \cdot \mathcal{X}_{[-\frac{\pi}{h},\frac{\pi}{h}]^n}(\xi)$

$f(x) = (2\pi)^{-n/2} \sum_{k\in\mathbb{Z}^n} f(k,\ell) \frac{\pi}{h} \cdot \mathrm{sinc}\left(\frac{\pi}{h}\cdot(x - k\ell)\right)$

f) Let f, g Ω- bandlimited, $h \leq \frac{\pi}{\Omega}$.

Then $f\cdot g$ is 2Ω-bandlimited (Remark 2, before 4.1).

$\Rightarrow \sum_{k\in\mathbb{Z}^n} (f\cdot g)(x) = h^n \sum_{k\in\mathbb{Z}^n} f(k,\ell)\, g(k,\ell)$ (Trapezoidal rule exact)

g) The sum converges arbitrarily slow.

It can be sped up by choosing $h \ll \frac{\pi}{\Omega}$.

Construction of a new CTF scheme

Requirements: We want to see objects
down to size $\delta \Rightarrow \eta$
Recover the Fourier transform $\hat{f}(\xi)$
of the image f for $\|\xi\| \leq \frac{\pi}{\delta}$.

Assumption: f is band-limited with
band-limit π/δ (we always do this).

equivalently

Question: Assume that we can measure
$b_j(\theta_j, s_j)$, $\theta_j = \theta(\gamma_j) = \begin{pmatrix} \cos\gamma_j \\ \sin\gamma_j \end{pmatrix}$. Possible sampling
$b = p_0 - p_1$ $\gamma_j = \frac{j\pi}{p_1}$, $s_j = \frac{k}{q} \cdot \frac{\pi}{\delta} - q_1 - q_0$.
How do we have to choose p, q,
such that f is uniquely determined
by the measurements? (band-limit)

Idea: f is band-limited $\Rightarrow \hat{f}(\xi) = 0$ for $\|\xi\| \geq \frac{\pi}{\delta}$
$\Rightarrow$ Sampling theorem, finite # of measurements.

Follow the book by Natterer.

Start from Theorem 3.3, Recall the FBP (2 dimensions):

$V_\psi f = R^\#(v \cdot \psi f)$, $V = R^\# v$,
for any $v \in \mathcal{F}(C)$ Choose v wisely.

Claim: we have that $V = R^\# v \in \tilde{S}$
$\Rightarrow V_\psi f = \tilde{f} \psi$ is smooth & rapid decay.
Problem: on filtered backprojection
$\widetilde{(\text{discrete theorem})}$.
$V_\psi f = (V_\psi f)^\sim = (\widetilde{V \cdot \hat{f}}) \cdot (2\pi)^{n/2}$.
$V \in \tilde{S}$ in $\tilde{V} = (2\pi)^{-n/2} \cdot (v)$
If f is bandlimited:
Take any function $\tilde{V}(\xi) \hat{f}(2\pi)^{n/2}$
for $\|\xi\| \leq \frac{\pi}{\delta}$
$= (\widetilde{V_\psi f}) \cdot (2\pi)^{n/2} = \widetilde{V \cdot \hat{f}}$
$\Rightarrow$ Formula exact if

Idea: Choose v such that
$\tilde{V}(\xi) = \begin{cases} \frac{\chi_{B(0,\pi/\delta)}}{(2\pi)^{n/2}}, & \forall \xi \notin \Omega \\ 0, & \forall \xi \in \Omega \end{cases}$ How do we choose v? ??

Theorem: Let $\psi \in \mathcal{F}(C)$, $V = R^\# v$,
$v(\theta, s) = V(s) * v(s)$. Then (new)
$\tilde{V}(\xi) = 2(2\pi)^{(n-1)/2} \tilde{V}(\xi) \tilde{\psi}(\xi)$
(in distributional sense).

$T_{R^\# v}(\hat{f}) = \langle R^\# v, \hat{f} \rangle_{L^2(\mathbb{R}^n)}$
$= \langle v, R \hat{f} \rangle_{L^2(C)}$ adjoint
$= \langle \tilde{v}, \widehat{R\hat{f}} \rangle_{L^2(\mathbb{S}^1 \times \mathbb{R})}$ Parseval w.r.t. 2nd argument
$= \int_{\mathbb{S}^1} \int_{-\infty}^{\infty} \overline{\widehat{Rf}} \, \tilde{v} \, |\xi|^{n-1} d\xi \, d\theta$ Fourier slice
$= (2\pi)^{(n-1)/2} \sum_{\pm 1} \int_{\mathbb{S}^1} \int_0^\infty \hat{f}(\theta, |\xi|\theta) \, \tilde{v}(\pm\theta, |\xi|) \, |\xi|^{n-1} d\xi \, d\theta$ Polar coordinates
$= \sum_{\pm 1} 2 (2\pi)^{(n-1)/2} \int_{\mathbb{R}^n} \hat{f}(\xi) \, \tilde{v}(\pm\theta, |\xi|) \, |\xi|^{n-1} d^n\xi$ $\quad \}$ use $\tilde{v}$ is even
$\hat{V}(\xi) = 2\tilde{V}(\xi) \hat{f}(\xi)$ $\forall \xi \in \mathbb{R}^n$
$\Rightarrow$ Requirement on $\tilde{V}$ to Requirement on $\hat{f}$

Remarks: 1) Let $\tilde{V}(\xi) = (2\pi)^{-n/2}$. Then the formula is exact.
$V_\psi f = R^\# (v * \psi)$
$V = R^\# v$
$\hat{V}(\xi) = (2\pi)^{-n} \Rightarrow \hat{V}(\xi) \hat{\psi}(\xi) = \frac{2}{(2\pi)^{n-1}} |\xi|^{n-1} \hat{\psi}(\xi) = (2\pi)^{-n/2}$
$\Rightarrow \tilde{v}(\pm\theta, s) = \frac{2}{(2\pi)^n} (2\pi)^{n/2} \delta(s) = \frac{1}{2\pi} \delta(s)$
$R^\# (\chi_{B(0,\pi/\delta)}) = R^\# \left(\frac{2}{(2\pi)^n} \hat{\chi}_{B(0,\pi/\delta)} \right) = \frac{1}{2\pi} R^{-1} \chi_{\left[-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right]}$ (Riesz),
$\Rightarrow$ choosing $V = v_{\text{ideal}}$ gives FBP back (see top).

2) Let $\psi \in \mathcal{F}$ band-limited, let
$\psi(s)_{\text{ideal}} = \begin{cases} \sigma, & |s| \leq \alpha \\ 0, & |s| > \alpha \end{cases}$. Set $\tilde{V}_{\psi_{\text{ideal}}}(\xi) = (2\pi)^{n/2} \hat{\rho}^{(n)}(\frac{\alpha}{2\pi}\xi)$, $\forall \xi \in \mathbb{R}^n$
$V_{\psi_{\text{ideal}}} = R^\# v_{\psi_{\text{ideal}}}$
$\Rightarrow$ $f = V_{\psi_{\text{ideal}}} * (f * \psi)$ $\Leftarrow$ This is what happens with filtered backprojection.
$\sigma^{(n)}$: $\hat{\rho} = R^\# (v_{\psi_{\text{ideal}}} * \rho)$ $\Leftarrow$ Convolution & filtering

Note: $v_{\psi_{\text{ideal}}}$ is always $\mathcal{F}$-band-limited even if $\hat{\rho}$ is not.
$\Rightarrow$ Projection onto the set of band limited functions.

$\hat{v}_{\psi_{\text{ideal}}}(\xi, s) = \frac{2}{\alpha} (2\pi)^{(n-1)/2} \hat{\rho}(\xi) \delta^{(n-1)}(2\pi \cdot s)$ $(\forall \xi \neq 0)$
$= \frac{2}{\alpha} \sqrt{2\pi} \cdot (2\pi)^{n-1} \hat{\rho}(\xi) \delta^{(n-1)}\left(\frac{2\pi}{\alpha} s\right)$
Inverse: $v_{\psi_{\text{ideal}}}(s) = \frac{2}{\alpha} (2\pi)^{n-1} \int_{-\sqrt{n}}^{\sqrt{n}} |\xi|^{n-1} \hat{\rho}\left(\frac{2\pi}{\alpha}\right) e^{i 2\pi \xi s} d\xi$

a) Let $\hat{\rho}(\xi) = \begin{cases} \sigma, & |\xi| \leq \alpha \\ 0, & |\xi| > \alpha \end{cases}$.

$n=2$. $v_{\psi_{\text{ideal}}}(s) = \frac{2}{\alpha} (2\pi)^{n-1} \int_{-\sqrt{n}}^{\sqrt{n}} |\sigma| e^{i 2\pi \sigma s} d\sigma$

$= \frac{\sigma}{\pi\alpha} \int_0^{\sqrt{n}} \sigma \cdot \cos(i 2\pi \sigma s) \, d\sigma$
$= \frac{\sigma}{\pi\alpha} \left(\left[\sigma \cdot \sin(i 2\pi \sigma s) \frac{1}{i 2\pi s} \right]_0^{\sqrt{n}} - \int_0^{\sqrt{n}} \frac{1}{i 2\pi s} \sin(i 2\pi \sigma s) d\sigma \right)$
$= \frac{\sigma}{\pi\alpha} \left(\sqrt{n}^2 \cdot \frac{\sin(i 2\pi \cdot \sqrt{n})}{i \cdot 2\pi} + \frac{\sigma}{i 2\pi} \cdot \left(\cos(i 2\pi \cdot \sqrt{n}) - 1 \right) \right)$
$= \frac{\sigma}{\pi\alpha} \left(\sin(\sqrt{n}) + \frac{\sigma}{i 2\pi s} \cdot \left(\cos(i 2\pi \cdot \sqrt{n}) - 1 \right) \right)$
$\left(\frac{\sigma}{2\pi} \text{ for } s = 0 \right)$

(Riemann-Lebesgue: $\lim_{|s| \to \infty} \int_{-\sqrt{n}}^{\sqrt{n}} \sigma \cdot \cos(i 2\pi \sigma s) \, d\sigma = 0$)
Gives band-limited version of δ (Mexican hat)

b) Consult (book) literature for derivation:
Shepp/Logan, $\hat{\rho}(\sigma) = \sin(\sigma \frac{\pi}{2})$ $|\sigma| \leq 1$,
0, else.

2b) Implementation:
Select a specific function $\hat{\rho}$, $\hat{v}_{\psi_{\text{ideal}}}$, $v_{\psi_{\text{ideal}}}$.
Measure data.
Compute $\hat{\rho} * v_{\psi_{\text{ideal}}}$.
Compute $R^\# (\hat{\rho} * v_{\psi_{\text{ideal}}})$.
Note: $R^\# (\hat{\rho} * v_{\psi_{\text{ideal}}})$ is continuous ($\hat{\rho}$ $\hat{v}_{\psi_{\text{ideal}}}$)

Implementation of possible geometry

$g(\theta_i, s_e) = (Rf)(\theta_i, s_e)$,

$\theta_i = \theta(f_u),\ \gamma_i = 4\frac{\pi}{p},\ i = 0 \ldots p-1$

$s_e = e \cdot L,\ L = \frac{2}{q},\ e = -q \ldots q$.

supp $f \subset k_1(0)$

f (essentially) Ω-band limited.

v_Ω is $\mathbb{R}$sm-Lsb $(\mathcal{F}_{[-\Omega,\Omega]^2} \cdot (2\pi)^{-n/2} = \check{v}_\Omega)$

Implement: $\mathcal{R}^*(v_\Omega * g)$ such that it is exact.

1) v_Ω is Ω-band limited by definition.

