Hölder spaces

- $\Omega \subset \mathbb{R}^n$ open, bounded
- $u \in C^0(\overline{\Omega})$
- $\gamma \in [0,1]$

$$[u]_\gamma := \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

Definition (Hölder space)

For $u \in C^k(\overline{\Omega})$ define the *Hölder norm*

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C^0(\overline{\Omega})} + \sum_{|\alpha|=k} [D^\alpha u]_\gamma .$$

The function space

$$C^{k,\gamma}(\overline{\Omega}) = \left\{ u \in C^k(\overline{\Omega}) \,\middle|\, \|u\|_{C^{k,\gamma}(\overline{\Omega})} < \infty \right\}$$

is called the *Hölder space* with exponent γ.

- $C^{k,0} = C^k$
- $C^{0,1}$ = space of Lipschitz-continuous functions

Theorem (Hölder space)

The Hölder space with the Hölder norm is a Banach space, i. e.

- $C^{k,\gamma}(\overline{\Omega})$ *is a vector space,*
- $\|\cdot\|_{C^{k,\gamma}(\overline{\Omega})}$ *is a norm,*
- *any Cauchy sequence in the Hölder space converges.*

Proof.

Homework! □

Weak derivative

- $u, v \in L^1_{\mathrm{loc}}(\Omega)$
- α a multiindex
- $C_c^\infty(\Omega)$ = infinitely smooth functions with compact support in Ω

Definition

v is called the α^{th} *weak derivative* of u,

$$D^\alpha u = v\,,$$

if

(1) $$\int_\Omega u D^\alpha \psi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\psi \,\mathrm{d}x$$

for all test functions $\psi \in C_c^\infty(\Omega)$.

Remark

- (1) $\widehat{=}$ k *times integration by parts*
- u *smooth* $\Rightarrow$ $v = D^\alpha u$ *is classical derivative*

Weak derivative

Example (on $\Omega = (0, 2)$)

1. - $u(x) = \begin{cases} x \text{ if } 0<x\leq 1 \\ 1 \text{ if } 1<x<2 \end{cases}$
   - $v(x) = \begin{cases} 1 \text{ if } 0<x\leq 1 \\ 0 \text{ if } 1<x<2 \end{cases}$

   $v = Du$, since for any $\psi \in C_c^\infty(\Omega)$

$$\int_0^2 u\psi' \,\mathrm{d}x = \ldots = -\int_0^2 v\psi \,\mathrm{d}x,$$

2. - $u(x) = \begin{cases} x \text{ if } 0<x\leq 1 \\ 2 \text{ if } 1<x<2 \end{cases}$

   u does not have a weak derivative, since

$$-\int_0^2 v\psi \,\mathrm{d}x = \ldots = -\int_0^1 \psi \,\mathrm{d}x - \psi(1)$$

cannot be fulfilled for all $\psi \in C_c^\infty(\Omega)$ by any $v \in L^1_{\mathrm{loc}}(\Omega)$

Lebesgue spaces

Definition (Lebesgue space)

Let $p \in [1, \infty]$.

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u|^p \, \mathrm{d}x\right)^{1/p} & (p < \infty) \\ \operatorname{esssup}_\Omega |u| & (p = \infty) \end{cases}$$

The Lebesgue space with exponent p is

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \,\middle|\, u \text{ measurable with } \|u\|_{L^p(\Omega)} < \infty \right\}.$$

Theorem (Lebesgue space)

$L^p(\Omega)$ is a Banach space.

Sobolve spaces

Definition (Sobolev space)

Let $p \in [1,\infty]$, $k \in \mathbb{N}_0$. The space

$$W^{k,p}(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) \,\middle|\, \text{weak derivative } D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \le k \right\}$$

with
$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p \,\mathrm{d}x\right)^{1/p} & 1 \le p < \infty \\ \sum_{|\alpha|\le k} \operatorname{esssup}_\Omega |D^\alpha u| & p = \infty \end{cases}$$

is called a *Sobolev space*.

Theorem (Sobolev space)

$W^{k,p}(\Omega)$ is a Banach space.

Remark

- $W^{0,p}(\Omega) \equiv L^p(\Omega)$
- $W_0^{k,p}(\Omega) =$ *closure of* $C_c^\infty(\Omega)$ *in* $W^{k,p}(\Omega)$
- $H^k(\Omega) \equiv W^{k,2}(\Omega)$ *are Hilbert spaces (what is inner product?)*

Properties of Lebesgue and Sobolev functions

Theorem (Hölder's inequality)

- $p, p^* \in [1, \infty]$ *with* $\frac{1}{p} + \frac{1}{p^*} = 1$
- $f \in L^p$, $g \in L^{p^*}$

$$\left.\begin{array}{l} p, p^* \in [1,\infty] \text{ with } \frac{1}{p}+\frac{1}{p^*}=1 \\ f \in L^p,\ g \in L^{p^*} \end{array}\right\} \quad \Rightarrow \quad \int_\Omega |fg|\,\mathrm{d}x \le \|f\|_{L^p(\Omega)} \|g\|_{L^{p^*}(\Omega)}$$

Theorem (Trace theorem)

Let $\Omega \subset \mathbb{R}^n$ *bounded,* $\partial\Omega$ *Lipschitz. There exists a continuous linear operator* $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, *the trace, with*

(i) $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$,

(ii) $\|Tu\|_{L^p(\partial\Omega)} \le C \|u\|_{W^{1,p}(\Omega)}$,

(iii) $Tu = 0 \quad \Leftrightarrow \quad u \in W_0^{1,p}(\Omega)$.

Theorem (Poincaré's inequality)

$\Omega \subset \mathbb{R}^n$ *bounded, open, connected,* $\partial\Omega$ *Lipschitz.* $\exists C = C(n, p, \Omega)$

$$\|u - ⨍_\Omega u\,\mathrm{d}x\|_{L^p(\Omega)} \le C \|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W^{1,p}(\Omega)$$

$$\|u\|_{L^p(\Omega)} \le C \|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W_0^{1,p}(\Omega)$$

Embedding theorems

Theorem (Sobolev embedding)

$\Omega \subset \mathbb{R}^n$ open, bounded, $\partial\Omega$ Lipschitz, $m_1, m_2 \in \mathbb{N}_0$, $p_1, p_2 \in [1, \infty)$. If

$$m_1 \geq m_2 \text{ and } m_1 - \frac{n}{p_1} \geq m_2 - \frac{n}{p_2}$$

then $W^{m_1,p_1}(\Omega) \subset W^{m_2,p_2}(\Omega)$ and there is a constant $C > 0$ s. t.

$$\|u\|_{W^{m_1,p_1}(\Omega)} \leq C\|u\|_{W^{m_2,p_2}(\Omega)} \qquad \forall u.$$

If the inequalities are strict, $W^{m_1,p_1}(\Omega) \hookrightarrow W^{m_2,p_2}(\Omega)$ compactly.

Theorem (Hölder embedding)

$\Omega \subset \mathbb{R}^n$ open, bounded, $\partial\Omega$ Lipschitz, $m, k \in \mathbb{N}_0$, $p \in [1, \infty)$, $\alpha \in [0, 1]$. If

$$m - \frac{n}{p} \geq k + \alpha \text{ and } \alpha \neq 0, 1$$

then $W^{m,p}(\Omega) \subset C^{k,\alpha}(\overline{\Omega})$ and there is a constant $C > 0$ s. t.

$$\|u\|_{W^{m,p}(\Omega)} \leq C\|u\|_{C^{k,\alpha}(\overline{\Omega})} \qquad \forall u.$$

If $m - \frac{n}{p} < k + \alpha$, $W^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$ compactly.