- $|\tilde{H}(x,p)-\tilde{H}(x,q)| \le C|p-q|$ *for a* $C>0$,
- $|\tilde{H}(x,p)-\tilde{H}(y,p)| \le C|y-x|(1+|p|)$.

Then there is at most one viscosity solution of $0 = H((t,x),(u_t(t,x),\nabla u(t,x))) = u_t + \tilde{H}(x,\nabla u)$ *with given boundary data at* $t=0$.

Proof. See e. g. Evans, "PDEs", p. 587 □

(Semilinear) partial differential equations of second order

In this section we consider semilinear equations of the form

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j}(x) + c(\nabla u(x), u(x), x) = 0 \tag{19}$$

with the matrix $A = (a_{ij})_{ij}$ symmetric (if u is twice continuously differentiable, the matrix can be symmetrised due to Schwarz' theorem).

Definition 36 (Classification)**.** *The second order partial differential equation* (19) *is called*

- *elliptic, if A has n strictly positive eigenvalues,*
- *parabolic, if A has a zero eigenvalue,*
- *hyperbolic, if A has one positive and $n-1$ negative eigenvalues.*

Since multiplication with -1 does not change the equation, positive and negative may be exchanged in the above. Nonlinear PDEs can be classified locally by their linearisation with respect to the second order derivatives. Since A may vary spatially, the PDE may change its type.

Elliptic PDEs

Laplace's equation

$$\Delta u = 0 \tag{20}$$

physically describes the equilibrium of a diffusing quantity such as heat.

temperature:	$u:\Omega\to\mathbb{R}$ (in a piece of material Ω)
conductivity:	$a>0$ (material parameter)
heat flux:	$F=-a\nabla u$ (in direction of negative temperature gradient)
equilibrium:	net flux $\int_{\partial V} F\cdot\nu\,\mathrm{d}x$ into $V\subset\Omega$ is zero, hence

$$0 = \int_{\partial V} F\cdot\nu\,\mathrm{d}x = \int_V \operatorname{div} F\,\mathrm{d}x = -a\int_V \Delta u\,\mathrm{d}x,$$

and (22) follows since V is arbitrary.

Definition 37 (Harmonic function)**.** *A twice continuously differentiable function u satisfying* (22) *is called a* harmonic function.

Harmonic functions exhibit a number of convenient properties, which all more or less relate to the smoothing properties of Laplace's equation.

Theorem 38 (Mean value formula). *If u is harmonic in Ω,*

$$u \text{ is harmonic in } \Omega \quad \Leftrightarrow \quad u(x) = \fint_{\partial B_r(x)} u(\tilde{x})\,\mathrm{d}\tilde{x} \ \ \& \ u \in C^2 \tag{21}$$

for any ball $B_r(x) \subset \Omega$.

Proof. "⇒" Define $f(r) = \fint_{\partial B_r(x)} u(\tilde x)\,d\tilde x = \fint_{\partial B_1(0)} u(x + rz)\,dz$.

$f(0) = u(x)$ & $f'(r) = \fint_{\partial B_1(0)} \nabla u(x+rz)\cdot z\,dz = \fint_{\partial B_r(x)} \nabla u(\tilde x)\cdot \nu(\tilde x)\,d\tilde x = \frac{1}{|\partial B_r(x)|}\underbrace{\int_{B_r(x)} \Delta u\,d\tilde x}_{=0}$

"⇐" Assume $\Delta u(x) > 0$, i.e. $\exists$ a ball $B_r(x)$ with $0 < \int_{B_r(x)} \Delta u\,d\tilde x$,

but $0 = f'(r) = \frac{1}{|\partial B_r(x)|}\int_{B_r(x)} \Delta u\,d\tilde x > 0$ ↯ □

Bemerkung 39. *Using $\int_{B_r(x)} u(\tilde{x})\,\mathrm{d}\tilde{x} = \int_0^r \left(\int_{\partial B_s(x)} u(\tilde{x})\,\mathrm{d}\tilde{x}\right)\mathrm{d}s = u(x)\int_0^r \left(\int_{\partial B_s(x)} \mathrm{d}\tilde{x}\right)\mathrm{d}s = u(x)$ we obtain a second mean value formula.*

Theorem 40 (Strong maximum principle). *A non-constant harmonic function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ has no ~~strict~~ local maxima in Ω. In particular,*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(The same holds for minima by replacing u with $-u$.)

Proof. Suppose $u(x_0) = M := \max_{\Omega} u$ for an $x_0 \in \Omega$. However, by Thm 38 / remark 39, $M = u(x_0) = \fint_{B_r(x_0)} u\,dx < M$ unless $u = M$ on all of $B_r(x_0)$. By repeating the argument for all points on $\partial B_r(x_0)$ and then for the new boundary points and so on, $u = M$ on all of Ω. □

Theorem 41 (Harnack's inequality). *For each connected open set $U \subset\subset \Omega$ there exists a constant $C = C(U)$ such that*

$$\sup_U u \leq C \inf_U u$$

for all nonnegative harmonic functions u in Ω.

Proof. Let $r := \frac{1}{4}\,\mathrm{dist}(U, \partial\Omega)$ and $x, y \in U$ arbitrary with $|x-y| \le r$. Then

$u(x) = \fint_{B_{2r}(x)} u\,d\tilde x \ge \frac{1}{2^n}\fint_{B_r(y)} u\,d\tilde x = \frac{1}{2^n}u(y)$.

Since U is connected & $\overline{U}$ compact, we can cover $\overline{U}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$ of radii $\frac{r}{2}$ and $B_i \cap B_{i+1} \neq \emptyset$. □

$\Rightarrow u(x) \ge \left(\frac{1}{2^n}\right)^{N+1} u(y) \quad \forall x, y \in U$.

Theorem 42 (Smoothness of harmonic functions). *A harmonic function $u \in C^0(\overline{\Omega})$ is infinitely often differentiable inside* Ω.

Proof. Define the mollifier $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{|x|}{\varepsilon}\right) = \tilde{\eta}_\varepsilon(|x|)$ for $\eta(|x|) = \begin{cases} C e^{\frac{1}{|x|^2-1}} & \text{for } |x|<1, \\ 0 & \text{else} \end{cases}$,

with C such that $\int_{\mathbb{R}^n} \eta(|x|)\,dx = 1$. We have $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$.

Set $u_\varepsilon(x) = (u * \eta_\varepsilon)(x) = \int_\Omega u(y)\,\eta_\varepsilon(x-y)\,dy \in C^\infty(\Omega_\varepsilon)$ for $\Omega_\varepsilon = \{x \in \Omega \mid \mathrm{dist}(x,\partial\Omega) \geq \varepsilon\}$.

Then $u_\varepsilon = u$ on Ω_ε:

$$u_\varepsilon(x) = \int_\Omega u(y)\,\tilde{\eta}_\varepsilon(|x-y|)\,dy = \int_{B_\varepsilon(x)} \tilde{\eta}_\varepsilon(|x-y|)\,u(y)\,dy$$

$$= \int_0^\varepsilon \tilde{\eta}_\varepsilon(r) \underbrace{\left(\int_{\partial B_r(x)} u(\tilde{x})\,d\tilde{x}\right)}_{u(x)\,|\partial B_r(x)|} dr = u(x) \underbrace{\int_{B_\varepsilon(0)} \eta_\varepsilon(\tilde{x})\,d\tilde{x}}_{= \int_{B_1(0)} \eta(|x|)\,dx = 1} = u(x).$$

□

Poisson's equation

$$-\Delta u(x) = f(x) \tag{22}$$

is Laplace's equation with a heat source term. Notwithstanding the theorem by Cauchy and Kovalevskaya, it transpires that adequate boundary conditions for elliptic PDEs are one condition on all of $\partial\Omega$, e. g. the function value (*Dirichlet boundary conditions*, leading to the *Dirichlet problem*)

$$u = g \quad \text{on } \partial\Omega \tag{23}$$

or the normal derivative (*Neumann boundary conditions*, leading to the *Neumann problem*)

$$\partial u / \partial \nu = g \quad \text{on } \partial\Omega. \tag{24}$$

Theorem 43 (Uniqueness). *The solution to the Dirichlet problem* (22) *with* (23), *if it exists, is unique. The Neumann problem* (22) *with* (24) *can only have a solution under the solvability condition*

$$\int_\Omega f(x)\,\mathrm{d}x = -\int_{\partial\Omega} g(x)\,\mathrm{d}x.$$

If it exists, it is unique up to a constant.

Proof. Let u, v be two solutions, then $w = u - v$ solves $\Delta w = 0$ and by Thm 40 takes its maximum & minimum on $\partial\Omega$. For case (23) we have also $w = 0$ on $\partial\Omega$ so that $w \equiv 0$.
For case (24), $\partial w/\partial\nu = 0$ on $\partial\Omega$ so that $w = \text{const}$.
Note that for case (24) we have $\int_\Omega f(x)\,dx = \int_\Omega -\Delta u(x)\,dx = -\int_{\partial\Omega} \nabla u \cdot \nu\,dx = -\int_{\partial\Omega} g(x)\,dx$. □

We will establish a handy formula for the solution of Poisson's equation. To this end, we first look for simple (radial) solutions of Laplace's equation. Let $r = |x|$ and $v(r) = u(x)$ be a solution of Laplace's equation in $\mathbb{R}^n \setminus \{0\}$. Using $\partial r / \partial x_i = x_i / r$, i. e. $u_{x_i} = v'(r)x_i/r$ and $u_{x_i x_i} = v''(r)x_i^2/r^2 + v'(r)(1/r - x_i^2/r^3)$, we obtain

$$v''(r) + \frac{n-1}{r} v'(r) = 0,$$

which has the solution $v'(r) = \frac{a}{r^{n-1}}$ for some constant a.