$(2\pi)^{(n-1)/2}\, \hat{f}(\sigma\cdot\theta) = \hat{g}(\theta,\sigma)$ (Fourier slice)

$|\sigma| \geq \Omega \Rightarrow \hat{g}(\theta,\sigma) = \hat{f}(\sigma\cdot\theta) = 0$ (f is Ω-band(limited))

$\Rightarrow g$ is Ω-band limited wrt 2nd var.

$(v_\Omega * g)(\theta,s) = \int_{-\infty}^{\infty} v_\Omega(s-s')\, g(\theta,s')\, ds'$

6, 7, 8: Trapezoids rule exact if $h \in \frac{\pi}{\Omega}\mathbb{Z} \Leftrightarrow q \geq \frac{\Omega}{\pi}$ Condition (1)

(2) Compute $\mathcal{R}^*(v_\Omega * g)(x)$

$= \int_0^{2\pi} (v_\Omega * g)(\theta(\varphi), x\cdot\theta(\varphi))\, d\varphi$

Let $h(\varphi) = (v_\Omega * g)(\theta(\varphi), x\cdot\theta(\varphi))$.

Without loss of generality, let $x = 0$
(otherwise, shift $f_{\delta x}(y) = f(x+y)$)

$h(\varphi) = (v_\Omega * g)(\theta,0) = \widetilde{(v_\Omega * g)}(\theta,0)$

$= \sqrt{2\pi}\, (\check{v}_\Omega \cdot \hat{g})^{\vee}(\theta,0)$ Fallunterscheidung

$= \int_{-\Omega}^{\Omega} \check{v}_\Omega(\sigma)\, \hat{g}(\theta,\sigma)\, e^{i\sigma 0}\, d\sigma$ IFT

$= \sqrt{2\pi} \int_{-\Omega}^{\Omega} \check{v}_\Omega(\sigma) \int (\sigma\cdot\theta)\, d\sigma$ Fourier slice

$= \sqrt{2\pi} \int_{-\Omega}^{\Omega} \check{v}_\Omega(\sigma) \int_{\|y\|<1} e^{-i\sigma\theta y} f(y)\, dy\, d\sigma$ FT

$= \sqrt{2\pi} \int_{\|y\|<1} f(y) \int_{-\Omega}^{\Omega} \check{v}_\Omega(\sigma)\, e^{-i\sigma\theta y}\, d\sigma\, dy$ (Change order of integration)

$h_i = \frac{\pi}{2\pi} \int_{-\pi}^{\pi} h(\varphi)\, e^{-i4\varphi}\, d\varphi$

$= (2\pi)^{-3/2} \int_{\|y\|<1} f(y) \int_{-\Omega}^{\Omega} \check{v}_\Omega(\sigma) \int_{-\pi}^{\pi} e^{-i\sigma\|y\|\cdot\cos(\varphi-\Psi(y))} e^{-i4\varphi}\, d\varphi\, d\sigma\, dy$

$y = \|y\| \cdot \theta(\Psi(y))$

Lemma 4.24: $= 2\pi J_4(\sigma\|y\|)$; $e^{-i4\Psi}$ was missing in the script

4.23 Debye's relation, $J_4(x) \sim 0$ for $|x| < |4|$.

$\sim 0, \ |4| > \Omega$.

That is for $x = 0$: If $x \in S_1 : y \to y \cdot x, \sim 0$ for

$|4| > 2\Omega$.

$\Rightarrow$ Fourier coefficients of the (2π)-periodic function vanish for $|4| \geq 2\Omega$.

$\Rightarrow$ Trapezoids rule for h_i is exact

for $\frac{2\pi}{2p} < \frac{2\pi}{2\Omega}$

$\int_{-\pi}^{\pi} h(\varphi)\, d\varphi = \frac{\pi}{p} \sum_{j=0}^{2p-1} h(\varphi_j)$.

Condition 2: $p > \Omega$!

All in all:

$(V_\Omega * f) = 2\frac{\pi}{p}\frac{1}{q} \sum_{i=0}^{p-1} \sum_{e=-q}^{q} v_\Omega(x\cdot\theta_i - s_e)\, g(\theta_i, s_e)$.

(Note: Implementation / Literature,
Efficient sampling,
3D,
Attenuated Radon Transform,
Algebraic Methods,
fan beam,
efficient implementation)

3D scanning

Idea:



Measure a complete X-Ray. Move the source along a path. From each position, the integrals along all lines through this position are recorded.

Cone Beam - Transform:

$$(\mathcal{D}f)(a,x) = \int_0^\infty f(a+x\cdot t)\,dt \quad (= \tfrac{1}{2}\dots)$$

Theorem: $\widehat{\mathcal{D}f}(a,\xi) = \int_0^\infty \rho^{n-2} e^{-i\rho a\cdot\xi} \hat{f}(\rho\xi)\,d\rho$

In distr. sense: $\hat{T}_{\mathcal{D}f}(\varphi) = T_{\mathcal{D}f}(\hat{\varphi})$

$$= \int_{\mathbb{R}^n} \mathcal{D}f(a,\xi)\,\hat{\varphi}(\xi)\,d\xi$$

$$= \int_{\mathbb{R}^n}\int_0^\infty f(a+r\xi,\xi)\,\hat{\varphi}(\xi)\,dr\,d\xi$$

$$= \int_0^\infty \int_{\mathbb{R}^n} r^{-n} e^{ia\cdot\xi/r} \hat{f}\left(\tfrac{\xi}{r}\right)\hat{\varphi}(\xi)\,d\xi\,dr \quad \text{(Parseval, 4.2)}$$

$$= \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \int_0^\infty r^{-n} e^{ia\cdot\xi/r} \hat{f}\left(\tfrac{\xi}{r}\right) dr\,d\xi \quad \rho = \tfrac{1}{r},\ r = \tfrac{1}{\rho},\ dr = \tfrac{1}{\rho^2}\,d\rho$$

$$= \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \underbrace{\int_0^\infty \rho^{n-2} e^{ia\cdot\xi\rho} \hat{f}(\rho\xi)\,d\rho}_{\widehat{\mathcal{D}f}(a,\xi)}\,d\xi$$

Now choose $n=3$. Assume that the radioactive source travels along $a(\lambda)$, $\lambda\in[0,1]$.

Tuy's condition:

Let $x\in\operatorname{supp} f$. Let $\theta\in S^2$.

Let $E_{x,\theta} = \{y : y\cdot\theta = x\cdot\theta\}$ (plane normal to θ through x).

The curve a satisfies Tuy's condition, iff

$\forall x,\theta\ \exists\,\lambda(\theta,x) = a(\lambda(\theta,x)) \in E$, (Every plane that hits the support of f has one point on the curve)

E is not tangential to a in $\lambda(\theta,x)$

$\Leftrightarrow a'(\lambda(\theta,x))\cdot\theta \neq 0$.

Example:



a is a circle around a 3D object.

a does not satisfy Tuy's condition (horizontal plane)

together with generic: Tuy satisfied.

Theorem: If Tuy is satisfied, then f can be reconstructed from $(\mathcal{D}f)(a(\lambda),x)$, $\lambda\in[0,1]$.

Proof: Let $x\in\operatorname{supp} f$. We prove this:

$$f(x) = (2\pi)^{-3/2} \overset{!}{=} \int_{S^2} \frac{1}{a'(\lambda(\theta,x))\cdot\theta} \frac{d}{d\lambda}\,(\widehat{\mathcal{D}f})(a(\lambda),\theta)\Big|_{\lambda=\lambda(\theta,x)}\,d\theta$$

$$f(x) = \check{\hat{f}}(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \hat{f}(\xi)\,e^{ix\cdot\xi}\,d\xi \qquad \xi = r\cdot\theta,\ r>0,\ \theta\in S^2,\ d\xi = r^2\,dr\,d\theta$$

$$= (2\pi)^{-3/2} \int_{S^2}\int_0^\infty \rho^2 e^{ix\cdot\rho\theta} \hat{f}(\rho\cdot\theta)\,d\rho\,d\theta$$

$$= (2\pi)^{-3/2} \int_{S^2} \underbrace{\int_0^\infty \rho^2 e^{i\rho\,a(\lambda(\theta,x))\cdot\theta} \hat{f}(\rho\cdot\theta)\,d\rho}_{}\,d\theta$$

$$\frac{d}{d\lambda}\,(\widehat{\mathcal{D}f})(a(\lambda),\theta) = \frac{d}{d\lambda}\int_0^\infty \rho\, e^{i\rho a(\lambda)\cdot\theta} \hat{f}(\rho\cdot\theta)\,d\rho$$

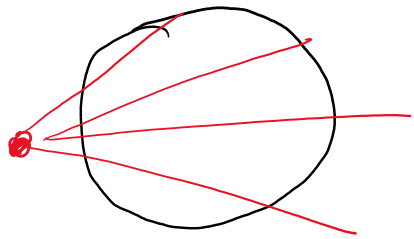
$$= ia'(\lambda)\cdot\theta \int_0^\infty \rho^2 e^{i\rho\,a(\lambda)\cdot\theta} \hat{f}(\rho\cdot\theta)\,d\rho$$

$$\Rightarrow f(x) = (2\pi)^{-3/2} \cdot i \int_{S^2} \frac{1}{a'(\lambda(\theta,x))\cdot\theta} \frac{d}{d\lambda}\,(\widehat{\mathcal{D}f})(a(\lambda),\theta)\Big|_{\lambda=\lambda(\theta,x)}\,d\theta$$

Note: This is not implemented.

Note: For the typical geometry we need something else (Approximate inversion, FDK) ↝ Feldkamp / Davis / Kress.

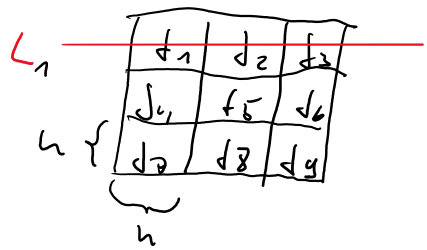
Analytic inversion : Fan-beam



Not parallel
⇒ Our reconstruction algorithm does not work

Redo all the work
Or Use interpolation to map measurements onto parallel measurements

New idea: We want to compute an image with finite resolution.



3×3. f is constant in each pixel.

$$\int_{L_1} f(x)\,dx = g_1 \Rightarrow h\cdot f_1 + h\cdot f_2 + h\cdot f_3 = g_1$$

for any line L_k:

$$\sum_p a_{k,p} f_p = g_k, \quad a_{k,p} = |L_k \cap \text{Pixel}_p|$$

$$\Leftrightarrow Af = g, \quad A = (a_{k,p}),\ g = (g_k),\ f = (f_p)$$

So: We have a simple matrix equation!
Independent of any geometrical considerations!
A is large. Need iterative algorithm.

efficient sampling

$g(\theta_k, s_\ell) = (Rf)(\theta_k, s_\ell)$,

$\theta_k = \theta(\Gamma_\alpha)$, $\gamma_q = 4\frac{\pi}{P}$, $k = 0 \ldots P-1$

$s_\ell = \ell \cdot 4$, $\ell = \frac{d}{q}$, $\ell = -q \ldots q$.

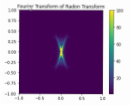
$q \geq \frac{\Omega}{\pi}$, $P \geq \Omega$.

Engineers: We can improve the resolution by measuring $(Rf)(\theta_k, s_\ell \pm \varepsilon_\ell)$ $\varepsilon_\ell = \begin{cases} 0, & \text{even} \\ \frac{4}{2}, & \text{odd} \end{cases}$ motivation

Idea: $R_f(\theta(\gamma), s) = g(\gamma, s)$.

Take $\hat{g}(k, \ell)$ wrt both variables.

In the first variable: FT of periodic function (see Exercises).



Support of the Fourier Transform of a band limited f:



very long story (Natterer, Theorem 3.1)

$K := \text{supp}\ \hat{g}$

$\subseteq \left\{ (k,s) : |s| < \Omega, |k| < |s| \right\}$ simplified

Petersen / Middleton:

$k\eta$ $K + 2\pi W^{-T}(\frac{k}{\ell}) = \emptyset \Rightarrow g(W k)$ uniquely determines g.

$W = \begin{pmatrix} \frac{\pi}{P} & 0 \\ 0 & \frac{\pi}{\Omega} \end{pmatrix}$ (possible scaling)

$\Rightarrow 2\pi W^{-T} = \begin{pmatrix} 2P & 0 \\ 0 & 2\pi q \end{pmatrix}$



shift in each direction $\geq 2\Omega$ to satisfy Petersen/Middleton

$\Rightarrow 2P \geq 2\Omega$, $2\pi q \geq 2\Omega$

$\Rightarrow P \geq \Omega$, $q \geq \frac{\Omega}{\pi}$.

But:



lots of empty space $\Rightarrow$ inefficient.



Covers $\mathbb{R}^2$ perfectly?

$2\pi W^{-T}$: $(0,1) \Rightarrow (\Omega, \Omega)$,

$(1,0) \Rightarrow (0, 2\Omega)$

$\Rightarrow 2\pi W^{-T} = \begin{pmatrix} \Omega & 0 \\ \Omega & 2\Omega \end{pmatrix} = \Omega \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

$\Rightarrow W = \frac{\pi}{\Omega} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$

Sample at $W\begin{pmatrix} k \\ \ell \end{pmatrix} = \left(\frac{2\pi}{\Omega} k - \frac{\pi}{\Omega} \ell, \ell \frac{\pi}{\Omega} \right)$

$= \left(\frac{\pi}{\Omega} (2k - \ell), \ell \frac{\pi}{\Omega} \right)$

$P = \Omega$, $q = \Omega/\pi$ $= \left(\frac{\pi}{P} (2k - \ell), \frac{\ell}{q} \right)$

For 1 pixel, spacing of is $\frac{2\pi}{P}$, before $\frac{\pi}{P}$ $\Rightarrow$ Half the dots must be measured.

$\Rightarrow$ Optimal scanning leads to less measured values $\Rightarrow$ less radioactive exposure for the patient.

Parallel Sampling - Remarks

$g(\theta_i, s_\ell) = (Rf)(\theta_i, s_\ell)$,

$\theta_i = \theta(f_m)$, $f_m = \frac{m}{P}$, $m = 0 \dots P-1$

$s_\ell = \ell \cdot h$, $h = \frac{2}{Q}$, $\ell = -q \dots q$.

$q \geq \frac{\Omega}{\pi}$, $P \geq \Omega$.

1) Let p, q given. Then the max. frequency, that can be reconstructed, is $\Omega = \min(p, \pi \cdot q)$.
⇒ It does not help if we increase only p or q.
Optimal ratio: $\frac{p}{q} = \pi$.

2) Numerical effort:

$$(V_\Omega \ast f) = 2 \frac{\pi}{P} \frac{1}{q} \sum_{j=0}^{P-1} \sum_{\ell=-q}^{q} v_\Omega(x \cdot \theta_j - s_\ell)\, g(\theta_j, s_\ell).$$

f is Ω-band limited.

Sampling theorem ⇒ f should be sampled with $h = \frac{\pi}{\Omega}$.
⇒ $\mathcal{O}(\Omega^2)$ values.

$q/p = \mathcal{O}(\Omega)$ ⇒ $\mathcal{O}(\Omega^2)$ operations per value
⇒ Numerics (effort $\mathcal{O}(\Omega^4)$)

Better idea: Fix $\theta = \theta_j$. ← inequal, not equidistant
In the inner sum, we compute $h(x \cdot \theta_j)$ where FFT does not help.

$h(s) = (V_\Omega \ast g)(s)$.

$\sqrt{2\pi}\, \underbrace{V_\Omega \cdot \hat{g}}_{=} \Rightarrow \Omega$-band limited.

⇒ Sample h on a grid of resolution $< \frac{\pi}{\Omega} = \frac{1}{q}$
$s = s_\ell$

$$h(s_\ell) = (V_\Omega \ast g)(s_j) = \frac{1}{2q} \sum_{\ell=-q}^{q} v_\Omega(s_j - s_\ell)\, g(\theta_i, s_\ell).$$

Compute $h(s)$ by interpolation.
For correct results: use Shannon to reconstruct h.
In practice: linear interpolation (Radon in MatLab, 1st parameter 'linear').

Can use Fourier Transform
⇒ Upper limit: For each θ: $\mathcal{O}(\Omega \log \Omega)$
$\Rightarrow \mathcal{O}(\Omega^2 \cdot \log \Omega)$

⇒ For each evaluation, $\mathcal{O}(\Omega)$
$\Rightarrow \mathcal{O}(\Omega^3)$

⇒ Numerical effort $\mathcal{O}(\Omega^3)$

3) Revisit Ram-Lak:

$$V_\Omega(s) = \frac{\Omega^2}{4\pi^2} \left(\operatorname{sinc}(s\Omega) + \frac{1}{\Omega^2 s^2} \left(\cos(s\Omega) - 1 \right) \right)$$

$(\frac{1}{2}$ for $s = 0)$

According to 2): Evaluate only for $s_\ell - s_{\ell-1} = s_{\ell-1} - (\ell-1)h = \frac{1}{q}$.
According to 1): $q = \frac{\Omega}{\pi}$

$$V_\Omega(0) = \frac{\Omega^2}{4\pi^2} \cdot \frac{1}{2}$$

$$\ell \neq 0: V_\Omega(s_\ell) = \frac{\Omega^2}{4\pi^2} \left(\underbrace{\operatorname{sinc}(\ell \cdot \Omega \cdot \tfrac{\pi}{\Omega})}_{=0} + \frac{1}{\Omega^2 \ell^2 \frac{\pi^2}{\Omega^2}} \left(\underbrace{\cos(\ell \cdot \Omega \cdot \tfrac{\pi}{\Omega})}_{=(-1)^\ell = -1} - 1 \right) \right)$$

$$= \begin{cases} 0, & \ell \text{ even} \\ \frac{\Omega^2}{2\pi^2 \ell^2}, & \ell \text{ odd} \end{cases}$$

...and this is the definition Ram-Lak actually gave. The definition of filtfilt is what is used (matlab, python...).

All in all:

$$(V_\Omega \ast f) = 2 \frac{\pi}{P} \frac{1}{q} \sum_{j=0}^{P-1} \sum_{\ell=-q}^{q} v_\Omega(x \cdot \theta_j - s_\ell)\, g(\theta_j, s_\ell).$$

(Note: ~~Interpolation, Filterinterpolation~~, ~~Efficient sampling~~, 3D, • Attenuated Radon Transform, Algebraic Methods, fan beam, ~~efficient implementation~~)

Lecture 21: Convergence analysis for the fully discrete case

Notation: Let R_k for as above, $k=0:p-1$, $R_k : X \to Y_k$, $R_k^* : Y_k \to X$.

Let $C_k : X \to X$ continuous and positive definite and $C_k \geq R_k^* R_k$ in the sense that $(C_k x, x) \geq (R_k^* R_k x, x) = (R_k x, R_k x)$
$= \|R_k x\|_{Y_k}^2 \quad \forall x \in X$. $(x, y$ adjoint $= C_k x)$
$\Rightarrow \sigma_{\min}(C_k) \geq SV(R_k)$

$R = \begin{pmatrix} R_0 \\ \vdots \\ R_{p-1} \end{pmatrix}$, $g = \begin{pmatrix} g_0 \\ \vdots \\ g_{p-1} \end{pmatrix}$. Typically, C_k simple (diagonal).

Assume $R f = g$ is consistent $\Rightarrow \exists f : Rf = g$. Let $\dim X = n$.

Let $f^{(\infty)} \in X$, $f^{(\infty)} = f^{(\infty)} + \omega R_k^* \underbrace{C_k^{-1}}_{\text{k=1 mod p}} (g_k - R_k f^{(\infty)})$.

Let $f^{(0)} \in \ker(R)^\perp$ (e.g. $f^{(0)} = 0$).

Then $f^{(kp)} \xrightarrow[k \to \infty]{} R^+ g$.

Proof: Assume X, Y_k are finite-dimensional.

Let $f^* = R^+ g \Rightarrow R(f^* - g, f^* \in \ker(R)^\perp$
$= \text{Range}(R^*)$

$f^{(p)}$ is the approximation after 1 sweep (using old data only).

$f^{(kp)}$ approximation after k sweeps.

To compute 1 Landweber-step:
Apply $R_0 : R_{p-1} \; R_0 : R_{n-1}$.

Same as for 1 Kaczmarz-sweep as iterate $f^{(p)}$.

$e_k = f^{(kp)} - f^*$. Show that $e_k \to 0$.

$Q_k = (I - \omega R_k^* C_k^{-1} R_k)$. $(X \to X)$

1 sweep: $Q_k \begin{pmatrix} f^{(k)} \\ \vdots \\ f^{(k)} \end{pmatrix} = \begin{pmatrix} f^{(k)} \\ \vdots \\ f^{(k)} \end{pmatrix} - \omega R_k^* C_k^{-1} \begin{pmatrix} R_0 f^{(k)} \\ \vdots \\ R_{p-1} f^{(k)} \end{pmatrix} - \begin{pmatrix} g_0 \\ \vdots \\ g_{p-1} \end{pmatrix}$

$= f^{(k+1)} - f^*$.

$Q_k \begin{pmatrix} f^{(k)} - f^* \\ \vdots \\ f^{(k)} - f^* \end{pmatrix} = f^{(k+1)} - f^* - \omega R_k^* C_k^{-1} R_k f^{(k)} - g_k$

$\Rightarrow Q = Q_{p-1} \cdots Q_0 \Rightarrow f^{(kp)} - f^* = Q^k (f^{(0)} - f^*) \Rightarrow e_{k+1} = Q e_k$

$\Rightarrow \|e_k\| \leq \|Q\|^k \|e_0\|$

If $\|Q\| < 1 \Rightarrow$ Done.

Problem: Take $u \in \ker(R) \Rightarrow u \in \ker(R_k)$
$\Rightarrow Q_k u = u$
$\Rightarrow Q u = u$
$\Rightarrow \|Q\| \geq 1$.

(1) $\Rightarrow \ker(R)$ is an invariant subspace of Q, $Q|_{\ker(R)} = I$.

(2) Q_k self-adjoint $\Rightarrow Q^* = Q_0 \cdots R_{p-1}$.
$Q^*|_{\ker(R)} = I$ (see above).
Let $x \in \ker(R)^\perp, y \in \text{Range}(R^*) = \ker(R)^\perp$
$\Rightarrow (x, Qy) = (Q^* x, y) = (x, y) = 0 \Rightarrow Qy \in \ker(R)^\perp$
$\Rightarrow \ker(R)^\perp = \text{Range}(R^*)$ is an invariant subspace of Q

Since $f^{(\infty)} \in \ker(R)^\perp$: $Q(f^{(\infty)} - f^*)$
$Q : \ker(R)^\perp \to \ker(R)^\perp$. Show that $\|Q\| < 1$.

