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The Wonderful World of Bregman Distances

Skiseminar, Zafernahütte

wissen.leben
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› Outline

› Fundamentals

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- Many problems in applied math can be modelled as inverse problems like

Inverse Problem with Exact Data

$$K\tilde{u} = g \tag{1}$$

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with a (linear) operator K, and given *exact data* g and the unknown *exact solution* $\tilde{u}$, or - since the exact data is usually not available - like

Inverse Problem with Noisy Data

$$Ku = f \tag{2}$$

with given *noisy data* f and unknown u

- Since the operator K is usually not (continuously) invertible and since g is usually not available, it is common to look for approximate solutions $\hat{u}$ via the variational framework

Variational Minimization Scheme

$$\hat{u} \in \arg\min_{u \in \mathrm{dom}(J)} \left\{ E(u) \right\} = \arg\min_{u \in \mathrm{dom}(J)} \left\{ H_f(Ku) + \alpha J(u) \right\} \tag{3}$$

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- The fidelity term measures the deviation between $K\hat{u}$ and f, usually considering the distribution of the noise in the data
- The regularization term incorporates a-priori knowledge on the desired solution $\hat{u}$

› Fidelities and Regularization Energies: Examples

- Typical fidelities are L^p-norms, i.e. $H_f(Ku) = \|Ku - f\|_{L^p(\Sigma)}$, usually to the power of p (i.e. $H_f(Ku) = \frac{1}{p}\|Ku - f\|^p_{L^p(\Sigma)}$), for $p \geq 1$ (in particular $p = 2$)

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  - And many many more...
- For us of particular interest:

ROF (Rudin-Osher-Fatemi)

$$\hat{u} \in \arg\min_{u \in \text{BV}(\Omega)} \left\{ \frac{1}{2}\|Ku - f\|^2_{L^2(\Sigma)} + \alpha \text{TV}(u) \right\}$$

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Compressed Sensing

$$\hat{u} \in \arg\min_{u \in \ell^1} \left\{ \frac{1}{2}\|Ku - f\|^2_{\ell^2} + \alpha\|u\|_{\ell^1} \right\}$$

- Another way to obtain approximate solutions $\hat{u}$ is via constrained minimization

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- The goal of this talk is to show that under appropriate conditions on the data f and on the operator K, and with the right tool - namely the Bregman distance - we are able to guarantee $\hat{u} = \tilde{u}$

› Bregman Distance

- The Bregman distance of a functional J is defined as

Bregman Distance

$$D_J^p(u, v) := J(u) - J(v) - \langle p, u - v \rangle \, , p \in \partial J(v) \, , \tag{5}$$

with $\partial J(v)$ denoting the subdifferential at position v, i.e.

$$\partial J(v) = \{ p \in \mathcal{X}^* \mid J(u) - J(v) - \langle p, u - v \rangle_{\mathcal{X}} \geq 0 \, , \forall u \in \mathcal{X} \} \tag{6}$$

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Subdifferential $\partial|u|$

$$\partial|v| = \text{sign}(v) = \begin{cases} \{1\} & \text{for } v > 0 \\ [-1, 1] & \text{for } v = 0 \\ \{-1\} & \text{for } v < 0 \end{cases}$$

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  - $v < 0$: $|u| \geq pu - (1 + p)v$ is fulfilled for each $u \in \mathbb{R}$ iff $p = -1$

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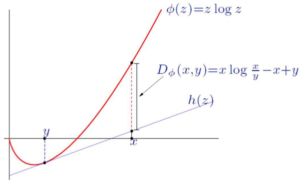

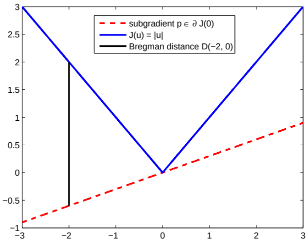

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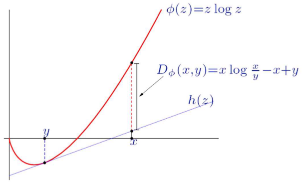

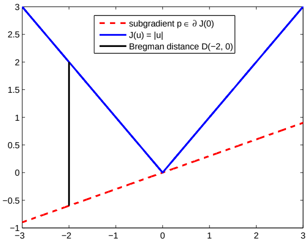

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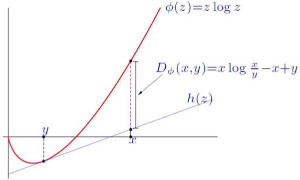

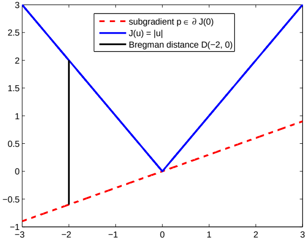

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- For convex J the Bregman distance is always non-negative; for strictly convex J we even have $D_J(u, v) = 0$ iff $u = v$
- The Bregman distance is no metric; it is usually not symmetric and does not satisfy a triangular inequality

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- Replacing the regularization term $J(u)$ with the Bregman distance $D_J^{p_k}(u, u_k)$, for $p_k \in \partial J(u_k)$, implies the following iterative procedure

Bregman Iteration

$$u_k = \arg\min_{u \in \mathrm{dom}(J)} \left\{ H_f(Ku) + \alpha D_J^{p_{k-1}}(u, u_{k-1}) \right\} \tag{7}$$

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- Considering the optimality condition $K^* H_f'(Ku_k) + \alpha(p_k - p_{k-1}) = 0$ as a backward-Euler-discretization with stepsize α, Bregman iteration can be seen as the discrete counterpart of

Inverse Scale Space Flow

$$\partial_t p = -K^* H_f'(Ku) \tag{8}$$

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- For the sake of simplicity we want to focus on $H_f(Ku) = \frac{1}{2}\|Ku - f\|^2_{L^2(\Sigma)}$; the variational scheme (3) and the inverse scale space formulation (8) therefore modify to

Bregman Tools for the Remainder of this Talk

$$\hat{u} \in \arg\min_{u \in \mathrm{dom}(J)} \left\{ \frac{1}{2}\|Ku - f\|^2_{L^2(\Sigma)} + \alpha J(u) \right\} \tag{9}$$

$$\partial_t p = K^* \left(f - Ku \right) \tag{10}$$

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Groundstates and Eigenfunctions

Let $J : \mathrm{dom}(J) \subseteq L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ be a convex functional and $K : L^2(\Omega) \to L^2(\Sigma)$ a linear operator. Then, an Eigenfunction $\hat{u}$ with Eigenvalue λ satisfies

$$\lambda K^* K \hat{u} \in \partial J(\hat{u}) \,. \tag{11}$$

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- Eigenfunctions of $J(u) = \mathrm{TV}(u)$ are more complicated to discover; let us therefore consider the subdifferential $\partial\mathrm{TV}$ first

TV-Subdifferential

$$\partial\mathrm{TV}(u) = \{\mathrm{div} q \mid q \in C_0(\Omega), \|q\|_\infty = 1, \langle u, \mathrm{div} q\rangle = \mathrm{TV}(u)\} \tag{13}$$

- In order to show that a particular function $\hat{u}$ is an Eigenfunction of TV with Eigenvalue λ, we need to show that there exists a q with $\operatorname{div} q = \lambda K^* K \hat{u}$ that satisfies all the subgradient properties of (13)

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- Example:

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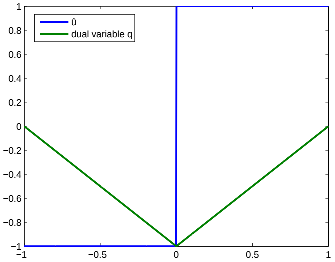

Eigenfunction-Example for $J(u) = \text{TV}(u)$

The function $q(x) = (|x| - L)/L$ satisfies

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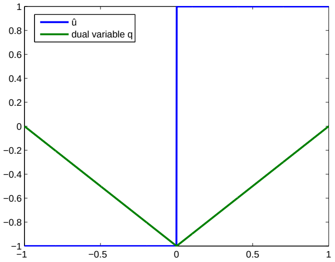

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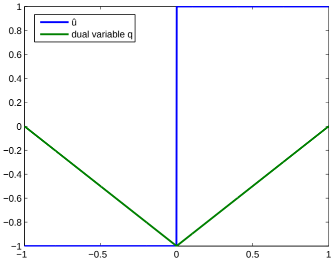

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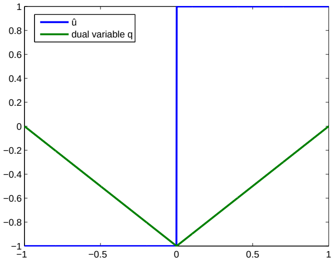

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- $\|q\|_\infty = 1$
- $\langle \hat{u}, q' \rangle = \text{TV}(\hat{u}) = 2$

- As a last example we want to consider $J(u) = \|u\|_{\ell^1}$; for an element $q \in \partial\|u\|_{\ell^1}$ we have $q_j = \text{sign}(u_j), \ \forall j \in \{1, \dots, n\}$

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Clean Data

Let $J : \mathrm{dom}(J) \subseteq L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ be a convex and one-homogeneous functional and let $K : L^2(\Omega) \to L^2(\Sigma)$ be a linear operator. Furthermore, let $\hat{u}$ be an Eigenfunction of J with corresponding Eigenvalue λ. Then, if the data f is given via $f = \gamma K \hat{u}$ for a positive constant γ, the solution of (9) is $u = c\hat{u}$ with

$$c = \gamma - \alpha\lambda\,, \tag{14}$$

if $\gamma > \alpha\lambda$ is satisfied.

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Proof

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\begin{aligned}
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&= \arg\min_u \left\{ \frac{1}{2}\|Ku - cK\hat{u}\|^2 + \alpha J(u) - \alpha J(c\hat{u}) \right. \\
&\qquad \left. - \frac{\gamma - c}{\lambda}\langle \lambda K^* K\hat{u}, u - c\hat{u}\rangle \right\}
\end{aligned}
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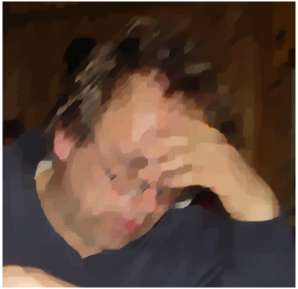

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u = \arg\min_u \left\{ \frac{1}{2} \|Ku - cK\hat{u}\|^2 + \alpha D_J^q(u, c\hat{u}) \right\} \Rightarrow u = c\hat{u}
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Noisy Data

Let the same assumptions hold as in the previous theorem. Furthermore, the data f is assumed to be corrupted by noise n, i.e. $f = \gamma K\hat{u} + n$ for a positive constant γ, such that there exist positive constants μ and η with

$$\mu K^* K\hat{u} + \eta K^* n \in \partial J(\hat{u}) \,. \tag{15}$$

Then, the solution of (9) is given via $u = c\hat{u}$ for

$$c = \gamma - \alpha\lambda + \frac{\lambda - \mu}{\eta} \,, \tag{16}$$

if γ satisfies the SNR-condition $\gamma > \mu/\eta$ and if $1/\eta \leq \alpha < \gamma/\lambda + 1/\eta$ holds.

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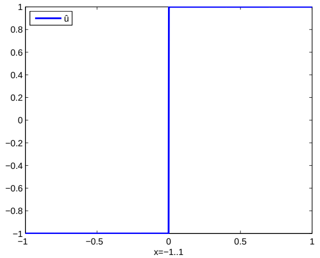

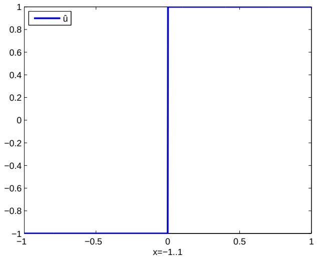

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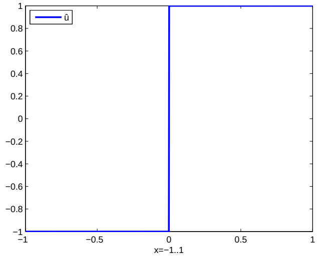

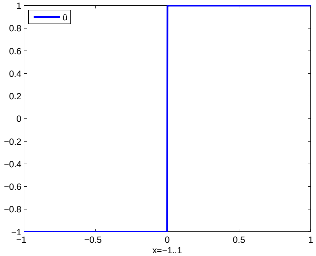

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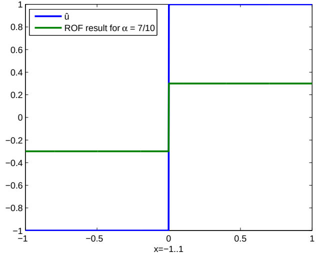

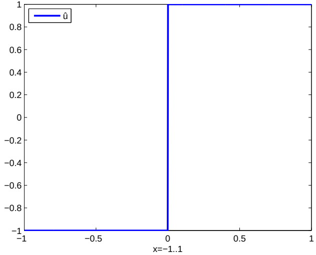

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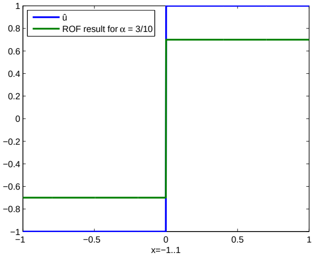

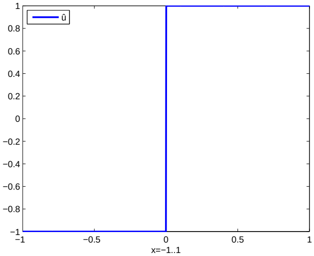

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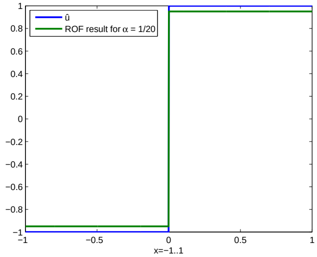

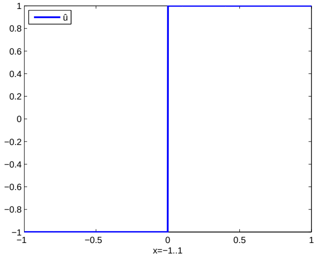

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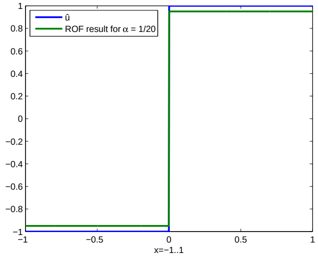

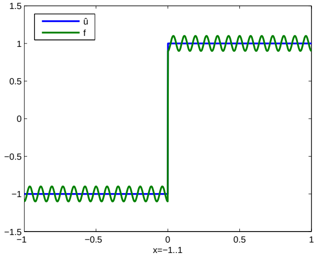

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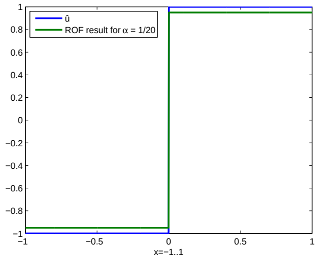

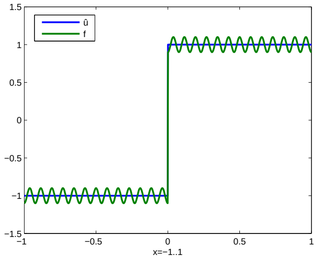

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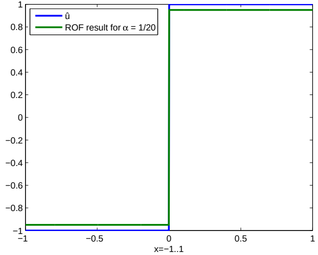

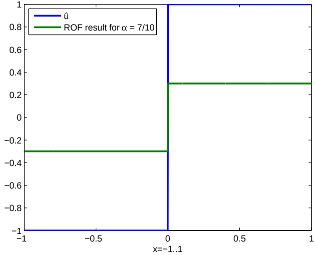

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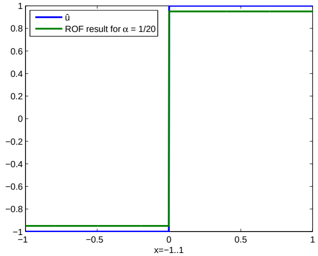

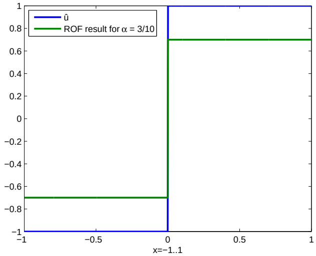

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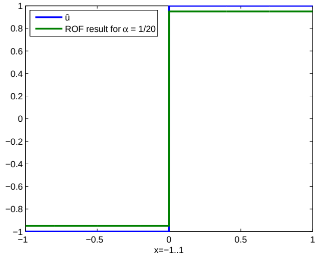

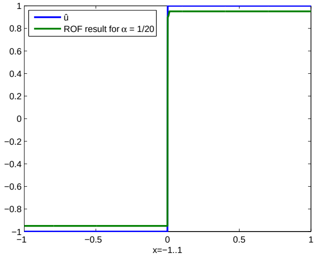

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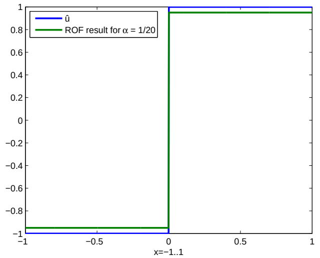

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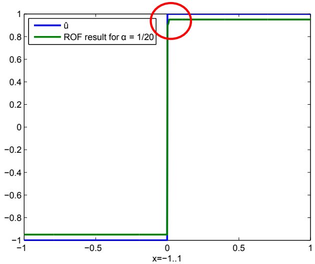

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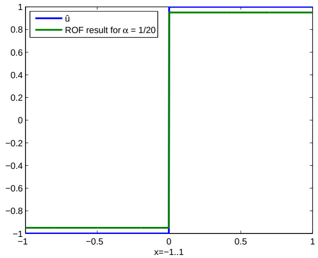

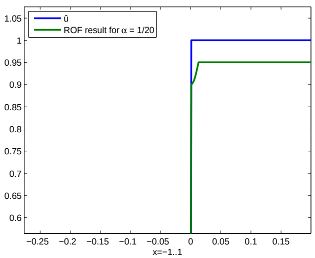

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Let the same assumptions hold as in the previous theorem. Furthermore the data f is assumed to be corrupted by noise n, i.e. $f = \gamma K\hat{u} + n$ for a positive constant γ, such that there exist positive constants μ and η that satisfy (15). Then, the solution of the Inverse Scale Space Flow (10) for time $t_* = (\lambda\eta)/(\gamma\eta + \lambda - \mu) \leq t < t_{**}$ is given via $u(t) = c\hat{u}$ for

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- Remarkable: assume $\gamma = 1$; for $\lambda = \mu$ we have $u(t) = \hat{u}$, no matter what value η takes (as long as η and μ satisfy (15))

Essen ist fertig!

› Examples

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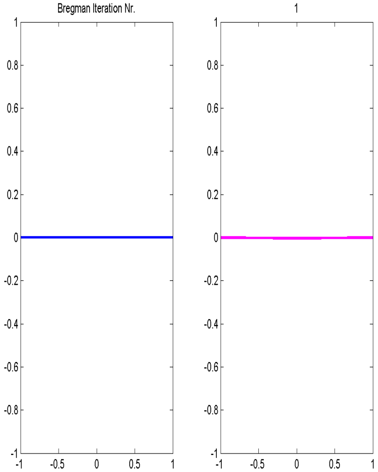
Bregman Iteration Nr.
1
1
0.8
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0
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-0.4
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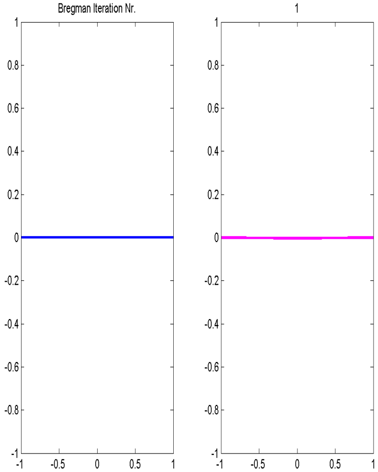
-0.8
-1
-0.5
0.5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- If we consider the related Inverse Scale Space Flow, i.e. $\partial_t p_i = (K^T(f - Ku))_i$, $(p \in \ell^\infty)$ we can also think of CS setup 2 as

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Determining $u(t_1)$ from $p(t_1)$

We can determine $u(t_1)$ from

$$u = \arg\min_u \left\{ \|KP_{I_1}u - f\|_{\ell^2}^2 \right\} \tag{20}$$

subject to $P_{I_1^c}u = 0$ and $u_i p_i \geq 0$ for $i \in I_1$, with I_1^c denoting the complement of I_1

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in order to satisfy $\partial_t p_i = \left(K^T (f - Ku(t_1))\right)_i$ and $p_i(t_1) = \left(K^T f\right)_i / \left\|K^T f\right\|_{\ell^\infty}$

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$$p_i(t) = p_i(t_1) + (t - t_1) \left(K^T \left(f - Ku(t_1)\right)\right)_i \tag{21}$$

in order to satisfy $\partial_t p_i = \left(K^T \left(f - Ku(t_1)\right)\right)_i$ and $p_i(t_1) = \left(K^T f\right)_i / \left\|K^T f\right\|_{\ell^\infty}$

- Note that we have $P_{I_1} p(t) = P_{I_1} p(t_1)$ and therefore $\|P_{I_1} p(t)\|_{\ell^\infty} = 1$, while for small $t - t_1$ we have $\|P_{I_1^c} p(t)\|_{\ell^\infty} \leq 1$, since $\|P_{I_1^c} p(t_1)\|_{\ell^\infty} < 1$
- Idea: find t_2 as the minimal time t such that

$$\left\|P_{I_1^c} \left(p(t_1) + (t - t_1) K^T \left(f - Ku(t_1)\right)\right)\right\|_{\ell^\infty} = 1 \tag{22}$$

- We can summarize the previous considerations to the following algorithm

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Adaptive Inverse Scale Space for Compressed Sensing

Algorithm 2 Adaptive Inverse Scale Space for Compressed Sensing

1. **Parameters:** K, f, $\delta \geq 0$
2. **Initialization:** $t_1 = 1/\left\|K^T f\right\|_{\ell^\infty}$, $p(t_1) = t_1 K^T f$, $I_1 = \{i \mid |p_i(t_1)| = 1\}$

while $\|Ku - f\|_{\ell^2} \leq \delta$ **do**

  Compute $u(t_k)$ from (20) with P_{I_k}
  Obtain t_{k+1} as the minimal time for which (22) holds
  Update the dual variable via (21) with $t = t_{k+1}$
  Compute $I_{k+1} = \{i \mid |p_i(t_{k+1})| = 1\}$

end while
return $u(t_k)$

› Computational Results

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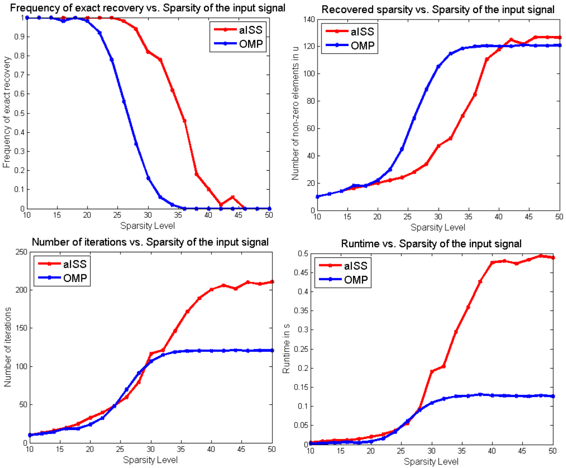
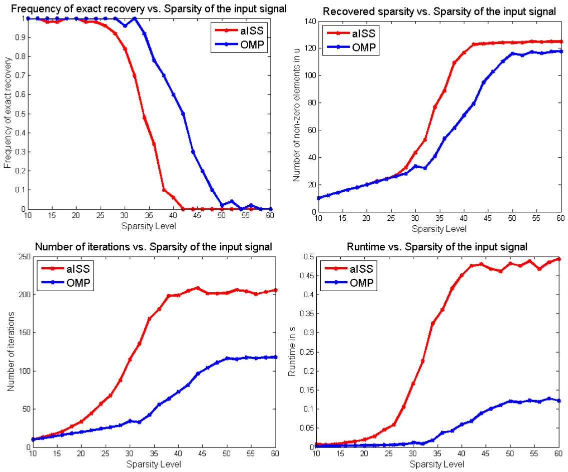
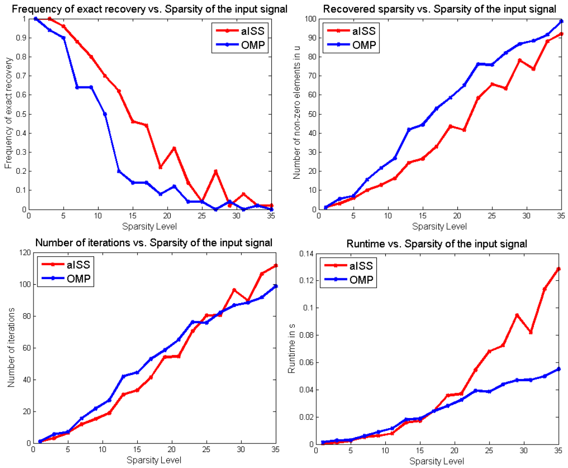


› Computational Results



› Computational Results



Thank you for attention!