supp fest

Theorem 29 (Hopf–Lax formula). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $g : \partial\Omega \to \mathbb{R}$, $g(x) - g(y) \leq \delta(y, x)\ \forall x, y \in \partial\Omega$. Then*

$$u(x) = \inf_{y \in \partial\Omega} \{g(y) + \delta(y, x)\}$$

is a Lipschitz-continuous viscosity solution of $H(x, \nabla u(x)) = 0$ in Ω with $u = g$ on $\partial\Omega$.

Bemerkung 30.

- *The theorem implies existence of a viscosity solution.*
- *For $H(x, p) = |p| - \frac{1}{v(x)}$ and $g \equiv 0$, $u(x)$ is the arrival time of a seismic wave starting from $\partial\Omega$.*
- *$g(x) - g(y) \leq \delta(y, x)$ means that the wavefront cannot arrive at x later than the time it needs from y to x.*

Lemma 31. *(i) L_x is convex*

(ii) $L_x(w) \leq C|w|$ for a constant C independent of w

(iii) $L_x(\overset{w}{\lambda}) = \lambda L_x(w) \forall \lambda > 0$

Proof. (i) $L_x(tq + (1-t)w) = \sup_{H(x,p)\leq 0} [tq + (1-t)w] \cdot p = \sup_{H(x,p)\leq 0} tq \cdot p + (1-t)w \cdot p \leq t \sup_{H(x,p)\leq 0} q \cdot p + (1-t) \sup_{H(x,p)\leq 0} w \cdot p = tL_x(q) + (1-t)L_x(w)$ $\forall t \in [0,1]$

(ii) $L_x(w) = \sup_{H(x,p)\leq 0} w \cdot p \leq \sup_{H(x,p)\leq 0} |w| \cdot |p| \leq (\sup_{H(x,p)\leq 0} |p|)|w|$

(iii) $L_x(\lambda w) = \sup_{H(x,p)\leq 0} \lambda w \cdot p = \lambda \sup_{H(x,p)\leq 0} w \cdot p$

□

Bemerkung 32. *The previous lemma implies that, if $H(x, p) = H(x, -p)$, δ is a pseudometric.*

Proof of Thm. 29 for the case $H(x, p) \equiv H(p)$ and $H(p) = H(-p)$. Note that in this case we have $\delta(x, y) = L(y - x)$ since L is independent of x.

a) u is Lipschitz continuous and $u(x) \leq u(y) + \delta(x, y)\ \forall x, y \in \overline{\Omega}$:

$$\begin{aligned} u(x) - u(y) &= \inf_{z_1 \in \partial\Omega} (g(z_1) + \delta(x, z_1)) - \inf_{z_2 \in \partial\Omega} (g(z_2) + \delta(y, z_2)) \\ &\leq \sup_{z_2 \in \partial\Omega} g(z_2) + \delta(x, z_2) - g(z_2) - \delta(y, z_2) \\ &\leq \delta(x, y) = L(y - x) \overset{(ii)}{\leq} C|y - x| \end{aligned}$$

b) Subsolution:
Let $u - \phi$ have a local maximum in x, i. e. $u(x') - \phi(x') \leq u(x) - \phi(x)$ or equivalently

$$\phi(x') - \phi(x) \geq u(x') - u(x) \geq -\delta(x, x').$$

Taking $x' = x + s\zeta$, ζ arbitrary,

$$\frac{\phi(x+s\zeta) - \phi(x)}{s} \geq \frac{-\delta(x, x+s\zeta)}{s} \overset{L \text{ indep. of } x}{=} \frac{-L(s\zeta)}{s} \overset{(iii)}{=} -L(\zeta) \quad \Rightarrow \quad \nabla\phi \cdot \zeta \geq -L(\zeta).$$

Analogously, $\nabla\phi \cdot \zeta \leq L(\zeta) = \sup_{H(p)\leq 0} \zeta \cdot p$, i. e. $|\nabla\phi \cdot \zeta| \leq L(\zeta)\ \forall \zeta \in \mathbb{R}^n$. Now assume $H(\nabla\phi) > 0$. Since H is convex and continuous, there is a hyperplane (whose normal shall be v) separating $\{p \,|\, H(p) \leq 0\}$ from $\nabla\phi$. We thus have $|\nabla\phi \cdot v| > \sup_{H(p)\leq 0} p \cdot v = L(v)$, a contradiction, i. e. $H(\nabla\phi) \leq 0$.

c) Supersolution:
Let $u(x) = g(\bar{y}) + \delta(x, \bar{y}) = g(\bar{y}) + L(x - \bar{y})$ for some $\bar{y} \in \partial\Omega$ and define $c(t) = \bar{y} + t(x - \bar{y})$. We have

$$\begin{aligned} g(\bar{y}) + L(x - \bar{y}) = u(x) &\leq u(c(t)) + \delta(c(t), x) = u(c(t)) + (1 - t)L(x - \bar{y}) \\ &\leq g(\bar{y}) + \delta(\bar{y}, c(t)) + (1 - t)L(x - \bar{y}) = g(\bar{y}) + L(x - \bar{y}) \end{aligned}$$

so that $u(c(t)) = g(\bar{y}) + tL(x - \bar{y}) = g(\bar{y}) + L(c(t) - \bar{y})$. Now let $u - \phi$ have a local minimum in x, i. e. $\phi(x) - \phi(x') \geq u(x) - u(x')$, and set $x = c(1)$, $x' = c(1 - s)$. We obtain

$$\begin{aligned} &\tfrac{\phi(x) - \phi(c(1-s))}{s} \geq \tfrac{u(x) - u(c(1-s))}{s} \\ \Rightarrow \quad &\nabla\phi(x) \cdot (x - \bar{y}) = \nabla\phi(x) \cdot \dot{c}(1) \geq \nabla u \cdot \dot{c}(1) = L(x - \bar{y}) = \sup_{H(p) \leq 0} (x - \bar{y}) \cdot p. \end{aligned}$$