Let $\|Q\| = 1$. $\|Q x_k\|^2 = \|x - \omega R_k^* C_k^{-1} R_k x\|^2$
$= (x - \omega R_k^* C_k^{-1} R_k x, x - \omega R_k^* C_k^{-1} R_k x)$
$= \|x\|^2 - \omega (R_k^* C_k^{-1} R_k x, x) - \omega (R_k^* C_k^{-1} R_k x, R_k^* C_k^{-1} R_k x)$
$\leq \|x\|^2 - \omega (2 - \omega) (R_k x, C_k^{-1} R_k x)$
(C_k pos. definite, ≥ 0)

Assume $x \in \ker(R)^\perp$, $x \neq 0 \Rightarrow \exists k : R_k x \neq 0$.
Let k be the smallest index with this property,
$R_0 x \ldots R_{k-1} x = 0 \Rightarrow Q_{k-1} x = x$, $R_k x \neq 0$.
$\|Q_k \cdots Q_0 x\| = \|Q_k x\| < \|x\|$
$\Rightarrow \|Q x\| = \|Q_{p-1} \cdots Q_0 x\| < \|x\|$.
$\|Q\|$, $\|Q\|_{\ker(R)^\perp}$ attains its maximum (finite dimensions)

$\Rightarrow \sup_{\substack{x \neq 0 \\ x \in \ker(R)^\perp}} \frac{\|Qx\|}{\|x\|} < 1 \Rightarrow \|Q\|_{\ker(R)^\perp} < 1$.

Things are much more difficult in the infinite-dimensional case.

ART / Kaczmarz: Examples

1) $R = \begin{pmatrix} a_0^t \\ \vdots \\ a_{p-1}^t \end{pmatrix}$, $g = \begin{pmatrix} g_0 \\ \vdots \\ g_{p-1} \end{pmatrix}$. $Rf = g$

Solve $R_k f = a_k^t \cdot f = g_k$, $k = 0 \dots p-1$. $k = k(\ell)$.

$$f^{(\ell+1)} = f^{(\ell)} + \omega a_k \left(a_k^t a_k\right)^{-1} \left(g_k - a_k^t f^{(\ell)}\right)$$

$$= f^{(\ell)} + \omega \frac{g_k - a_k^t f^{(\ell)}}{\|a_k\|^2} \cdot a_k$$

(Kaczmarz 1937), $(k = 4)$.

2) Parallel geometry, semidiscrete case.

$(R_k f)(s) = Rf(\theta_k, s) = g_k(s)$, $k = 0 \dots p-1$.

$R_k : L^2(K_1(0)) \to L^2([-1,1])$, restriction of R on unit circle.

R_k^* is backprojection:

$$(R_k^* g)(x) = g(x \cdot \theta_k) \quad [\|x\| < 1 \text{ ?}]$$

$$(R_k R_k^* g)(s) = R(R_k^* g)(\theta_k, s)$$

$$= \int_{\substack{x \cdot \theta_k = s \\ \|x\| < 1}} g(x \cdot \theta_k) \, dx$$

$$= 2\sqrt{1 - s^2}\, g(s) \qquad \left[R_k^* (R_k R_k^*)^{-1} (g_k - R_k f)\right]$$

$$\Rightarrow f^{(\text{neu})}(x) = f^{(\text{alt})}(x) + \omega \frac{1}{2\sqrt{1 - (x \cdot \theta_k)^2}} \left(g_k(x \cdot \theta_k) - Rf(\theta_k, x \cdot \theta_k)\right)$$

Discretization ($\mathbb{R}^2$) ART - Algebraic Reconstruction Techniques

$Rf(\theta,s) = g(\theta,s)$ (continuous form.)
$Rf(\theta_i,s_j) = g_{ij}$ (discrete form.)
$Rf(\theta_i,s) = g_i(s)$ (semi-discrete form)

Discretize: $f(x) \approx \sum_k f_k \chi_k(x)$, χ_k Ansatz-functions

Typically χ_k := Char. fct. of Pixel/Voxel
or $e^{-\lambda\|x-x_k\|^2/2}$

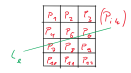
Fully discrete:

$$g_{ij} = (Rf)(\theta_i,s_j) = \sum_k \int_{L_{ij}} f(x)\,dx \qquad (L_{ij} = (\theta_i,s_j))$$
$$= \sum_k f_k \underbrace{\int_{L_{ij}} \chi_k(x)\,dx}_{a_{ijk}}$$

$\Rightarrow \sum_k a_{ijk} f_k = g_{ij}$

$\Rightarrow Rf = g$, $R = (a_{ijk})$, $f = (f_k)$, $g = (g_{ij})$.
$\Rightarrow$ linear equation for f.

In the case of pixels:



$a_{ijk} = \int_{L_{ij}} \chi_k(x)\,dx = \|L_{ij} \cap \operatorname{supp}\chi_k\|$
= length of intersection of line L_{ij} and Pixel k.

Almost always 0 (max 2n, typically $\le 4n$)
$\Rightarrow A$ is sparse

Sidham / Bresenham: a_{ijk} can be computed very fast (follow the line)
$\Rightarrow A$ is never stored, but computed on-the-fly

Typical size: $n = 1024 \times 1024$ unknowns
$\sim 10^6$, $\sim 10^6$ equations
$\Rightarrow$ least squares solution

Iterative Methods ♡

Landweber: $f^{(k+1)} = (I - \omega R^T R)(f^{(k)} - R f^{(k)}))$
2 applications of R in each step $\Rightarrow$ expensive
- The same equations, the same difficulty
- We work hard, only slowly the iterates are bad in the beginning
- reduce resolution / equations in the beginning?
(Multi-Resolution)
- Problem with ill-conditioned matrices

Alternative approach: (Redefine g_i)
example for parallel sampling
$g(\theta_i,s)$ measured, $i = 0 \dots p-1$.
$(R_i f)(s) = (Rf)(\theta_i,s) = g_i(s) = g(\theta_i,s)$
Solve $(R_i f) = g_i$, $i = 0 \dots p-1$.
Very common structure. Mostly source position.
Idea: For $f^{(0)} \to f^{(1)}$, use only R_0.
$f^{(1)} \to f^{(2)}$, use only R_1.
$\vdots$
$f^{(p-1)} \to f^{(p)}$, use only R_{p-1}.
} sweep
$f^{(p)} \to f^{(p+1)}$, use only R_0 (mod p)

$R_k \underbrace{f^{(k+1)}}_{\in \mathbb{R}^n} = \underbrace{g_k}_{\in \mathbb{R}^1}$ underdetermined ♡

Difference to $f^{(k)}$ should be minimized
$f^{(k+1)} = f^{(k)} + d$
$R_k(f^{(k)} + d) = g_k \Rightarrow R_k d = g_k - R_k f^{(k)}$
$\Rightarrow d = R_k^* \underbrace{(R_k R_k^*)^{-1}}_{\text{if } R_k \text{ is surjective}} (g_k - R_k f^{(k)})$

Kaczmarz-Method:
Task: Solve $R_k f = g_k$, $k = 0 \dots p-1$, (discrete, semi-...)
$R_k : X \to Y_k$, Hilbert spaces. $f^{(0)} \in X$,
$f^{(k)}$ ($\to f$): $k = kl(k)$.
$f^{(k+1)} = f^{(k)} + R_k^* (R_k R_k^*)^{-1} (g_k - R_k f^{(k)})$.
orthogonal Projection of $f^{(k)}$ onto $\{R_k^* f = g_k =: P_k^{(k)}\}$ (affine)
$f^{(k+1)} = f^{(k)} + \omega R_k^* (R_k R_k^*)^{-1} (g_k - R_k) f^{(k)} = P_k^\omega f^{(k)}$

Much cheaper, because in each step only R_k is applied.
Typically: Time for 1 sweep $\cong$ time for 1 Landweber step.
Problem: Convergence?
Invention 2

Noise (statistical view), Regularization

$g = Rf$, $\tilde{g} = g + n$, $n(\theta,s)$ is random variable

$E n(\theta,s) = 0$

$\underbrace{E(n(\theta,s)\, n(\theta',s'))}_{\text{cov } n} = \begin{cases} \sigma^2, \theta=\theta', s=s' \Rightarrow \sigma^2 \cdot \delta(\theta-\theta')\delta(s-s') \\ 0, \text{ sonst} \end{cases}$

uncorrelated noise (think of it discretized)

$\tilde{f}(x) = \int_{S^{n-1}} \int_{\mathbb{R}} \tilde{g}(\theta,s)\, v(x\cdot\theta - s)\, ds\, d\theta$ is random variable

$(E\tilde{f})(x) = \int_{S^{n-1}} \int_{\mathbb{R}} g(\theta,s)\, v(x\cdot\theta - s)\, ds\, d\theta$

Variance: $\sigma^2_{recon}(x) = E\left(\left(\tilde{f}(x) - E\tilde{f}(x) \right)^2 \right)$

$= E\Big(\int_{S^{n-1}} \int_{\mathbb{R}} n(\theta,s)\, v(x\cdot\theta - s)\, ds\, d\theta \cdot \int_{S^{n-1}} \int_{\mathbb{R}} n(\theta',s')\, v(x\cdot\theta' - s')\, ds'\, d\theta' \Big)$

$= \int\int\int\int \underbrace{\left[E\, n(\theta,s)\cdot n(\theta',s') \right]}_{\sigma^2\cdot\delta(\theta-\theta')\delta(s-s')} v(x\cdot\theta - s)\, v(x\cdot\theta' - s')\, ds\, ds'\, d\theta\, d\theta'$

$= \int_{S^{n-1}} \int_{\mathbb{R}} v^2(x\cdot\theta - s)\, \sigma^2(\theta,s)\, ds\, d\theta$

$\sigma^2(\theta,s) = \sigma^2$

$= |S^{n-1}| \int_{\mathbb{R}} v^2(s)\, ds \cdot \sigma^2$

$= |S^{n-1}| \int_{\mathbb{R}} |\hat{v}(\tau)|^2 d\tau \cdot \sigma^2 \downarrow$ 6.10

$= \sigma^2 \cdot |S^{n-1}| \cdot C \cdot \int_{-\Omega}^{\Omega} |\tau|^{2n-2} |\hat{\phi}(\tfrac{\tau}{\Omega})|^2 d\tau$

Shannon-Lobb, $n=2$: $\frac{\Omega^3}{24\pi^2} \cdot \sigma^2$

Regularization is of Projection-Type.

Regularization Parameter is the bandwidth.

We project onto the space of band limited functions.

Ω large $\to$ good reconstruction, big noise amplification

Ω small $\to$ worse reconstruction, smaller noise amplification

} Trade-off

In the L^2-framework: (supp $\hat{f} \subset K_\Omega(0)$):

Let $f \in S(\mathbb{R}^n)$, consider $\|\cdot\|_2$. Reconstruction $V_\Omega * f$, Shannon-Lobb.

$f = f_\Omega + f_\Omega^*$, $f_\Omega(x) = (2\pi)^{-n/2} \int_{\|\xi\|\le\Omega} \hat{f}(\xi) e^{ix\xi} d\xi$

(Ω-band limited version of f),

f_Ω^* "the rest". Parseval: $\|f_\Omega^*\|_2^2 = \int_{\|\xi\|\ge\Omega} |\hat{f}(\xi)|^2 d\xi \to 0$, $\Omega\to\infty$

$g = Rf$, $\tilde{g} = Rf + n$, $n(\theta,s)$ measurement noise $\in \mathcal{Y}$.

$= R(f_\Omega + f_\Omega^*) + n$

$V_\Omega * \tilde{g} = V_\Omega * \left(R(f_\Omega + f_\Omega^*) + n \right)$

$\|V_\Omega * \tilde{g} - f\|_2^2 = \|V_\Omega * \left(R(f_\Omega + f_\Omega^*) + n \right) - (f_\Omega + f_\Omega^*)\|_2^2$ f_Ω is band limited

$\le \|V_\Omega * R f_\Omega^*\|_2^2 + \|f_\Omega^*\|_2^2 + \|V_\Omega * n\|_2^2$

Shannon-Lobb $= \int_{S^{n-1}} \int_{-\Omega}^{\Omega} |\sigma|^2 |(R f_\Omega^*)(\theta,\sigma)|^2 d\sigma\, d\theta$ + $\int_{|\xi|>\Omega} |\hat{f}(\xi)|^2 d\xi$ + $\int_{S^{n-1}} \int_{-\Omega}^{\Omega} |\hat{n}(\theta,\sigma)|^2 |\sigma|\, d\sigma\, d\theta$

Parseval: model error; noise amplified by $\int_{-\Omega}^{\Omega} |\sigma|\, d\sigma$ from $\|n\|_2^2$; see below

$= 0$?

Fourier slice: Since $\hat{f}_\Omega^*(\sigma\cdot\theta) = 0$ for $|\sigma| \le \Omega$, $(R f_\Omega^*)(\theta,\sigma) = 0$, $|\sigma| \le \Omega$

However: This is not implemented.

Discrete convolution not exact

$\Rightarrow$ Aliasing (sound example)

However: $\forall \Omega^2 \|f_\Omega^*\|^2 \to 0$, since $f \in \mathcal{Y}$.

$\to 0$, $\Omega \to \infty$ (noiseless case)

continuous (sampling theorem bounded)

$\Rightarrow V_\Omega * g$ is Regularization with regularization parameter $(\frac{1}{\Omega})$

$V*g$: $\int_{S^{n-1}} \int_{\mathbb{R}} (v * g)(\theta,s)^2\, ds\, d\theta$

$= \int_{S^{n-1}} \int_{\mathbb{R}} (\hat{v}\cdot\hat{g})(\theta,\sigma)^2\, d\sigma\, d\theta$

$= c \int_{S^{n-1}} \int_{-\Omega}^{\Omega} |\sigma|^2 \hat{g}(\theta,\sigma)^2\, d\sigma\, d\theta$

$\le c\cdot\Omega^2 \cdot \|g\|_2^2$

$R^*h = \int_{\mathbb{R}^2} \int_{S^{n-1}} h(\theta, x\cdot\theta)^2\, d\theta\, dx$

$= \int_{S^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} h(\theta, (s\cdot\theta + t\cdot\theta^\perp)\cdot\theta)\, ds\, dt\, d\theta$ (supp $h \subset K_\Omega(0)$)

$= 2 \int_{S^{n-1}} \int_{-\Omega}^{\Omega} h(\theta,s)^2\, ds\, d\theta$

$= 2\|h\|^2$

Emission Tomography

→ Safety: Radioactive nuclear decay

→ X-Ray source inside the body

→ Idea: Inject a radioactive substance ("tracer") into the body

→ Gamma Camera (Hospital Gamma Camera)

1950 Rudi Kaplan (Texas) with radioactive Iod. Which sort knows, that the Iod will be in a specific tissue in the blood. Measure the distribution in the blood by mapping it input in a specific organ. Cerebral vessels?

In medicine CAT (X-Ray in CAT) we see morphology, or function, and part of the heart will look a malfunction.

Q: How can we identify (quick) tumors?

Idea: Feed the patient with radioactive sugar (Iodine!). The sugar will be where it is needed to fuel growth. Tumor need lots of sugar to survive (obvious metabolism, functional tomography)

Can only use low dose radioactivity. Instead of a constant flow we have single particles. Poisson, Binomial Distribution

Goal: Find the distribution $f(x)$ of the tracer inside the body, measuring the radioactivity outside the body.

Idea: (1) prepare tracer to be of specific metabolism

Problem: That works, in theory, and it is a random variable. (Not what I)

→ Detektorgraphy / PET-Scans





Just measuring the photons would be easy



Collimation

early detection of photons after leaving the body

Problem: 99% of the radioactivity is wasted.

Mathematical model:

Assume the simplified Poisson model for single rates. Now I make the detection

→ $\sum_{k=0}^{\infty} f(x)\,dx = 0$ measured (rarely included ?)

Problem: Assuming perfect collimation ($\Theta(x,\omega)$ in CT). Take a particle with varying unknown probability $p(x,\omega)$ (usually in the CT case)

→ $\sum_{k=0}^{\infty} \frac{1}{k!} p(x,\omega)^k \cdot e^{-p(x,\omega)} f(x)\,dx$ is now measured

→ Attenuated Radon Transform

Definition: Let $f: \mathbb{R}^2 \to [0,\infty)$ be compactly supported and $\Theta(x,\omega) = \int_0^\infty p(x+t\omega)\,dt$ (ω unit vector)

$(R_p f)(\theta,s) = \int_{x\cdot\theta=s} e^{-\Theta(x,\omega)} f(x)\,dx$ (attenuated Radon transform)

$\theta = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$, $\theta^\perp = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$ $\Rightarrow \frac{d}{ds}$ rotation

Note: $(R_p f)(\theta,s) = \int_{x\cdot\theta=s} e^{-\int_0^\infty p(x+t\theta^\perp)\,dt} f(x)\,dx$

$(R_p f)(-\theta,-s) = \int_{x\cdot\theta=s} e^{-\int_0^\infty p(x-t\theta^\perp)\,dt} f(x)\,dx$

$\neq$

→ Scanning image equivalent to an X-Ray, with different math modelling. Since problem's dimensions $(R_p f)(\theta,s)$, θ known too!

NO INVERSE PROBLEM

(2) SPECT - Single Photon Emission CT

From now: Add a 2D $\frac{1}{r}$ plane through the body.

Rotate around body.

receive decay of isotope (commonly an $Al(OH)_3$)

Combined with CT: Use it to compute p (attenuated transform Θ)

$R(R_p f)(\theta,s)$ is measured

Problem: attenuated transform → Non analytic inversion.

Problem: Still first problem. Radioactively washed.

(3) Positron Emission CT

2D again. Different isotope, (O^{15})

Ring of Detectors around the body, without collimators.

Whistler, combined with CT (see 10.6)

Radioactive decay: Two photons in opposite direct that are emitted (actually a positron, which immediately decays).

Up to light speed, the two photons are measured at the same time.

→ Delay time when $\frac{|D_1 - D_2|}{c}$

→ No need for collimators (everything in the plane is measured)

Forward Case $L: \sum_k f(x)\,dx$ = Number of L with attenuation. Pair of opposing D_1, D_2 of Quality D_1

$\Rightarrow \sum_k p(x) e^{\int_0^\infty p} \sum_k f(x)\,dx = \sum_k f(x)\,dx$

$\Rightarrow \frac{e^{-\int_0^\infty p(t)\,dt}}{D_1 D_2} \sum_{x\in L} f(x)\,dx$

If p is known, $\Theta(\theta,s)$ can be computed

$\Rightarrow \sum_{x\in L} f(x)\,dx = \frac{1}{A(\theta,s)}$ Measurements

Radon Transform without attenuation

One Main Example from now:

→ PET, modelled by the Radon transform

→ Difference to CT: Statistical model

→ Now.

Poisson statistics

Very heuristic derivation.

$P_k(t_1,t_2)$ = probability that k particles decay (in a volume, on a line, whatever) in time $[t_1,t_2]$

Assume decay is independent of time,

$P_k(t_1,t_2) = P_k(0, t_2-t_1) = P_k(t_2-t_1)$.

This is not completely true; in the end, the radioactivity is weaker (but that can be accounted for).

Exp. Value Today: (# of decays in $[0,t]$) $= t \cdot c$.

dt very small: $P_0(dt) \approx (1-P_1-\dots)$.

$\Rightarrow P_1(dt) \approx dt \cdot c$, $P_0(dt) \approx 1 - dt \cdot c$.

$P_0(t+dt) = P_0(0,t+dt) = P_0(0,t) \cdot P_0(t,t+dt)$

$= P_0(t) \cdot (1 - dt \cdot c)$

$dt \to 0$, $\frac{d}{dt}$: $P_0'(t) = P_0(t) \cdot c \Rightarrow P_0(t) = e^{-ct}$, $P_0(0) = 1$.

$P_{n+1}(t+dt) \sim P_{n+1}(t) P_0(\ dt) + P_n(t) P_1(dt)$

$\approx P_{n+1}(t)\,(1 - c\,dt) + P_n(t) \cdot (c\,dt)$

$\Rightarrow P_k(t) = \frac{(ct)^k}{k!} e^{-ct}$.

Poisson distribution.

Fix $t = T_0$ (measurement time), $\lambda = c \cdot T_0$,

X = number of decayed particles (random variable),

$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$.

Poisson distribution. Values are integers.

$E(X) = \left(\sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!}\right) \cdot e^{-\lambda} = \left(\sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!}\right) e^{-\lambda} = \lambda e^{\lambda} e^{-\lambda} = \lambda$

$V(X) = \lambda$.

Apply for Tomography, (PET, Radon Transform, no attenuation)

We will use discrete models (as in ART (Kaczmarz)).



X = # of particles measured on l

= # of particles emitted on l_3 + # of particles on l_2

Radioactivity constant in each Pixel, $l_3 = T \cdot c_3$, ...

Expected value for X: $\int_{l_3} 1\,dl + \int_{l_2} 1\,dl = \lambda$.

X is Poisson-distributed.

$\Rightarrow P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$.

General case: N Pixels $P_0 \sim P_{N-1}$, M lines $l_0 \sim l_{M-1}$.

$a_{ij} = |l_i \cap P_j|$, $A = (a_{ij})$, $g = (g_i)$.

$\lambda_i = \sum_j A_{ij} f_j = (Af)_i$. γ_i random variable for line l_i.

$P(\gamma_i = g_i) = e^{-(Af)_i} \cdot \frac{(Af)_i^{g_i}}{g_i!}$

= probability of measuring g_i particles on line i if (f_j) is the tissue distribution.

(γ_i independent, $\gamma = (\gamma_i)$, $g = (g_i)$)

$\Rightarrow P(\gamma = g) = \prod_i \left(e^{-(Af)_i} \frac{(Af)_i^{g_i}}{(g_i)!}\right) =: L(f)$

$\Rightarrow$ probability (likelihood) of measuring g when f is the tissue distribution.

Assume that a vector g has been measured.

① Maximum Likelihood:

Find the $f = f_{ML}$, that maximizes $L(f)$,

for which the probability of measuring g is the highest.

② Bayes-estimate (Apotemi):

View f as a random variable, distribution

$p(f \mid \gamma = g)$.

$E(f)$ is a Bayes-estimator.

Bayes - Estimate : $Af = g$

f is n-dim. ($\vec{f}, F$) normally distributed

Drawn from Pool of images

e.g. $\vec{f} = 0, n$ $\vec{f}$ typical image: Mean X-ray of knee?

Covariance F: Values of neighboring pixels in an image tells us how to be similar

$\Rightarrow$ f not uncorrelated

$\Rightarrow$ models for the covariance of images exist.



n noise. in $(0, \sigma^2 I)$ - normally distributed

σ^2 - noise strength, n_i is independent

Assume f, n uncorrelated (noise distribution does not change with f is changed).

$\Rightarrow \begin{pmatrix} f \\ n \end{pmatrix} \sim \left(\begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}, \begin{pmatrix} F & 0 \\ 0 & \sigma^2 I \end{pmatrix} \right)$ - normally distributed.

Observation: $Af + n = g$, A observation operator (Radon transform ...)

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} f \\ n \end{pmatrix} \Rightarrow g \text{ is random variable.}$$

From the definition of normal distribution:

X normally distributed $\Rightarrow BX$ normally distributed for any B

From (1), (2): $E(BX) = B\,E(X)$

$\operatorname{cov}(BX) = B\,\operatorname{cov}(X)\,B^t$

$\Rightarrow \begin{pmatrix} f \\ g \end{pmatrix}$ normally distributed

$$E\begin{pmatrix} f \\ g \end{pmatrix} = E\left(\begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} f \\ n \end{pmatrix} \right)$$

$$= \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix} = \begin{pmatrix} \vec{f} \\ A\vec{f} \end{pmatrix}$$

$$\operatorname{cov}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & \sigma^2 I \end{pmatrix} \begin{pmatrix} I & A^t \\ 0 & I \end{pmatrix} \quad \left| (I_n, I_m) \right.$$

$$= \begin{pmatrix} F & 0 \\ AF & \sigma^2 I \end{pmatrix} \begin{pmatrix} I & A^t \\ 0 & I \end{pmatrix} = \begin{pmatrix} F & FA^t \\ AF & AFA^t + \sigma^2 I \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$$

$$E(f \mid g = \tilde{g}) = \vec{f} + k_{12} k_{22}^{-1} \left(\tilde{g} - A\vec{f} \right)$$

$$= \vec{f} + FA^t \left(AFA^t + \sigma^2 I \right)^{-1} \left(\tilde{g} - A\vec{f} \right)$$

Assuming f is uncorrelated: $F = I$

$$f_{Bayes} = \vec{f} + A^t \left(AA^t + \sigma^2 I \right)^{-1} \left(\tilde{g} - A\vec{f} \right)$$

$$A^t \left(AA^t + \sigma^2 I \right) = \left(A^t A + \sigma^2 I \right) A^t$$

$$\Rightarrow \left(A^t A + \sigma^2 I \right)^{-1} A^t = A^t \left(AA^t + \sigma^2 I \right)^{-1}$$

$$\Rightarrow \qquad = \left(A^t A + \sigma^2 I \right)^{-1} A^t \left(\tilde{g} - A\vec{f} \right)$$

Tikhonov for the 2-Norm for

$A(f + \vec{f}) = g$, $f_{Prior} - F \neq I$ $\Rightarrow$ different norm

(usual Tikhonov for $\vec{f} = 0$). Regression loss weighing

In this particular case, stochastic modeling gives a generalization of Tikhonov.

This is a common thing: Normally distributed RV yield a variant of Tikhonov.

This is somewhat disappointing.

To get something that's new, we need a different distribution for the noise.

$\Rightarrow$ Emission Tomography.

Maximum Likelihood for Emission Tomography

For normal distribution: $j_{ML} = j_{Bayes}$

$Aj = \tilde{y}$ — no noise, $\tilde{y}$ one realization of the random variable y.
y_i is Poisson-distributed, $\lambda_i = (Aj)_i$.

$$P_j(y = \tilde{y}) = \prod_i \left(e^{-(Aj)_i} \frac{(Aj)_i^{\tilde{y}_i}}{(\tilde{y}_i)!} \right) =: L(j) \quad \text{"likelihood"} \quad j_{ML} = \operatorname{argmax}_j L(j)$$

If viewing j as a variable: $L(j) := p(j)$

$-L$ convex $\Rightarrow$ All maxima are global. 1 extremal point.
$\log$ monotonous $\Rightarrow$ L has maximum $j \Leftrightarrow \log L$ has maximum in j

$$l(j) = \log L(j) \quad \text{"log-likelihood"}$$

$$= \underbrace{\sum_i \tilde{y}_i \log (Aj)_i - (Aj)_i}_{} - \underbrace{\sum_i \log(\tilde{y}_i!)}_{\text{independent of } j}$$

$$\frac{d}{dj_s} : \tilde{y}_i \frac{a_{is}}{(Aj)_i} - a_{is}$$

$$\Rightarrow \nabla l(j) = A^t \begin{pmatrix} \frac{\tilde{y}}{Aj} - 1 \end{pmatrix} \quad \text{vector of ones}$$

pointwise division

In emission tomography, $j \geq 0$. ($j \geq 0$)

j_{ML} maximizes $l(j)$. $\quad$ j unrestricted $\Rightarrow \nabla l(j_{ML}) = 0$.

Motivation Kuhn-Tucker:

$\mathbb{R}^2$, x_1, x_2, h

Minimum in x: $\nabla h(x) = 0$.
Maximum in $x_1, x_2 = 0$:

$$\frac{\partial h}{\partial x_2} = 0, \quad \frac{\partial h}{\partial x_1} \leq 0$$

Necessary conditions: $\quad \frac{\partial h}{\partial x_k} = 0$ or $x_k = 0$.

$\Rightarrow x \cdot (\nabla h)(x) = 0$ pointwise multiplication

For our model:

$$0 = j_{ML} (\nabla l)(j_{ML})$$

$$= j_{ML} A^t \left(\frac{\tilde{y}}{Aj_{ML}} - 1 \right)$$

$$= j_{ML} \cdot A^t \frac{\tilde{y}}{Aj_{ML}} - j_{ML} \cdot A^t 1$$

$$\Rightarrow j_{ML} = j_{ML} \frac{1}{A^t 1} A^t \frac{\tilde{y}}{Aj_{ML}}$$

pointwise

$\Rightarrow j_{ML}$ is a Fixpoint of the r.h.s.

Fixpoint alg.:

$$j^{(k+1)} = j^{(k)} \cdot \frac{1}{A^t 1} A^t \frac{\tilde{y}}{Aj^{(k)}}$$ Expectation Maximization

converges $\to j_{ML}$ (Shepp/Vardi 1985, Dempster/Laird/Rubin 1977)

Very technical proof using Jensen's inequality.

k pos. def., $\Sigma = k\xi^t = D$

$$(\xi D^{-1/2})\, k\, (D^{-1/2}\xi^t) = I$$

$\mu = 0$, $k = I$: $\int_{\mathbb{R}^n} (2\pi)^{-n/2} e^{-\frac{1}{2}\|y\|_2^2} = 1$

comp.: $\int_{\mathbb{R}^n} y_i \cdot p(x=y)\, dy = (2\pi)^{-n/2} \int_{\mathbb{R}^n} y_i\, e^{-\frac{\|y\|^2}{2}} dy$ (odd in y_i)

$= 0$

$cov(x_i, x_j) = \int_{\mathbb{R}^n} y_i y_j e^{-\frac{\|y\|^2}{2}} dy$ (odd in y_i) $= 0$

$Var(x_i) = E(x_i^2) = \int_{\mathbb{R}^n} y_i^2 e^{-\frac{\|y\|^2}{2}} dy \cdot (2\pi)^{-n/2}$

$= \int_{\mathbb{R}} y_i \cdot y_i e^{-\frac{y_i^2}{2}} dy_i \cdot \frac{1}{\sqrt{2\pi}}$

$= \int_{\mathbb{R}} 1 \cdot e^{-\frac{y_i^2}{2}} dy_i \cdot \frac{1}{\sqrt{2\pi}} = 1$

$\Rightarrow$ Exp. value 0, $cov(X) = I$.

$Y = B \cdot X + c$, B pos. def., $B = Q^t D Q$

$\Rightarrow E(Y) = B \cdot E(X) + c$

$E((Y - E(Y))(Y - E(Y))^t)$

$= E((BX - BE(X))(X - E(X))^t B)$

$= B\, cov(X)\, B^t$

X standard

$Y = B^{-1/2} X + \mu \Rightarrow E(Y) = \mu$, $cov(Y) = Q^t (D^{1/2})^2 Q = B$

$p(Y=y) = p(B^{-1/2} X + \mu = y)$

$= p(X = B^{1/2}(y - \mu))$

$E(Ax) = \int_{\mathbb{R}^n} Ax\, p(X=x)\, dx$

$= \int_{\mathbb{R}^n} Ax \cdot p(X=x)\, dx$

$= A \cdot \int_{\mathbb{R}^n} x \cdot p(X=x)\, dx$

$= A \cdot E(X)$

Statistics - Basics

Multidimensional Distributions

continuous distributions:
X random variable with distribution
$\int_{\mathbb{R}^n} p(x) = 1$

1) $E(X) = \int_{\mathbb{R}^n} x \cdot p(X=x)\, dx$

$E(X+\mu) = E(X) + \mu$
$E(AX) = A \cdot E(X)$

2) $cov(X) = E((X - E(X))(X - E(X))^T)$
Diagonal: Variance, variance of one entry
Odd: Diagonal: $\neq 0 \Rightarrow X_i$ uncorrelated.
X_i independent $\Rightarrow X_i$ uncorrelated.
$cov(X)$ pos. definite

$cov(AX+\mu) = E((AX - AE(X))(AX - AE(X))^T)$
$= A\, cov(X)\, A^T$

3) Normal distribution:

$p(X=x) = (2\pi)^{-n/2} e^{-\frac{\|x\|^2}{2}}, \quad x \in \mathbb{R}^n$

$\int_{\mathbb{R}^n} p(x) = \prod_i \int_{\mathbb{R}} x \cdot p(x)\, dx = 0 + E(x_i)$
odd function, integral
$\int_{\mathbb{R}^n} x_i\, p(x_i x_j)\, dx = 0$
odd in x_i, int.

$(2\pi)^{-n/2} \int_{\mathbb{R}^n} x_i \cdot e^{-\frac{\|x\|^2}{2}} dx = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{\|x\|^2}{2}} dx = 1$
$x_i e^{-\|x\|^2/2}$ partial integrable, integral $= 1$

$cov(X) = I$ $\quad X \sim (0, I) =$ normally distributed, mean 0, cov unit

Let Σ positive definite, $\Sigma = Q\, \Omega\, Q^T$, $\Sigma^{1/2} = Q\, \Omega^{1/2} Q^T$
$Y = \Sigma^{1/2} X + \mu \Rightarrow E(Y) = \mu$, $cov(Y) = \Sigma^{1/2} \cdot I\, (\Sigma^{1/2})^T = \Sigma$

$p(X+\mu) = P(X \in \Sigma^{-1/2}(y - \mu)) \overset{\text{def}}{=} \frac{1}{(2\pi)^{n/2} \sqrt{\det \Sigma}} \int_{\Sigma^{-1/2}(y-\mu)} e^{-\frac{1}{2} x^T x}\, dx$
$= p(\Sigma^{-1/2}(x+\mu-y))$
$\Sigma^{-1/2}(y-\mu) \in \mathbb{R}^n$

$\Rightarrow Y \sim (\mu, \Sigma)$ normally distributed.

4) Conditions (probability): $Y = (Y_1, Y_2)$

$p(Y_1 = y \mid Y_2 = g) = \frac{p(Y_1 = y, Y_2 = g)}{\int_{\mathbb{R}^n} p(Y_1 = y, Y_2 = g)\, dy}$
$p(Y_2 = g)$

5) Let $Y \sim (\mu, k)$ normally distributed,
$Y = (F, G)$, $\mu = \begin{pmatrix} f \\ g \end{pmatrix}$, $k = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$, $F \in \mathbb{R}^m$, $G \in \mathbb{R}^n$.

Then $p(F = y \mid G = g)$ is normally distributed (w.r.f)

with Mean $f + K_{12} K_{22}^{-1} (g - g)$
Cov $K_{11} - K_{12} K_{22}^{-1} K_{21}$

Proof: Write $g - g = 0$.

$p(Y_1 = y \mid Y_2 = g) = \frac{e^{-\frac{1}{2}(y-f)^T K^T (y-f)} \cdot \frac{1}{\sqrt{\det K}}}{\int_{\mathbb{R}^m} e^{-\frac{1}{2}(y-f)^T K^T (y-f)} dy}$ $\quad K^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

$\frac{1}{2}(y-f)^T K^{-1} \begin{pmatrix} y-f \\ g-g \end{pmatrix} = \frac{1}{2}\left((y-f)^T A_{11} (y-f) + 2 (y-f)^T A_{12} 0 + 0^T A_{21} 0 \right)$

$\left(\begin{pmatrix} I & A_{11}^{-1} A_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} y-f \\ g-g \end{pmatrix} \right)^T A_{11} \left(\begin{pmatrix} I & A_{11}^{-1} A_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} y-f \\ g-g \end{pmatrix} \right)$
$= (y-f)^T A_{11} (y-f) + 2 (y-f)^T A_{11} A_{11}^{-1} A_{12} 0 + 0^T A_{21} A_{11}^{-1} A_{11} A_{11}^{-1} A_{12} 0$
independent of g / y

$\Rightarrow p(Y_1 = y \mid Y_2 = g) = \frac{e^{-\frac{1}{2}(y-\mu)^T A_{11} (y-\mu)}}{\int_{\mathbb{R}^m} e^{-\frac{1}{2}(y-\mu)^T A_{11} (y-\mu)} dy}$
$\mu = A_{11}^{-1} A_{12} g$ $\quad (2\pi)^{m/2} \cdot (\det A_{11})^{-1/2}$

$\Rightarrow (Y_1 \mid Y_2 = g) \sim (-A_{11}^{-1} A_{12} g, A_{11}^{-1})$ normally distributed

$\begin{pmatrix} K_{11} & K_{12} \\ K_{21}^T & K_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$

$K_{11} A_{11} + K_{12} A_{12}^T = I \Rightarrow A_{11}^T K_{11}^T + K_{12} A_{12}^T = I$
$K_{21}^T A_{11} + K_{22} A_{12}^T = 0 \Rightarrow A_{11}^T K_{21} + A_{12} K_{22}^T = 0$
$\Rightarrow A_{12} K_{22}^{-T} = -A_{11}^T K_{21}^T \Rightarrow A_{12} = -A_{11}^T K_{21}^T K_{22}^{-T}$

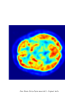
$A_{11}^T K_{11}^T + K_{12} A_{12}^T = I$
$A_{11}^T K_{11}^T - K_{12} K_{22}^{-T} K_{21}^T A_{11}^T = I$
$(K_{11} - K_{12} K_{22}^{-T} K_{21}^T) A_{11} = I$

$A_{11} = I - K_{12} K_{22}^{-T} K_{21}^T A_{11}$
$= I - K_{12} K_{22}^{-T} \left(K_{21} A_{11}^T + K_{22} A_{12}^T \right)$

$\Rightarrow A_{11}^{-1} = K_{11} - K_{12} K_{22}^{-T} K_{21}^T$
$A_{11}^{-1} A_{12} = -K_{12} K_{22}^{-1}$



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"A sagittal plane magnetic resonance imaging (MRI) of a normal human brain" by Jmarchn is licensed under CC BY-SA 3.0. Unported license. Attribution: Jmarchn



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Simple Explanation:

Particle model for fly decay on
Cosmic rays rotating protons with a
⇒ gyroscope
main idea: precession

A gyroscope are kept upright by gravitation
a perpendicular push will cause a precession
the frequency will depend on the gravitational strength

MRI: Apply an (unbelievably strong) magnetic field ($3T$)



⇒



magnetic field (along the body main axis), z-axis.



Problem: proton field can be oriented upwards and downwards

But, there is a slight net addition in the direction of the field, which becomes strong with the strength of the field.

Assume from time on the net protons, all pointing upwards.

Induce precession with a perpendicular (electromagnetic) push

The push is a wave, it will have a wavelength only if the frequency ≈ the wave matches the precession frequency.

⇒  ⇐ small push - frequency as most a side, long frequency

⇒ protons start precessing, we say "tip over".
e.g. B_0: strength of magnetic field (Larmor-frequency).

Signal will decay on the depending on the material ⇒ allows classification of materials.

We restrict our view to an x-y-plane.

Coil moving in circular paths
⇒ moving magnets
⇒ generate alternating electric field
⇒ measured in coils around the patient

Measurable: $\frac{\partial}{\partial t}\int_\Omega \rho(x)\,e^{i\omega t}\cdot\frac{1}{|x-x_0|}\,dx$ (ρ = independent of x)

supp $\frac{1}{g}$ = partial, so location of the coil away from supp $\frac{1}{g}$ & from density.

Tomographic idea: Move the coils around the patient.

$g(\vec{x}) = \int_\Omega f(x)\,k(x,\vec{x})\,dx$

Since $\vec{x}$ away from the patient;
best ⇒ Regarding x of $\frac{1}{g}$ for $\vec{x}$ is exponentially ill-posed.
Thus will not work.

New idea (Stefan Kunis, Berkeley 2003):
Make ω dependent on $\omega(x,y,z)$.

Constant magnetic field:
field strength/direction is one direction, on top of B_0, e.g. z-direction.
Since $\omega = \frac{1}{g}$, the Larmor frequency now depends on ω:

$B = B(x,y,z,t)$, for one constant we x,y,z t.
$\omega = \omega(x,y,z,t)$.

Assume, that the perpendicular push has frequency ω_0. Only protons were $\omega(x,y,z) = \omega_0$ are affected by the push, for ≈ 2.

⇒ $g(t) = \sum_{k\in\mathbb{Z}} f(x,y)\,e^{i\omega t}\,dx\,dy$ ← Fourier-coefficient

Still, this goes with $e^{i\omega t}$ since B_0 is constant in time.

Assume a proton is circulating with $e^{i\omega t}$ (after the push).
If we rotate the magnetic field (for a short time) the it will circulate faster (for a short time), the left hand to ω_0 (small Fourier-difference $e^{i(\omega t - (\omega_0 - \omega))}$

Now assume that this field is a gradient field in y-direction $e^{i\omega_0 t + y\cdot(\omega_0 - \omega)}$

While we measure, apply a gradient in x-direction ⇒ $e^{i[\omega_0 t + y\cdot(\omega_0 - \omega) + x\cdot(\omega_0 - \omega)]}$

⇒ $g(t) = \frac{1}{g}\int f(x,y)\,\rho(x,y)\,e^{i\omega_0 t}\,e^{-i\omega_0 x}\,e^{-i\omega_0 y}\,dx\,dy$

$\Rightarrow \mathcal{F}(x^{-1}\cdot\int_{R^2} f(x,y,z)\,\rho(x,y)\,e^{i(-\omega_0 x + \omega_0 y)}\,dx\,dy$ ← Fourier-coefficient

$\int f(\omega_0 t, -\omega_0, -\omega_0)$ ⇒ We measure the FT of f

⇒ Inverse Transform ⇒ f.
Result: $\mathbb{R}^3 \leftarrow \mathbb{R}^3 \Rightarrow \mathbb{R}^4$.

But, Typically we assumed: Efficient measurable geometry?

Radial sampling: $\int_\Omega (\theta_1, \theta_2)$

$(B_0^{-1})(\theta_1, \theta_2)$

⇒ Typically, parallel geometry

⇒ Different projection

Simon Magnetic spin in 3 dimension

Overview: Imaging with sound & ultrasound

Inner ear (cochlea)

Brain tissue

Reflections

→ sound waves
→ satisfy the wave equation

$c(x)^2 \Delta_x u(x,t) = u_{tt}(x,t)$

Measure u (sound pressure) away from the object

Reconstruct c.

Applications: → 1D-Ultrasound
→ Geophysical Problems (Wave into the ground, reflection measured)
→ Inverse Radar problem
→ Ultrasound Tomography

Set transducer, emitter/receiver

reflected wave going up when object is deep → less time to travel down+up

Tissue 1 | Tissue 2 | Tissue 3

Depth: Depth 1, Depth 2

Pressure, Transducer

ψ time

→ Holes start
→ Images are ... not optimal

Model: Waves are only moving up + down.



Idea: Do Breast Ultrasound Tomography to get rid of Mammography.

Project with Technion / SICK (Lake City)

Goal: Make ultrasound specific.

Restrict ourselves to 2D.



- Send in a plane wave of frequency ω: $e^{i\omega t} u^i(x)$ ($u^i(x)$: incident wave)
- produces a scattered wave $u^s(x,t) e^{i\omega t} u^s(x)$.
- Model: wave $u = u^i + u^s$ satisfies $c(x)^2 \Delta u(x,t) = u_{tt}(x,t)$, u^i small in some sense.

$\Rightarrow \Delta u + \frac{\omega^2}{\left(\frac{c(x)}{\omega}\right)^2} u = 0$ + B.C. radiativity (Radiation condition) $\Rightarrow \Delta u + k(x)^2 u = 0$

For $c(x)$ constant, there is no scattered wave

$\Rightarrow u^i$ must satisfy $\Delta u^i + k^2 u^i = 0$.

Plane wave: $u^i(x) = e^{ik x \cdot \theta}$, parallel incoming wave in direction θ



Tomographic idea:



Radiate waves through the unknown medium and measure the scattered waves.

$\Rightarrow$ Inverse scattering problem.

Unknown quantity $c(x)$. $q^2(x) \cdot \frac{\omega^2}{c(x)^2} = k^2 (= q(x))$

$k = k(x)$ in the exterior of the body.

$q(x)$ contains inside the body.

Measurement operator:

For incident wave with direction θ, measure the scattered wave on a circle around the object (S^2).

$\Rightarrow M : L^2(K_q(0)) \to L^2(S^2 \times S^2)$,

$(Mq)(\theta, \psi) = u_\theta^s(\psi)$, u_θ^s scattered wave for incoming wave $e^{ix\cdot\theta}$.

θ, ψ on the unit circle $\Rightarrow$ 2 degrees of freedom

q 2D-quantity

Problem: $\Delta u + k^2 (1+q) u = 0$

This is linear in q, and linear in u

But both are unknown

u depends on q in a non-linear way (2 u is not the solution for 2 q)

$\Rightarrow M$ is non-linear.

Idea: Linearize.

If q is small, then u^s is small.

$\Delta u + k^2 (1+q(x)) u = 0 \quad u = u^i + u^s, \ \Delta u^i + k^2 u^i = 0$

$\Rightarrow \Delta u^s + k^2 u^s = -k^2 q(x) u$

$\Rightarrow u^s(x) = -k^2 \int_\Omega q(y) u(y) G(x,y) dy$ ($u = u^i + u^s$)

$G(x,y)$ Green's function $\Delta u^s + k^2 u^s \delta(x-y)$

Model problem

$\sim -k^2 \int_\Omega q(y) u^i(y) G(x,y) dy$. Linear in q!

Born approximation

Nonlinear Problems: Ekström

Inverse Scattering is a nonlinear problem.
In most situations, the Born approximation is not valid.
⇒ need to solve a nonlinear problem.

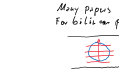
Idea:

- For each incoming wave, we have a measurement operator R_q and measured data g_q.
- Find q: $R_q(q) = g_q$, $q = 0, \dots, n-1$.
- Looks much like Kaczmarz: $q^{(0)}$, $q^{(1)} = q^{(0)} + \omega R_q^{\dagger}(g_q - R_q q^{(0)})$ (Landweber, simplified Gauss-Newton (Kalman))
- Problem: R_q is not linear, what is $R_q^{\dagger}$?
- Newton: $f(x) = 0 \Rightarrow 0 = f(x^{(0)} + dx) \approx f(x^{(0)}) + f'(x^{(0)}) \cdot dx$
- Same idea: $R_q(q^{(0)} + dq) = g_q$

$$\Rightarrow g_q = R_q(q^{(0)}) = R_q(q^{(0)} + dq) = R_q(q^{(0)}) + R_q'(q^{(0)}) \cdot dq$$

$$\Rightarrow dq = R_q'(q)^{\dagger} \underbrace{\left(R_q'(q^{(0)})\, R_q'(q^{(0)})^*\right)^{-1}}_{\text{Replace by } \omega I \text{ (as in ART)}} \left(g_q - R_q q^{(0)}\right)$$

Needed: The derivative of R_q and its adjoint.
Many papers in IP dealing with this problem.
For bilinear problems, the linearization is simple.



$\Delta u + k^2(1+q)u = 0$
u satisfies $u + d\omega \frac{\partial u}{\partial \nu} = f_0$ on Γ_1
u measured on Γ_1, using on Γ_2.

(gives rise to initial value problem, approximately solvable very efficiently)

$R(q) = u|_{\Gamma_2}$

Find q : $R(q) = g$

$\approx R(q^{(0)} + dq) \approx R(q^{(0)}) + R'(q^{(0)}) \cdot dq$

Compute $R'(q)$. q is fixed ∇

Let u solution to the forward problem, i.e.

$\Delta u + k^2(1+q)u = 0$, $u|_{\Gamma_0} = f_0$, $\frac{\partial u}{\partial \nu}|_{\Gamma_0} = f_1$, $R(q) = u|_{\Gamma_2}$.

$q \to q + dq$, $u \to u + du$: $\Delta(u+du) + k^2(1+q+dq)(u+du) = 0$

$\Rightarrow \Delta\, du + k^2(1+q)\, du = -k^2 dq \cdot u - \underbrace{k^2 dq\, du}_{\text{nonlinear part} \to 0 \text{ quadratically}}$

$\Delta \widetilde{du} + k^2(1+q)\widetilde{du} = -k^2 \cdot dq \cdot u$

$\widetilde{du}|_{\Gamma_0} = \frac{\partial \widetilde{du}}{\partial \nu}|_{\Gamma_0} = 0$

$\widetilde{du} \sim du$, $\widetilde{du}$ depends on dq linearly

$R(q+dq) = (u+du)|_{\Gamma_2} \sim \underbrace{u|_{\Gamma_2}}_{R(q)} + \underbrace{\widetilde{du}|_{\Gamma_2}}_{=R'(q)\cdot dq \text{ (linear)}}$

$\Rightarrow \left(R'(q)\right) \cdot dq = \widetilde{du}|_{\Gamma_2}$

$\Delta \widetilde{du} + k^2(1+q)\widetilde{du} = -k^2 u\, dq$, u solves $\Delta u + k^2(1+q)u = 0$,

$\widetilde{du}|_{\Gamma_0} = \frac{\partial \widetilde{du}}{\partial \nu}|_{\Gamma_0} = 0$

What is $R'(\Gamma_1)^*$? Ω = "Strip" between Γ_0, Γ_2.

$$\int_{\Omega} (\Delta u \cdot v - u \cdot \Delta v)\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot v - u \cdot \frac{\partial v}{\partial \nu}\, d\sigma \quad \text{(2 Green's identities Formula)}$$

$$\int_{\Omega} (\Delta u + k^2(1+q)u)v - u \cdot (\Delta v + k^2(1+q)v)\, dx$$

$R'(q)^*$: Measurement space $\to$ image space (q) (q)

$\left(R'(q)\cdot dq, g\right) = \left(dq, R'(q)^* g\right)$

$\widetilde{du}|_{\Gamma_2} \Rightarrow \int_{\partial\Omega}$ $\Rightarrow \int_{\Omega}$

Green, $u = \widetilde{du}$:

$$\int_{\Omega} -k^2 u \cdot dq \cdot v - \underbrace{du \cdot (\Delta v + k^2(1+q)v)}_{=0}\, dx = \int_{\Gamma_2} \frac{\partial \widetilde{du}}{\partial \nu} \cdot v - \widetilde{du} \cdot \frac{\partial v}{\partial \nu}\, d\sigma$$

$\Downarrow$ $R'(q)\cdot dq$

Choose v: $\frac{\partial v}{\partial \nu}|_{\Gamma_2} = g$, $v|_{\Gamma_2} = 0$.

$\Delta v + k^2(1+q)v = 0$ — v depends on g linearly by ∇

$$\Rightarrow \int_{\Omega} dq \cdot \underbrace{(-k^2 u \cdot v)}_{=: R'(q)\cdot g}\, dx = \underbrace{\int_{\Gamma_2} R_q'(q) dq\, g\, d\sigma}_{(R'(q)\cdot dq, g)}$$

Projection theorem for the Born approximation in the time-harmonic inverse scattering problem (diffraction tomography, Bang Ho Young & Mayko)

Inverse source:

Input artificial data that was produced using the assumptions for the solution.

Input artificial data from the same programs that are used for inversion.

$\Delta u + k^2 u = 0, \quad \frac{\partial u}{\partial n} - ik u = 0$ (radiation condition)

Green's function: $\Delta u + k^2 u = \delta + BC$, depend on (x,y)

$u = v(r)\cdot w(\theta)$ Separation of variables

$\Delta u = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} w_{\theta\theta}$

$\Rightarrow r^2 v'' w + r v' w + \frac{1}{r^2} v\, w'' + k^2 r^2 v w = 0 \quad \frac{1}{vw}$

$\Rightarrow r^2 \frac{v''}{v} + r \frac{v'}{v} + k^2 r^2 = -\frac{w''}{w} \qquad \text{Assume: } w = e^{i n \theta}$

$\Rightarrow r^2 v'' + r v' + (k^2 r^2 - n^2) v = 0$ (Bessel's equation)

Comes $\psi, \eta = J_n$ satisfies Bessel's equation, 2nd order. second solution Y_n with a singularity at 0 (Taylor series)

$H_n = J_n + i Y_n$

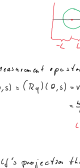
$\Rightarrow H_n(kr) e^{in\theta}$ is a solution to the unperturbed Helmholtz equation, satisfies BC

$w(0) \cdot H_n(kr)$: polar wave with a singularity at 0, Green's function.

$G(x,y) = -\frac{i}{4} H_0(k|x-y|)$.

Born: $u^B(x) = -\frac{k^2}{4} \int_D H_0(k|x-y|) q(y) v(y)\, dy$

Assume that u^B is measured on an infinite line before or after the obj and in direction $\theta^\perp$:



Measurement equation:

$g(\theta,s) = (R_q)(\theta,s) = u^B(s\cdot\theta + t\cdot\theta^\perp) \qquad (2.2)$

$= -\frac{ik^2}{8} \int_R H_0(k\sqrt{(s\cdot\theta - t\cdot\theta^\perp - y)^2}) q(y)\, e^{ikx\cdot\theta}\, dy$

$\sqrt{\ }$ could be negative or positive.

(reciprocity & translation invariance)

Weyl's projection theorem (2.3.3):

Let $\alpha(\theta) = \begin{cases} \sqrt{k^2 - \theta_1^2}, & |\theta| \le k \\ i\sqrt{\theta_1^2 - k^2}, & |\theta| \ge k \end{cases}$ — for $I > 0$

— for $I < 0$

Then

$\hat{(R_q)}(\theta,\sigma) = i\frac{\pi}{\alpha} e^{i\alpha s} \int_R e^{-i(\alpha(\theta)\cdot\theta + \sigma\cdot\theta^\perp)} \hat{q}\,(\alpha(\theta)\cdot\theta + \sigma\cdot\theta^\perp)$

Looks very much like Fourier slices

In fact: for waves, the sound wave behaves like an X-Ray. Fourier slices is a limiting case of this formula. Let $I > 0$ for now.

Morse/Feshbach, p. 822:

$H_n(z\sqrt{z^2-1}) \sim \frac{2}{\pi} \int_0^\infty e^{iz\alpha(\sigma) - in\sigma} \frac{d\sigma}{\alpha(\sigma)}\, d\sigma$

$\Rightarrow \hat{(R_q)}(\theta,s) = \frac{ik^2}{4} \int_R H_0(k\|(s\cdot\theta + t\cdot\theta^\perp - y)\|)\, q(y)\, e^{ikx\cdot\theta}\, dy$

$= \frac{ik^2}{4\pi} \int_R \int_0^{2\pi} \frac{1}{\alpha(\sigma)} e^{ik\alpha(\sigma)} \hat{q}\,(k(\alpha(\sigma)\cdot\theta + \sigma\cdot\theta^\perp))\, e^{ikx\cdot\theta}\, dy$

$\|(s\cdot\theta + t\cdot\theta^\perp - y)\|^2 = (s\cdot y\cdot\theta)^2 + (t\cdot y\cdot\theta^\perp)^2$

$= \frac{ik^2}{4\pi} \int_R \int_0^{2\pi} e^{-ik(\alpha\cdot\theta + \sigma\cdot\theta^\perp)\cdot x} \hat{q}\,(k(\alpha\cdot\theta + \sigma\cdot\theta^\perp)) \frac{1}{\alpha(\sigma)}\, q(y)\, e^{ikx\cdot\theta}\, dx\, dy$

$|y\cdot\theta| \le t < L \Rightarrow |t - y\cdot\theta| = L - y\cdot\theta$ (Morse: if $L < 0$, $-L - y\cdot\theta$)

Change order of integration

$= \frac{ik^2}{4\pi} \int_0^{2\pi} \frac{1}{\alpha(\sigma)} e^{ik\alpha(\sigma)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ik(\alpha\cdot\theta + \sigma\cdot\theta^\perp)\cdot x} \hat{q}\,(k(\alpha\cdot\theta + \sigma\cdot\theta^\perp))\, dy\, dx$

$\hat{q}\,((\alpha\cdot\theta)\theta + \sigma\cdot\theta^\perp)\, 2\pi$

Inverse FT of $\frac{ik^2}{4\pi} e^{ik\alpha(\sigma)} \hat{q}\,((\alpha(\sigma)\cdot\theta)\theta + \sigma\cdot\theta^\perp) \cdot 2\pi$

evaluated at s

$\Rightarrow$ Take FT in s: $\hat{(R_q)}(\theta,\sigma) = i\frac{k^2}{2} \frac{\pi}{\alpha(\sigma)} e^{ik\alpha(\sigma)} \hat{q}\,((\alpha(\sigma)\cdot\theta)\theta + \sigma\cdot\theta^\perp)$

$\alpha(\sigma) = \begin{cases} \sqrt{1 - |\sigma|^2}, & |\sigma| \le 1 \\ i\sqrt{|\sigma|^2 - 1}, & |\sigma| \ge 1 \end{cases}$

Discussion:

a) $\alpha(\sigma)$ is imaginary for $|\sigma| \ge k$

b) $e^{ik\alpha(\sigma)}$ decays exponentially

c) almost no scattered wave with frequency $\ge k$.

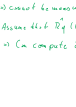
d) cannot be measured if $|\theta| \le k$, $\alpha(\sigma)$ is real.

Assume that $\hat{(R_q)}(\theta,\sigma)$ measured for θ fixed, $|\sigma| \le k$.

a) Can compute $\hat{q}\,((\sqrt{k^2 - \sigma^2} - k)\theta + \sigma\cdot\theta^\perp)$ $(I > 0)$

$\sqrt{(\sqrt{k^2 - \sigma^2} - k)^2 + \sigma\cdot\theta^\perp}$

$\Rightarrow$ on a circle of radius k around $-k\cdot\theta$. $\sigma = 0 \cdot \theta + 0 \cdot \theta^\perp$, $\sigma = k\theta + \sqrt{k^2 - k^2}\,\theta^\perp$



Can compute $\hat{q}\,((-\sqrt{k^2 - \sigma^2} - k)\theta + \sigma\cdot\theta^\perp)$ $I < 0$

$\Rightarrow$ on same circle. $\sigma = 0 \cdot \theta - 2k$, $\sigma = k\theta + \sqrt{k^2 - k^2}\,\theta^\perp$

$2k \ge |\sigma| \ge \sqrt{k^2}\, k$

Now rotate $\theta \Rightarrow$ $\hat{q}$ can be computed for $k \le |\sigma| \le 2k$. (diffraction limit)

Resolution limited by $2k$.

Choose k large (that the waves are diffracted).

Bad: if only transmitted data measured, the limit is $\sqrt{2}\, k$.

Even worse: if only reflected data is measured, only the frequency range $\sqrt{2}\, k - 2k$ can be computed

a) only edges are visible in the images, not quantitative.

Situation in geophysics & common referenced.