Definition 44 (Fundamental solution). *The function*

$$\Phi(x) = \begin{cases} -\frac{1}{2}|x| & (n=1) \\ -\frac{1}{2\pi}\log|x| & (n=2) \\ -\frac{1}{\cancel{n}(n-2)|\partial B_1(0)|}\frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases}$$

solves Laplace's equation on $\mathbb{R}^n \setminus \{0\}$ *and is called the* fundamental solution *of Laplace's equation.*

Definition 45 (Delta-distribution). *The linear operator* $\hat{\delta}: C^0(\mathbb{R}^n) \to \mathbb{R}$, $\hat{\delta}(u) = u(0)$, *is called the* δ-distribution. *One also uses the notation*

$$\hat{\delta}(u) = \int_{\mathbb{R}^n} \delta(x)u(x)\,\mathrm{d}x,$$

thinking of δ *like a function which is zero everywhere except at 0, where it is infinite, and such that* $\int_{\mathbb{R}^n}\delta(x)\,\mathrm{d}x = 1$.

Multiplying $\Delta\Phi$ with a smooth function ψ with $\psi = 0$ on $\partial\Omega$ and integrating by parts twice, we obtain

$$\int_\Omega \psi(x)\Delta\Phi(x)\,\mathrm{d}x = \int_\Omega \Phi(x)\Delta\psi(x)\,\mathrm{d}x.$$

The following is to be understood in this sense.

Theorem 46 (Fundamental solution). *We have*

$$-\Delta\Phi(x) = \delta(x)$$

for the δ*-distribution, i. e.* $\Delta\Phi(x) = 0$ *on* $\mathbb{R}^n\setminus\{0\}$ *and* $\int_{\mathbb{R}^n}\Phi(x)\Delta\psi(x)\,\mathrm{d}x = \psi(0)$ *for all smooth functions* ψ *with compact support.*

Proof. We already know $\Delta\Phi(x) = 0$ for $x \in \mathbb{R}^n \setminus \{0\}$.

Fix $\varepsilon > 0$. $\int_{\mathbb{R}^n} \Phi(x)\Delta\psi(x)\,dx = \underbrace{\int_{B_\varepsilon(0)} \Phi(x)\Delta\psi(x)\,dx}_{I_\varepsilon} + \underbrace{\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Phi(x)\Delta\psi(x)\,dx}_{J_\varepsilon}$

$$|I_\varepsilon| \le \|\psi\|_{C^2}\int_{B_\varepsilon(0)} |\Phi(x)|\,dx \le \begin{cases} C\varepsilon^2|\log\varepsilon| & \text{for } n=2 \\ C\varepsilon^2 & \text{for } n\neq 2 \end{cases}$$

$$J_\varepsilon = \int_{\partial B_\varepsilon(0)} \Phi(x)\frac{\partial\psi}{\partial\nu}(x)\,dx - \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \nabla\Phi\cdot\nabla\psi\,dx$$

$$= \underbrace{\int_{\partial B_\varepsilon(0)} \Phi(x)\frac{\partial\psi}{\partial\nu}(x)\,dx}_{\le \|\psi\|_{C^1}\int_{\partial B_\varepsilon(0)}|\Phi(x)|\,dx \le C\begin{cases} C\varepsilon|\log\varepsilon| & \text{for } n=2 \\ C\varepsilon & \text{for } n\neq 2\end{cases}} \underbrace{- \int_{\partial B_\varepsilon(0)} \psi(x)\frac{\partial\Phi}{\partial\nu}(x)\,dx}_{=:K_\varepsilon} + \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \underbrace{\Delta\Phi(x)}_{=0}\psi(x)\,dx$$

$$K_\varepsilon = \left.\begin{cases} -(\psi(-\varepsilon)+\psi(\varepsilon))/2 & \text{for } n=1 \\ -\int_{\partial B_\varepsilon(0)} \psi(x)\frac{1}{\varepsilon^{n-1}|\partial B_1(0)|}\,dx & \text{for } n>1 \end{cases}\right\} = -\fint_{\partial B_\varepsilon(0)} \psi(x)\,dx$$

Altogether, $\int_{\mathbb{R}^n} \Phi(x)\Delta\psi(x)\,dx = -\fint_{\partial B_\varepsilon(0)} \psi(x)\,dx + O(\varepsilon|\log\varepsilon|) \xrightarrow[\varepsilon\to 0]{} -\psi(0).$ □

Now assume $\partial\Omega$ is Lipschitz. In the same sense as before, consider the solution of the following problem,

$$\begin{cases} -\Delta G^y(x) = \delta(x-y) & \text{in } \Omega \\ G^y = 0 & \text{on } \partial\Omega. \end{cases} \tag{25}$$

Motivation: If we manage to find G^y for all $y \in \Omega$, then

$$u(x) = \int_\Omega G^x(y) f(y)\,\mathrm{d}y \tag{26}$$

satisfies (informally)

$$-\Delta u(x) = \int_\Omega -\Delta G^x(y) f(y)\,\mathrm{d}y = f(y)$$

Bemerkung 47. *Obviously, $G^y(x) = \Phi(x-y) - \phi^y(x)$ with ϕ^x a solution to*

$$\begin{cases} \Delta\phi^y = 0 & \textit{in } \Omega \\ \phi^y = \Phi(x-y) & \textit{on } \partial\Omega. \end{cases} \tag{27}$$

Theorem 48 (Green's formula). *If $u \in C^2(\overline{\Omega})$ solves the Dirichlet problem* (22) *with* (23), *then*

$$u(x) = -\int_{\partial\Omega} g(y) \frac{\partial G^x(y)}{\partial\nu}\,\mathrm{d}y + \int_\Omega f(y) G^x(y)\,\mathrm{d}y.$$

Proof. By Thm 46, for $x \in \Omega$,

$$\begin{aligned} u(x) &= \int_\Omega u(y)\,\Delta\Phi(y-x)\,dy \\ &= \int_\Omega u(y)\,\Delta G^x(y)\,dy \\ &= \int_{\partial\Omega} u(y)\nabla G^x(y)\cdot\nu\,dy - \int_\Omega \nabla u(y)\cdot\nabla G^x(y)\,dy \\ &= \int_{\partial\Omega} g(y)\frac{\partial G^x(y)}{\partial\nu}\,dy - \int_{\partial\Omega} \nabla u(y)\cdot\nu\,\underbrace{G^x(y)}_{=0}\,dy + \int_\Omega \underbrace{\Delta u(y)}_{=f(y)}\,G^x(y)\,dy. \end{aligned}$$

□

Beispiel 49 (Green's function for a half-space). *Green's function G^y for $y \in \Omega = \{x \in \mathbb{R}^n \mid x_n > 0\}$ is found by the* method of images*: Note that $\phi^y(x) = \Phi(x - \cancel{(-y)}\,y^*)$ satisfies* (27) *so that*

$$G^y(x) = \Phi(x-y) - \Phi(x \cancel{+} y^*).$$

As a specific example, in 2D, $G^y(x) = \frac{1}{4\pi}\log\left(\frac{|x-y|^2}{|x+y|^2}\right)$.

Beispiel 50 (Green's function for a disk). *Green's function G^y for $y \in \Omega = B_r(0)$ is found similarly: Note that for $\tilde{y} = \frac{r^2}{|y|^2}y$ the ratio $\frac{|x-y|}{|x-\tilde{y}|}$ is constant on $x \in \partial\Omega$ and given by $|y|/r$. Thus,*

$$G^y(x) = \Phi(x-y) - \Phi((x-\tilde{y})|y|/r).$$

An analogous approach can be taken for the Neumann problem.

Excursion: Hölder and Sobolev spaces (We just state the results & refer to literature for proofs)

To understand the existence and regularity of solutions to Poisson's or more general elliptic equations, we need to introduce some function spaces.

For a continuous function $u \in C^0(\overline{\Omega})$ on some open bounded $\Omega \subset \mathbb{R}^n$ and for $\gamma \in [0,1]$, define

$$[u]_\gamma = \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

Definition 51 (Hölder space). *For $u \in C^k(\overline{\Omega})$ define the* Hölder norm

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{C^0(\overline{\Omega})} + \sum_{|\alpha|=k} [D^\alpha u]_\gamma .$$

The function space

$$C^{k,\gamma}(\overline{\Omega}) = \{u \in C^k(\overline{\Omega}) \mid \|u\|_{C^{k,\gamma}(\overline{\Omega})} < \infty\}$$

is called the Hölder space *with exponent γ.*

Theorem 52 (Hölder space). *The Hölder space with the Hölder norm is a Banach space, i. e. $\|\cdot\|_{C^{k,\gamma}(\overline{\Omega})}$ is a norm, and any Cauchy sequence in the Hölder space converges.*

Proof. Homework! □

Notice $C^{k,0} = C^k$ and $C^{0,1}$ is the space of Lipschitz-continuous functions.

Next we introduce a weaker notion of differentiability.

Definition 53. *Let $u, v \in L^1_{\mathrm{loc}}(\Omega)$ and α be a multiindex. v is called the α^{th}* weak derivative *of u,*

$$D^\alpha u = v,$$

if

$$\int_\Omega u D^\alpha \psi \,\mathrm{d}x = (-1)^|\alpha| \int_\Omega v\psi \,\mathrm{d}x \tag{28}$$

for all test functions $\psi \in C_c^\infty(\Omega)$ (infinitely smooth functions with compact support in Ω).

Bemerkung 54. *If u is smooth,* (28) *is exactly the result of k times integrating by parts, and v is the classical derivative.*

Beispiel 55. *Set $\Omega = (0,2)$ and*

- $u(x) = \begin{cases} x & \text{if} 0<x\le 1 \\ 1 & \text{if} 1<x<2 \end{cases}$, $v(x) = \begin{cases} 1 & \text{if} 0<x\le 1 \\ 0 & \text{if} 1<x<2 \end{cases}$, *then $v = Du$, since for any $\psi \in C_c^\infty(\Omega)$*

$$\int_0^2 u\psi' \,\mathrm{d}x = \int_0^1 x\psi' \,\mathrm{d}x + \int_1^2 \psi' \,\mathrm{d}x = -\int_0^1 \psi \,\mathrm{d}x + \psi(1) - \psi(1) = -\int_0^2 v\psi \,\mathrm{d}x,$$

- $u(x) = \begin{cases} x & \text{if} 0<x\le 1 \\ 2 & \text{if} 1<x<2 \end{cases}$, *then u does not have a weak derivative, since*

$$-\int_0^2 v\psi \,\mathrm{d}x = \int_0^2 u\psi' \,\mathrm{d}x = \int_0^1 x\psi' \,\mathrm{d}x + 2\int_1^2 \psi' \,\mathrm{d}x = -\int_0^1 \psi \,\mathrm{d}x - \psi(1)$$

cannot be fulfilled for all $\psi \in C_c^\infty(\Omega)$ by any $v \in L^1_{\mathrm{loc}}(\Omega)$.

Theorem 52 (Hölder space). *The Hölder space with the Hölder norm is a Banach space, i. e. $\|\cdot\|_{C^{k,\gamma}(\overline{\Omega})}$ is a norm, and any Cauchy sequence in the Hölder space converges.*

Proof. Homework! □

Notice $C^{k,0} = C^k$ and $C^{0,1}$ is the space of Lipschitz-continuous functions.
Next we introduce a weaker notion of differentiability.

Definition 53. *Let $u, v \in L^1_{\text{loc}}(\Omega)$ and α be a multiindex. v is called the α^{th}* weak derivative *of u,*

$$D^\alpha u = v,$$

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$$\int_\Omega u D^\alpha \psi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\psi \,\mathrm{d}x \tag{28}$$

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Bemerkung 54. *If u is smooth,* (28) *is exactly the result of k times integrating by parts, and v is the classical derivative.*

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- $u(x) = \begin{cases} x & \text{if } 0<x\leq 1 \\ 1 & \text{if } 1<x<2 \end{cases}$, $v(x) = \begin{cases} 1 & \text{if } 0<x\leq 1 \\ 0 & \text{if } 1<x<2 \end{cases}$, *then $v = Du$, since for any $\psi \in C_c^\infty(\Omega)$*

$$\int_0^2 u\psi' \,\mathrm{d}x = \int_0^1 x\psi' \,\mathrm{d}x + \int_1^2 \psi' \,\mathrm{d}x = -\int_0^1 \psi \,\mathrm{d}x + \psi(1) - \psi(1) = -\int_0^2 v\psi \,\mathrm{d}x,$$

- $u(x) = \begin{cases} x & \text{if } 0<x\leq 1 \\ 2 & \text{if } 1<x<2 \end{cases}$, *then u does not have a weak derivative, since*

$$-\int_0^2 v\psi \,\mathrm{d}x = \int_0^2 u\psi' \,\mathrm{d}x = \int_0^1 x\psi' \,\mathrm{d}x + 2\int_1^2 \psi' \,\mathrm{d}x = -\int_0^1 \psi \,\mathrm{d}x - \psi(1)$$

  cannot be fulfilled for all $\psi \in C_c^\infty(\Omega)$ by any $v \in L^1_{\text{loc}}(\Omega)$.

Definition 56 (Sobolev space). *Let $p \in [1,\infty]$, $k \in \mathbb{N}_0$. Recall the Lebesgue space*

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ measurable with } \|u\|_{L^p(\Omega)} < \infty\}$$

with the norm $\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u|^p \,\mathrm{d}x\right)^{1/p} & (p<\infty) \\ \operatorname{esssup}_\Omega |u| & (p=\infty) \end{cases}$. *The space*

$$W^{k,p}(\Omega) = \{u \in L^1_{\text{loc}}(\Omega) \mid \text{the weak derivative } D^\alpha u \text{ exists for all } |\alpha| \leq k \text{ with } D^\alpha u \in L^p(\Omega)\}$$

with the norm

$$\|u\|_W^{k,p}(\Omega) = \begin{cases} \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p \,\mathrm{d}x\right)^{1/p} & 1 \leq p < \infty \\ \sum_{|\alpha|\leq k} \operatorname{esssup}_\Omega |D^\alpha u| & p = \infty \end{cases}$$

is called a Sobolev space.
$W_0^{k,p}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Note $W^{0,p}(\Omega) \equiv L^p(\Omega)$.

Theorem 57 (Sobolev space). *The Sobolev space space with the Sobolev norm is a Banach space.*

Proof. See e. g. Evans, "PDEs", p. 262. □

Bemerkung 58. *The spaces*

$$H^k(\Omega) \equiv W^{k,2}(\Omega)$$

are Hilbert spaces, i. e. their norm is induced by an inner product,

$$(u,v)_{H^k(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha u D^\alpha v \,\mathrm{d}x.$$

Theorem 59 (Hölder's inequality). *Let* $f \in L^p$, $g \in L^{p^*}$, $p, p^* \in [1,\infty]$ *with*

$$\tfrac{1}{p} + \tfrac{1}{p^*} = 1.$$

Then

$$\int_\Omega |fg| \,\mathrm{d}x \le \|f\|_{L^p(\Omega)} \|g\|_{L^{p^*}(\Omega)}.$$

Proof. See e. g. Alt, "Lineare Funktionalanalysis", p. 52. □

Theorem 60 (Trace theorem). *Let* Ω *be bounded and have Lipschitz boundary. There exists a continuous linear operator* $T: W^{1,p}(\Omega) \to L^p(\partial\Omega)$, *the* trace, *with*

(i) $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$,

(ii) $\|Tu\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$,

(iii) $Tu = 0 \quad \Leftrightarrow \quad u \in W_0^{1,p}(\Omega)$,

where the constant $C > 0$ *only depends on* p *and* Ω. *For simplicity, we will simply refer to* u *on* $\partial\Omega$ *when we mean its trace.*

Proof. See e. g. Evans, "PDEs", p. 272. □

Theorem 61 (Poincaré's inequality). *Let* $\Omega \subset \mathbb{R}^n$ *be bounded, open, connected with Lipschitz boundary. There exists a constant* $C = C(n,p,\Omega)$ *with*

$$\|u - ⨍_\Omega u \,\mathrm{d}x\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$ *and*

$$\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}$$

for all $u \in W_0^{1,p}(\Omega)$.

Proof. See e. g. Alt, "Lineare Funktionalanalysis", p. 171. □

Theorem 62 (Sobolev embedding). *Let* $\Omega \subset \mathbb{R}^n$ *open, bounded with Lipschitz boundary,* $m_1, m_2 \in \{0,1,2,\ldots\}$, $p_1, p_2 \in [1,\infty)$. *If*

$$m_1 \ge m_2 \text{ and } m_1 - \tfrac{n}{p_1} \ge m_2 - \tfrac{n}{p_2}$$

then $W^{m_1,p_1}(\Omega) \subset W^{m_2,p_2}(\Omega)$ *and there is a constant* $C > 0$ *s. t. for all* u

$$\|u\|_{W^{m_1,p_1}(\Omega)} \le C\|u\|_{W^{m_2,p_2}(\Omega)}.$$

even embeds compactly in

If the inequalities are strict, $W^{m_1,p_1}(\Omega)$ ~~*is even a compact subset of*~~ $W^{m_2,p_2}(\Omega)$.

Proof. See e. g. Alt, "Lineare Funktionalanalysis", p. 328. □

Theorem 63 (Hölder embedding). *Let $\Omega \subset \mathbb{R}^n$ open, bounded with Lipschitz boundary, $m, k \in \{0, 1, 2, \ldots\}$, $p \in [1, \infty)$, $\alpha \in [0, 1]$. If*

$$m - \tfrac{n}{p} \geq k + \alpha \text{ and } \alpha \neq 0, 1$$

then $W^{m,p}(\Omega) \subset C^{k,\alpha}(\overline{\Omega})$ and there is a constant $C > 0$ s. t. for all u

$$\|u\|_{W^{m,p}(\Omega)} \leq C\|u\|_{C^{k,\alpha}(\overline{\Omega})}.$$

If $m - \frac{n}{p} < k + \alpha$, $W^{m,p}(\Omega)$ even embeds compactly in ~~*is even a compact subset of*~~ *$C^{k,\alpha}(\overline{\Omega})$.*

Proof. See e. g. Alt, "Lineare Funktionalanalysis", p. 333. □

Weak solutions

For $\Omega \subset \mathbb{R}^n$ open and bounded, consider the elliptic Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{29}$$

with $f : \Omega \to \mathbb{R}$, $g \in H^1(\Omega)$, and

$$Lu(x) = -\operatorname{div}(A(x)\nabla u(x)) + b(x) \cdot \nabla u(x) + c(x)u(x)$$

for $A : \Omega \to \mathbb{R}^{n \times n}_{\text{symm}}$, $b : \Omega \to \mathbb{R}^n$, $c : \Omega \to \mathbb{R}$.

Definition 64 (Ellipticity). *The operator L is called (uniformly) elliptic, if there exists a constant $\theta > 0$ s. t.*

$$\xi^T A(x)\xi \geq \lambda|\xi|^2$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

To define a weak solution, we again multiply the PDE by a smooth function and integrate by parts, which motivates the following.

Definition 65 (Weak solution). *$u \in g + H^1_0(\Omega)$ is called a* weak solution *to* (29) *if*

$$B(u, v) := \int_\Omega \nabla v(x)^T A(x)\nabla u(x) + b(x) \cdot \nabla u(x)v(x) + c(x)u(x)v(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x \quad \forall v \in H^1_0(\Omega). \tag{30}$$

In the following assume there exist constants $\lambda, \Lambda, \nu > 0$ such that for all $x \in \Omega$, $\xi, \zeta \in \mathbb{R}^n$

- $\xi^T A(x)\xi \geq \lambda|\xi|^2$,
- $|\xi^T A(x)\zeta| \leq \Lambda|\xi||\zeta|$,
- $\lambda^{-2}|b(x)|^2 + \lambda^{-1}|c(x)| \leq \nu^2$,
- $c(x) \geq 0$.

We will next prove existence and uniqueness of weak solutions. As before, uniqueness is based on a maximum principle. Let us abbreviate $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$.