Hence, $(\nabla\phi(x) + \alpha(x - \bar{y})) \cdot (x - \bar{y}) > \sup_{H(p)\leq 0}(x - \bar{y}) \cdot p$ for all $\alpha > 0$ and thus $H(\nabla\phi(x) + \alpha(x - \bar{y})) > 0$ and $H(\nabla\phi) \geq 0$ by continuity.

d) Boundary data:
Let $x \in \partial\Omega$, then $g(x) - g(y) \leq \delta(y, x)$ implies $g(x) \leq g(y) + \delta(y, x)$ for all $y \in \partial\Omega$ and thus $g(x) \leq u(x)$. Furthermore, $u(x) \leq g(x) + \delta(x, x) = g(x)$ and hence $u(x) = g(x)$.

□

For uniqueness we require additional conditions on H, as the following example shows. Below we will state two possible uniqueness results.

Beispiel 33. *Let $\Psi \in C^1(\Omega)$, $\Psi = 0$ on $\partial\Omega$, and consider $H(x, p) = |p|^2 - |\nabla\Psi(x)|^2$. Then $u = \Psi$ and $u = -\Psi$ are both (classical and viscosity) solutions of $0 = H(x, \nabla u(x))$.*

Theorem 34 (Uniqueness via comparison). *Let $\Omega \subset \mathbb{R}^n$ be bounded and open and $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ continuous with*

- *$H(x, u, p) - H(x, v, p) > \gamma(u - v)$ for a $\gamma > 0$,*
- *$|H(x, u, p) - H(y, u, p)| \leq C|y - x|(1 + |p|)$ for a $C > 0$.*

If u is a viscosity subsolution and v a viscosity supersolution of $0 = H(x, u(x), \nabla u(x))$ with $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω. Hence, the viscosity solution is unique.

Proof idea. Suppose u and v are smooth and $u - v$ has a maximum at $x_0 \in \overline{\Omega}$ with $u(x_0) - v(x_0) > 0$. By the definition of viscosity super- and subsolutions, we have

$$\begin{aligned} H(x_0, u(x_0), \nabla v(x_0)) &\leq 0, \\ H(x_0, v(x_0), \nabla u(x_0)) &\geq 0. \end{aligned}$$

Since $\nabla(u - v)(x_0) = 0$, we have $\nabla u(x_0) = \nabla v(x_0)$ and thus

$$0 \geq H(x_0, u(x_0), \nabla v(x_0)) = H(x_0, u(x_0), \nabla u(x_0)) > H(x_0, v(x_0), \nabla u(x_0)) \geq 0,$$

a contradiction. Non-smooth u, v require more work. □

Theorem 35 (Uniqueness for Hamilton–Jacobi–Bellman equation). *Let $H : (\mathbb{R} \times \mathbb{R}^{n-1}) \times \mathbb{R}^n \to \mathbb{R}$, $H((t, x), (p^t, p^x)) = p^t + \tilde{H}(x, p^x)$ for $\tilde{H}$ continuous with*

- $|\tilde{H}(x,p) - \tilde{H}(x,q)| \leq C|p-q|$ *for a* $C > 0$,
- $|\tilde{H}(x,p) - \tilde{H}(y,p)| \leq C|y-x|(1+|p|)$.

Then there is at most one viscosity solution of $0 = H((t,x),(u_t(t,x),\nabla u(t,x))) = u_t + \tilde{H}(x,\nabla u)$ *with given boundary data at* $t = 0$.

Proof. See e. g. Evans, "PDEs", p. 587 □

(Semilinear) partial differential equations of second order

In this section we consider semilinear equations of the form

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j}(x) + c(\nabla u(x), u(x), x) = 0 \tag{19}$$

with the matrix $A = (a_{ij})_{ij}$ symmetric (if u is twice continuously differentiable, the matrix can be symmetrised due to Schwarz' theorem).

Definition 36 (Classification)**.** *The second order partial differential equation* (19) *is called*

- *elliptic, if A has n strictly positive eigenvalues,*
- *parabolic, if A has a zero eigenvalue,*
- *hyperbolic, if A has one positive and $n-1$ negative eigenvalues.*

Since multiplication with -1 does not change the equation, positive and negative may be exchanged in the above. Nonlinear PDEs can be classified locally by their linearisation with respect to the second order derivatives. Since A may vary spatially, the PDE may change its type.

Elliptic PDEs

Laplace's equation

$$\Delta u = 0 \tag{20}$$

physically describes the equilibrium of a diffusing quantity such as heat.

temperature:	$u : \Omega \to \mathbb{R}$ (in a piece of material Ω)
conductivity:	$a > 0$ (material parameter)
heat flux:	$F = -a\nabla u$ (in direction of negative temperature gradient)
equilibrium:	net flux $\int_{\partial V} F \cdot \nu \,\mathrm{d}x$ into $V \subset \Omega$ is zero, hence

$$0 = \int_{\partial V} F \cdot \nu \,\mathrm{d}x = \int_V \mathrm{div} F \,\mathrm{d}x = -a \int_V \Delta u \,\mathrm{d}x,$$

and (22) follows since V is arbitrary.

Definition 37 (Harmonic function)**.** *A twice continuously differentiable function u satisfying* (22) *is called a* harmonic function.