Hausaufgabe 9 (Abgabe bis Mittwoch, 18. Juni, 12 Uhr)

1. Consider the boundary value problem (in $\mathbb{R}^2$)

$$\begin{cases} \Delta u = c & \text{for } r = |x| < 1\,, \\ \frac{\partial u}{\partial r} = 2 & \text{on } r = 1\,. \end{cases}$$

   Show that there is no solution unless $c = 4$, and find solutions in this case. (In polar coordinates r, θ, $\Delta u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$.) (4 pt)

2. A function u satisfies $\Delta u - cu = f$ in a domain $\Omega \subset \mathbb{R}^2$ with $u = g$ on $\partial\Omega$. If u exists, show that it is unique if $c > 0$. For the case $c < 0$ find non-trivial solutions when $f = g = 0$ and Ω is the unit disk, stating for which values of c these solutions exist. (In polar coordinates r, θ, $\Delta u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$.) (4 pt)

3. Prove the weak maximum principle for the case $b \neq 0$ and dimension $n \geq 3$: This time, choose the test function $v = (u-k)^+$ for $\sup_{\partial\Omega} u^+ \leq k < \sup_\Omega u^+$ and derive

$$\lambda \int_\Omega |\nabla v|^2 \,\mathrm{d}x \leq \lambda\nu \int_U v|\nabla v| \,\mathrm{d}x \leq \lambda\nu \|v\|_{L^2(U)} \|\nabla v\|_{L^2(\Omega)}$$

   with $U = \mathrm{supp}(\nabla v)$. From this inequality, derive (using Poincaré's inequality, Sobolev embedding, and Hölder's inequality)

$$\|v\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C\|v\|_{L^2(\Omega)} \leq \tilde{C}\sqrt[n]{|U|}\|v\|_{L^{\frac{2n}{n-2}}(\Omega)}$$

   for some constants $C, \tilde{C} > 0$. Now letting $k \to \sup_\Omega u^+$, we see $u = \sup_\Omega u^+$ on a set $\tilde{U} \subset U = \mathrm{supp}\nabla v$ with $|\tilde{U}| \geq \frac{1}{\tilde{C}^n}$, however, this implies $\nabla v = 0$ on $\tilde{U}$, a contradiction. (7 pt)

Schließlich, wie in der Vorlesung versprochen, hier noch eine Anleitung, wie Existenz einer schwachen Lösung für $b \neq 0$ gezeigt werden kann (dies gehört nicht zur Hausaufgabe):
Prove existence of a weak solution to $Lu = f$ on Ω, $u = g$ on $\partial\Omega$ under the conditions from the lecture with $b \neq 0$:

- Show $B(u,u) \geq \frac{\lambda}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \lambda\nu^2 \int_\Omega u^2 \,\mathrm{d}x$. (Hint: You may need Young's inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$.)
- For any $u \in L^2(\Omega)$, $l_u : H_0^1(\Omega) \to \mathbb{R}, v \mapsto l_u(v) = \int_\Omega uv \,\mathrm{d}x$ is a continuous linear operator. Thus, by Riesz' theorem, there is an element $Iu \in H_0^1(\Omega)$ such that $l_u(v) = (Iu, v)_{H_0^1(\Omega)}$ $\forall v \in H_0^1(\Omega)$. