Numerical methods for shallow water wave equations

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Overview

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- Relaxation Model for Conservation Laws
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  - Cons. Laws with Source Terms
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- Dispersive water wave models
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  - Numerical Results
Finite volume schemes for Conservation Laws
We consider the initial value problem for scalar conservation laws

\[ u_t + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0 \]
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R} \]

Consider a uniform partition of \( \mathbb{R} \times \mathbb{R}^+ \) in cells \([x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]\)

\[ x_i = i\Delta x, \quad i \in \mathbb{Z}, \quad x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}, \quad t^n = n\Delta t \]
\[
0 = \int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [u_t + f(u)_x] \, dx \, dt \\
= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [u(x, t^{n+1}) - u(x, t^n)] \, dx + \int_{t^n}^{t^{n+1}} \left[ f(u(x_{i+\frac{1}{2}}, t)) - f(u(x_{i-\frac{1}{2}}, t)) \right] \, dt \\
= \text{Change of Mass} + \text{Difference of Fluxes in cell } [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]
\]

\[
U_i^n \sim \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t^n) \, dx, \quad F_{i+\frac{1}{2}}^n := F(U_{i+1}^+, U_{i+1}^-) \sim \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}}, t)) \, dt
\]

\(U_i^n\) approximates the average of \(u\) in \(C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]\) at time \(t^n\)

\(F_{i+\frac{1}{2}}^n\) approximates the average in \([t^n, t^{n+1}]\) at \(x = x_{i+\frac{1}{2}}\)

\(F\) is a \textbf{numerical flux} function

\(U_{i+1}^+\) some approximation of \(u(x_{i+\frac{1}{2}} - 0, t^n)\)

\(U_{i+1}^-\) some approximation of \(u(x_{i+\frac{1}{2}} + 0, t^n)\)
Basic FV scheme

\[ U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} \left( F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right) \]

- \( F \) numerical flux function:
  1. Consistency: \( F(u, u) = f(u) \)
  2. Monotonicity: \( \partial_u F > 0, \partial_v F < 0 \)
  3. Upwind, Lax-Friedrichs, Lax-Wendroff, Godunov, Central, Roe’s, ...

- CFL condition: \( \sup_u |f'(u)| \frac{\Delta t}{\Delta x} \leq 1 \)

- \( U_i^+, U_{i+1}^- \) Reconstruction process:
  1. Piecewise constants: \( U_i^+ = U_i, U_{i+1}^- = U_{i+1} \)
  2. Piecewise linear: \( U_i^+ = U_i + \frac{\Delta x}{2} S_i, U_{i+1}^- = U_{i+1} - \frac{\Delta x}{2} S_{i+1} \) where \( S_i \sim u_x(x_{i+\frac{1}{2}}), x \in C_i \) \( S_{i+1} \sim u_x(x_{i+\frac{1}{2}}), x \in C_{i+1} \). Limiters.
  3. Higher order polynomials are constructed using the cell averages \( U_i \).
Reconstruction process

Constant

\[ x_{i-\frac{3}{2}} \quad x_{i-\frac{1}{2}} \quad x_{i+\frac{1}{2}} \quad x_{i+\frac{3}{2}} \]

Linear

\[ x_{i-\frac{3}{2}} \quad x_{i-\frac{1}{2}} \quad x_{i+\frac{1}{2}} \quad x_{i+\frac{3}{2}} \]
Relaxation model for Conservation Laws


Relaxation Model for Scalar CL

\[ u_t + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \]
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \]

(1)

Relaxation system proposed by Jin & Xin 1995

\[ u_t + v_x = 0, \]
\[ v_t + c^2 u_x = -\frac{1}{\epsilon} (v - f(u)), \quad \epsilon \rightarrow 0 \]

(2)

This system can be viewed as a regularization of (1) by the wave operator

\[ u_t + f(u)_x = -\epsilon(u_{tt} - c^2 u_{xx}) + O(\epsilon^2). \]

Applying the Champan-Enskog expansion we get

\[ u_t + f(u)_x = \epsilon \partial_x \left( (c^2 - f'(u)^2) \partial_x u \right) + O(\epsilon^2). \]

If the subcharacteristic condition : \(|f'(u)| < c\) holds then a rigorous convergence analysis, for 1D scalar case, can be applied yielding at the relaxation limit \(\epsilon \rightarrow 0\) the conservation law (1). (JX, 1995)
For a conservation law with a source term

\[ u_t + f(u)_x = q(u), \quad x \in \mathbb{R}, \quad t > 0, \]
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \] (3)

a relaxation system considered takes the form

\[ u_t + v_x = q(u), \]
\[ v_t + c^2 u_x = -\frac{1}{\epsilon}(v - f(u)), \] (4)
yielding the following regularization of (3),

\[ u_t + f(u)_x = q(u) + \epsilon q(u)_t - \epsilon(u_{tt} - c^2 u_{xx}). \]

**Remarks**
1) Same subcharacteristic condition
2) Extra term : \( \epsilon q(u)_t \).
3) In general (4) does not preserve the steady states.
4) The time discretization of (4)
\[
\frac{U^{n+1} - U^n}{\Delta t} + V_x^n = q(U^{n+1}), \\
\frac{V^{n+1} - V^n}{\Delta t} + c^2 U_x^n = -\frac{1}{\epsilon}(V^{n+1} - f(U^{n+1})),
\]

is fully coupled system, not the case for the corresponding time discetization of (2).

An alternative approach: we consider the following relaxation system

\[
\begin{align*}
  u_t + v_x &= 0, \\
  v_t + c^2 u_x &= -\frac{1}{\epsilon}(v - f(u)) - \frac{1}{\epsilon} R(u),
\end{align*}
\]

where \( R(u) \) is an antiderivative of \( q(u) \),

\[
R(u(x)) = \int^x q(u(s))ds.
\]
In this case (6) provides exactly a wave-type regularization of (3),

\[ u_t + f(u)_x = q(u) - \epsilon(u_{tt} - c^2 u_{xx}). \]  \hspace{1cm} (7)

Also an implicit-explicit time discretization is now possible when we treat the source terms implicitly:

\[
\frac{U^{n+1} - U^n}{\Delta t} + V^n_x = 0,
\]

\[
\frac{V^{n+1} - V^n}{\Delta t} + c^2 U^n_x = -\frac{1}{\epsilon} (V^{n+1} - f(U^{n+1})) - \frac{1}{\epsilon} R(U^{n+1}).
\]  \hspace{1cm} (8)

If \( |f'(u)| < c \) from (7) we recover formally (3)

System (6) preserves steady states

Initial, Boundary Cond. : \( v_0 = f(u_0), \ v_b = f(u_b) \)
\[ \partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) = 0, \quad x \in \mathbb{R}^d, \quad u = u(x, t) \in \mathbb{R}^n, \quad t > 0 \]

\[ u(\cdot, 0) = u_0(\cdot) \]

Relaxation model

\[ \partial_t u + \sum_{j=1}^{d} \partial_{x_j} v_j = 0, \]

\[ \partial_t v_i + A_i \partial_{x_i} u = -\frac{1}{\epsilon} (v_i - F_i(u)), \quad i = 1, \ldots, d \]

it’s a regularization by a wave operator of order $\epsilon$, and $A_i$ are symmetric positive definite matrices with constant coefficients that are selected to satisfy the corresponding sub-characteristic conditions.
Shallow Water Equations (SWE)
Shallow water eqns (1D)

\[ h_t + (hu)_x = 0, \]
\[ (hu)_t + (hu^2 + \frac{g}{2} h^2)_x = -ghZ', \]  

(9)

General steady states:

\[ Q = hu = Cnst \]
\[ \frac{u^2}{2} + g(h + Z) = Cnst \]

SWE is a hyperbolic system with source term
SW Relaxation Models

Relaxation Model A

\[ \begin{align*}
    h_t + v_x &= 0 \\
    Q_t + w_x &= -ghZ' \\
    v_t + c_1^2 h_x &= -\frac{1}{\epsilon}(v - Q) \\
    w_t + c_2^2 Q_x &= -\frac{1}{\epsilon}(w - \left(\frac{Q^2}{h} + \frac{g}{2}h^2\right))
\end{align*} \]

Relaxation Model B

\[ \begin{align*}
    h_t + v_x &= 0 \\
    Q_t + w_x &= 0 \\
    v_t + c_1^2 h_x &= -\frac{1}{\epsilon}(v - Q) \\
    w_t + c_2^2 Q_x &= -\frac{1}{\epsilon}(w - \left(\frac{Q^2}{h} + \frac{g}{2}h^2\right)) + \frac{1}{\epsilon}R(Z; h)
\end{align*} \]
\[ R(Z; h)(x) = \int_{0}^{x} g(hZ')(y) \, dy \]

- \( c_1, c_2 \) are chosen according to sub-characteristic condition:
  \[ |\lambda_i(F')| < c_i, \quad i = 1, 2, \quad F' = \text{Jacobian of flux vector} \]

- For \( Z \equiv 0, \ (A) \equiv (B) \)
- For \( \epsilon \to 0 \) we recover the original SW system
- Both relaxation systems have linear principal part
- Implicit-explicit time discretizations for (B)
- System (B) have same steady states as the continuous problem
We consider the Relaxation Model B and let

\[
\mathbf{u} = \begin{bmatrix} h \\ Q \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v \\ w \end{bmatrix},
\]

our system can be rewritten as

\[
\begin{align*}
\mathbf{u}_t + \mathbf{v}_x &= 0, \\
\mathbf{v}_t + \mathbf{C}^2 \mathbf{u}_x &= -\frac{1}{\epsilon} (\mathbf{v} - \mathbf{F}(\mathbf{u})) - \frac{1}{\epsilon} \mathbf{S}(\mathbf{u}),
\end{align*}
\]

\[
\mathbf{F}(\mathbf{u}) = (Q, \frac{Q^2}{h} + \frac{g}{2} h^2)^T, \quad \mathbf{S}(\mathbf{u}) = (0, -\int_x gh(y)Z'(y)dy)^T
\]

where \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^2 \) and \( \mathbf{C}^2 \in \mathbb{R}^{2\times2} \) is a positive matrix.
Relaxation Schemes for SWE (1D)
We assume a uniform spaced grid with $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and a uniform time step $\Delta t = t^{n+1} - t^n$, $n = 0, 1, 2, \ldots$

$$u^n_i \sim \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t^n) \, dx, \quad u^n_{i+\frac{1}{2}} \sim u(x_{i+\frac{1}{2}}, t^n)$$

We start by considering the following one-step conservative system for the homogeneous case (no source term present)

$$\frac{\partial}{\partial t} u_i + \frac{1}{\Delta x} (v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}) = 0,$$

$$\frac{\partial}{\partial t} v_i + \frac{1}{\Delta x} C^2 (u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}) = -\frac{1}{\epsilon} (v_i - F(u_i)).$$

The linear hyperbolic part has two Riemann invariants (characteristic speeds) $v \pm Cu$ associated with the characteristic fields $\pm C$ respectively. The first order upwind approximation of $v \pm Cu$ is

$$(v + Cu)_{i+\frac{1}{2}} = (v + Cu)_i, \quad (v - Cu)_{i+\frac{1}{2}} = (v - Cu)_{i+1}.$$
Hence,

\[
\begin{align*}
    u_{i+\frac{1}{2}} &= \frac{1}{2}(u_i + u_{i+1}) - \frac{1}{2}C^{-1}(v_{i+1} - v_i), \\
    v_{i+\frac{1}{2}} &= \frac{1}{2}(v_i + v_{i+1}) - \frac{1}{2}C(u_{i+1} - u_i).
\end{align*}
\]

First order upwind semi-discrete approximation of the relaxation scheme:

\[
\begin{align*}
    \frac{\partial}{\partial t} u_i + \frac{1}{2\Delta x} (v_{i+1} - v_{i-1}) - \frac{1}{2\Delta x} C(u_{i+1} - 2u_i + u_{i-1}) &= 0, \\
    \frac{\partial}{\partial t} v_i + \frac{1}{2\Delta x} C^2(u_{i+1} - u_{i-1}) - \frac{1}{2\Delta x} C(v_{i+1} - 2v_i + v_{i-1}) &= -\frac{1}{\epsilon} (v_i - F(u_i)) \\
    &\quad - \frac{1}{\epsilon} S(u_i),
\end{align*}
\]

where

\[
    S(u_i) = \begin{bmatrix} 0 \\ - \int_{x_i+\frac{1}{2}}^{x_{i+1}} gh(y)Z'(y)dy \end{bmatrix}.
\]
We replace the piecewise constant approximation by a MUSCL piecewise linear interpolation: for the $k$–th component of $v \pm Cu$ we have:

$$(v + c_k u)_{i + \frac{1}{2}} = (v + c_k u)_i + \frac{1}{2} \Delta x s^+_i,$$

$$(v - c_k u)_{i + \frac{1}{2}} = (v - c_k u)_{i + 1} - \frac{1}{2} \Delta x s^-_{i + 1},$$

where $u, v$ are the $k$–th components of $v, u$ and the slopes $s^\pm$ in the $i$–th cell:

$$s^\pm_i = \frac{1}{\Delta x} (v_{i + 1} \pm c_k u_{i + 1} - v_i \mp c_k u_i) \phi(\theta^\pm_i)$$

$$\theta^\pm_i = \frac{v_i \pm c_k u_i - v_{i - 1} \mp c_k u_{i - 1}}{v_{i + 1} \pm c_k u_{i + 1} - v_i \mp c_k u_i},$$

where $\phi$ is a limiter function satisfying $0 \leq \phi(\theta) \leq \minmod(2, 2\theta)$.

MinMod (MM): $\phi(\theta) = \max(0, \min(1, \theta))$,

VanLeer (VL): $\phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}$,

Monotonized Central (MC): $\phi(\theta) = \max(0, \min((1 + \theta)/2, 2, 2\theta))$.
Second order semi-discrete relaxation scheme (componentwise form)

\[
\frac{\partial}{\partial t} u_i + \frac{1}{2\Delta x} (v_{i+1} - v_{i-1}) - \frac{c_k}{2\Delta x} (u_{i+1} - 2u_i + u_{i-1}) \\
- \frac{1}{4} (s_{i+1}^- - s_i^- + s_{i-1}^+ - s_i^+) = 0,
\]

\[
\frac{\partial}{\partial t} v_i + \frac{c_k^2}{2\Delta x} (u_{i+1} - u_{i-1}) - \frac{c_k}{2\Delta x} (v_{i+1} - 2v_i + v_{i-1}) \\
+ \frac{c_k}{4} (s_{i+1}^- - s_i^- - s_{i-1}^+ + s_i^+) = -\frac{1}{\epsilon} (v_i - F_k(u_i)) - \frac{1}{\epsilon} S_k(u_i),
\]

with \( S_k, F_k \) being the \( k \)-th components of \( S, F \) respectively.
A first order in time RK-type scheme, \((Z \equiv 0)\)

(A) Given \(u^n, v^n\) apply a finite volume method to update \(u, v\) over time \(\Delta t\) by solving the homogeneous linear hyperbolic system

\[
\begin{bmatrix}
    u \\
    v
\end{bmatrix}_t + \begin{bmatrix}
    0 & I \\
    C^2 & 0
\end{bmatrix} \begin{bmatrix}
    u \\
    v
\end{bmatrix}_x = \begin{bmatrix}
    0 \\
    0
\end{bmatrix},
\]

and obtain new values \(u^{(1)}, v^{(1)}\).

(B) Update \(u^{(1)}, v^{(1)}\) to \(u^{n+1}, v^{n+1}\) by solving the equations,

\[
\begin{align*}
    u_t &= 0, \\
    v_t &= -\frac{1}{\epsilon}(v - F(u)),
\end{align*}
\]

over time \(\Delta t\).
A second order in time RK-type scheme,

\[ u^{n,1} = u^n, \quad v^{n,1} = v^n + \frac{\Delta t}{\epsilon} (v^{n,1} - F(u^{n,1})) + \frac{\Delta t}{\epsilon} S(u^{n,1}); \]
\[ u^{(1)} = u^{n,1} - \Delta t D_+ v^{n,1}, \quad v^{(1)} = v^{n,1} - \Delta t C^2 D_+ u^{n,1}; \]
\[ u^{n,2} = u^{(1)}, \quad v^{n,2} = v^{(1)} - \frac{\Delta t}{\epsilon} (v^{n,2} - F(u^{n,2})) - \frac{2\Delta t}{\epsilon} (v^{n,1} - F(u^{n,1})) \]
\[ - \frac{\Delta t}{\epsilon} S(u^{n,2}) - \frac{2\Delta t}{\epsilon} S(u^{n,1}); \]
\[ u^{(2)} = u^{n,2} - \Delta t D_+ v^{n,2}, \quad v^{(2)} = v^{n,2} - \Delta t C^2 D_+ u^{n,2}; \]
\[ u^{n+1} = \frac{1}{2}(u^n + u^{(2)}), \quad v^{n+1} = \frac{1}{2}(v^n + v^{(2)}). \]

where

\[ D_+ w_i = \frac{1}{\Delta x} (w_{i+\frac{1}{2}} - w_{i-\frac{1}{2}}). \]
Choice of parameters

□ CFL Condition

1st order scheme: \[ \max\{c_1, c_2\} \frac{\Delta t}{\Delta x} \leq 1 \]

2nd order scheme: \[ \max\{c_1, c_2\} \frac{\Delta t}{\Delta x} \leq \frac{1}{2} \]

□ Choice of \( c_1, c_2 \): based on rough estimates of the eigenvalues: \( u \pm \sqrt{gh} \) and satisfy the subcharacteristic condition

\[ c_1 \geq \sup |u + \sqrt{gh}| \quad \text{and} \quad c_2 \geq \sup |u - \sqrt{gh}| \]

\[ c_1 = c_2 = \max \left\{ \sup |u + \sqrt{gh}|, \sup |u - \sqrt{gh}| \right\} \]

□ Choice of \( \epsilon \): \( \epsilon << \Delta x, \epsilon << \Delta t \)
SWE(1D) : Numerical Results

- No source term: $Z \equiv 0$: Dam Break problem
  - Subcritical, supercritical, strongly supercritical
  - Dry bed problem

- Source term: $Z \neq 0$:
  - Flow at Rest
  - Nontrivial Steady States
  - Drain on Non-Flat bottom
We consider a channel of length $L = 2000m$. A dam is located at $x_0 = 1000m$ and at time $t = 0$ the dam collapses. We compute the solution for time $T = 50s$ with initial conditions:

$$u(x, 0) = 0, \quad h(x, 0) = \begin{cases} h_1 & x \leq 1000, \\ h_0 & x > 1000, \end{cases}$$

with $h_1 > h_0$. This is the Riemann problem for the homogeneous problem. The flow consists of a shock wave (bore) travelling downstream and a rarefaction wave (depression wave) travelling upstream. The upstream depth $h_1$ was kept constant at 10m, while the downstream depth $h_0$ was different for each problem.

- $h_0/h_1 > 0.5$ : subcritical flow
- $h_0/h_1 < 0.5$ : subcritical upstream, supercritical downstream
- $h_0/h_1 << 0.5$ : strongly supercritical downstream
- $CFL = 0.5m$, $\Delta x = 20m$, $c_1 = 5$, $c_2 = 12$, $\epsilon = 1.E - 4$
Dam-break flow, $h_0/h_1 = 0.5$, (x) Upwind and (o) MUSCL with MC limiter.
Dam-break flow, $h_0/h_1=0.05$, (x)Upwind and (o)MUSCL with MC limiter.
Dam-break flow, $h_0/h_1 = 0.005$, (x) Upwind and (o) MUSCL with MC limiter.
Dry Bed problem, $h_0 = 0$

This a challenging problem as a result of the singularity that occurs at the transition point of the advancing front. We compute the solution at $T = 40s$

- No modifications to the scheme to incorporate the dry area
- Globally accurate results free of oscillations
- The water height and discharge remain positive
- The transition point between the wet and the dry zone is close to the exact one, but some difficulties appear on the velocity.
- Overall the solution is stable, monotone with no special front tracking techniques

- $CFL = 0.5 \ \Delta x = 10m, \ c_1 = 18, \ c_2 = 16, \ \epsilon = 1.E-4$
Dry bed dam-break flow \( (h) \), (o)MUSCL with MM limiter.
Dry bed dam-break flow \((q)\), (o)MUSCL with MM limiter.
Dry bed dam-break flow \((u)\), (o)MUSCL with MM limiter.
Flow at Rest, $Z \neq 0$

We consider a channel of length $L = 25m$ with a non-trivial bathymetry $Z$, with initial conditions

$$u(x, 0) = 0, \quad \forall x \in \mathbb{R},$$
$$h(x, 0) + Z(x) = H, \quad \forall x \in \mathbb{R},$$

Exact solution

$$u(x, t) = 0, \quad \forall x \in \mathbb{R}, t \geq 0,$$
$$h(x, t) + Z(x) = H, \quad \forall x \in \mathbb{R}, t \geq 0,$$

$$Z(x) = \begin{cases} 
0.2 - 0.05(x - 10)^2, & 8 \leq x \leq 12, \\
0, & \text{otherwise},
\end{cases}$$

with $H = 2m$, $\epsilon = 1.E - 5$, $c_1 = 4$, $c_2 = 4.5$, $CFL = 0.5$, $T = 200s$, $\Delta x = 0.125$
Flow at rest (water height): (+) standard source, (o) integral source
Flow at rest (discharge): (+) standard source, (o) integral source
Flow at rest: Magnified view of the discharge.
Non trivial steady states

We consider with the convergence towards steady flow over the parabolic hump in a channel of length \( L = 25m \).
Depending on the boundary conditions the flow maybe *subcritical*, *transcritical* with a shock or without a shock. In all cases we use MUSCL scheme with

\[
CFL = 0.5, \quad \Delta x = 0.125m, \quad T = 200s, \quad \epsilon = 1.E-5, \quad c_1 = 5, \quad c_2 = 7
\]

\[
u(x,0) = 0, \quad \forall x \in \mathbb{R},
\]

\[
h(x,0) + Z(x) = H_0, \quad \forall x \in \mathbb{R},
\]

where \( H_0 \) water level downstream.

- **Subcritical Flow**: \( q_{up} = 4.42m^2/s, \ H_0 = 2m \)
- **Transcritical Flow without shock**: \( q_{up} = 1.53m^2/s, \ H_0 = 0.66m \)
- **Transcritical Flow with shock**: \( q_{up} = 0.18m^2/s, \ H_0 = 0.33m \)
Transcritical flow with shock: Water Level

Transcritical flow with shock ($h$)
Transcritical Flow with shock: Discharge

Transcritical flow with shock ($q$)
Drain on a non-flat bottom.

□ Difficult problem since it involves the calculation of dry areas.

□ BC’s : Upstream reflective, Downstream dry bed

□ IC’s : $h + Z = 0.5m$ and $q = 0m^3/s$

□ Solution : a state at rest, on the left part of the hump with $h + Z = 0.2m$ with $q = 0m^3/s$ and a dry state (i.e. $h = 0$ and $q = 0m^3/s$) on the right of the hump.

□ MUSCL scheme with
   $\Delta x = 0.1m$, $CFL = 0.5$, $\epsilon = 1.E - 6$, $c_1 = c_2 = 3.5$

□ No modification of the method to overcome the dry area problem of zero depth and discharge.
Drain on a non-flat bottom: Water Level

Drain on a non-flat bottom ($h$)
Drain on a non-flat bottom: Discharge

\[ q \text{ (m}^2/\text{s}) \]

\[ x \text{ (m)} \]

\[ t=0s \quad t=10s \quad t=20s \quad t=100s \quad t=1000s \]

Drain on a non-flat bottom \((q)\)
The 2D Shallow Water Equations

\[ U_t + F(U)_x + G(U)_y = S(U); \quad (x, y) \in \Omega, \quad t \geq 0 \]

\[ U = \begin{pmatrix} h \\ hu_1 \\ hu_2 \end{pmatrix} = \begin{pmatrix} h \\ q_1 \\ q_2 \end{pmatrix}, \quad S(U) = \begin{pmatrix} 0 \\ -gh \frac{\partial Z}{\partial x}(x, y) - gh S_f^x \\ -gh \frac{\partial Z}{\partial y}(x, y) - gh S_f^y \end{pmatrix}, \]

\[ F(U) = \begin{pmatrix} q_1 \\ \frac{q_1^2}{h} + \frac{1}{2}gh^2 \\ \frac{q_1 q_2}{h} \end{pmatrix}, \quad G(U) = \begin{pmatrix} q_2 \\ \frac{q_1 q_2}{h} \\ \frac{q_2^2}{h} + \frac{1}{2}gh^2 \end{pmatrix}. \]

\[ S_f^x = n_m^2 u_1 \sqrt{u_1^2 + u_2^2 h^{-4/3}} \quad S_f^y = n_m^2 u_2 \sqrt{u_1^2 + u_2^2 h^{-4/3}}, \text{ where } n_m \text{ is the Manning roughness coefficient.} \]
Relaxation System for 2D SWE

\[
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}_t + \begin{bmatrix}
  0 & I & 0 \\
  C^2 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}_x + \begin{bmatrix}
  0 & 0 & I \\
  0 & 0 & 0 \\
  D^2 & 0 & 0
\end{bmatrix} \begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}_y = \begin{bmatrix}
  0 \\
  -\frac{1}{\epsilon}(v - F(u) + \tilde{S}(u)) \\
  -\frac{1}{\epsilon}(w - G(u) + \tilde{S}(u))
\end{bmatrix}
\]

\[
\tilde{S}(u) = \begin{bmatrix}
  0 \\
  -\frac{1}{2} \int_x^x gh(s, y) \frac{\partial Z}{\partial x}(s, y) ds \\
  -\frac{1}{2} \int_x^x gh(s, y) \frac{\partial Z}{\partial y}(s, y) ds
\end{bmatrix}
\]

\[
\tilde{S}(u) = \begin{bmatrix}
  0 \\
  -\frac{1}{2} \int_y^y gh(x, s) \frac{\partial Z}{\partial x}(x, s) ds \\
  -\frac{1}{2} \int_y^y gh(x, s) \frac{\partial Z}{\partial y}(x, s) ds
\end{bmatrix}
\]

Subcharacteristic condition:

\[
\frac{\lambda^2_i}{c^2_i} + \frac{\mu^2_i}{d^2_i} \leq 1, \quad \forall \ i = 1, 2, 3,
\]

with \(\lambda_i, \mu_i\) the eigenvalues of \(\partial F(u)/\partial u\) and \(\partial G(u)/\partial u\) respectively.
Fully Discrete Schemes

- **Upwind**: 1st order; **MUSCL**: 2nd order

- **CFL** condition, guarantees the TVD property of both schemes

\[
CFL = \max \left( \left( \max_i c_i \right) \frac{\Delta t}{\Delta x}, \left( \max_i d_i \right) \frac{\Delta t}{\Delta y} \right) \leq \frac{1}{2}.
\]

- **I.B. Cond.**: \( v_0 = F(u_0), w_0 = G(u_0), v_b = F(u_b), w_b = G(u_b) \)

- **Choice of** \( c_k, d_k, k = 1, 2, 3 \):

  1. rough estimates of the eigenvalues \((u_1, u_1 \pm \sqrt{gh})\) and \((u_2, u_2 \pm \sqrt{gh})\)

  2. calculate \( c \) and \( d \) locally at every cell as

     \[
     c_{i+\frac{1}{2},j} = \max_{u \in \{u_{i+\frac{1}{2},j}, u_{i-\frac{1}{2},j}\}} \left| \frac{\partial F(u)}{\partial u_k} \right|
     \]

     \[
     d_{i,j+\frac{1}{2}} = \max_{u \in \{u_{i,j+\frac{1}{2}}, u_{i,j-\frac{1}{2}}\}} \left| \frac{\partial G(u)}{\partial u_k} \right|
     \]

  3. global choice: \( c_k = d_k = \max_{i,j} (c_{i+\frac{1}{2},j}, d_{i,j+\frac{1}{2}}) \)

- \( \Delta t \gg \epsilon \) and \( \Delta y, \Delta x \gg \epsilon \)
No source $Z \equiv 0$ : Partial Dam Break, Circular Dam Break

Dam break in a channel with topography
The dam, located in the center of a channel breaks instantaneously.

No friction \((n_m = 0)\). \(h_u = 10m\) and \(h_d = 5, 0.1, 0m\)

Channel: \(200m \times 200m\), \(41 \times 41\) square grid.

The breach is 75m in length, 30m from the left bank, 95m from the right.

BC’s: \(x = 0\) and \(x = 200m\) transmissive and all other boundaries are reflective.

2nd order MUSCL scheme

\(\epsilon = 10^{-6}\) and \(c_1 = 10, c_2 = 6, c_3 = 11, d_1 = 10, d_2 = 5, d_3 = 11\).

\(T = 7.2s\)
2D Partial Dam-Break, $h_d = 5$
2D Partial Dam-Break, $h_d = 0.1m$
2D Partial Dam-Break, $h_d = 0m$
A two dimensional Riemann problem for the 2D SWEs

Two regions of still water separated by a cylindrical wall with radius $11m$ centered in a channel. The water depth within the cylinder is $10m$ and $1m$ outside.

The wall is removed instantaneously, the bore waves will spread and propagate radially and symmetrically.

There is a transition from subcritical to supercritical flow.

2nd order MUSCL scheme

Channel: $50 \times 50m$, $51 \times 51$ square grid

$\epsilon = 10^{-6}$ and $c_1 = c_3 = 12$, $c_2 = 7$, $d_1 = d_3 = 12$, $d_2 = 7$, $T = 0.69s$
at $t = 0.69\text{s}$, Water depth, depth contours, velocity field, (MM, VL limiter)
at $t = 0.69s$, Water depth, depth contours, velocity field, (MC, SB limiters)
Circular Dam Break, Dry Bed

at $t = 0.69s$, Water depth, depth contours, velocity field, (VL limiter)
Consider a channel 75 m long and 30 m wide.

A dam is situated at $x = 16$ m with initial water depth $h + Z = 1.875$ m while the rest of the channel is considered dry.

The topography consists of three mounds located in the channel bottom.

Manning coefficient $n_m = 0.018$, $c_i = d_i = 5$, $\epsilon = 1 \times 10^{-8}$.
Conclusions

- Relaxation Schemes for SW which combine
  - Simplicity
  - Robustness
  - Efficiency
  - Riemann solver free

- Novel ways to incorporate source terms

- Small errors of order of $\epsilon$ while preserving steady states.

- The benchmark tests show that the schemes provide accurate solutions in good agreement with well known analytical solutions.

- Comparable solutions with well known solvers

- Can be considered for practical applications?
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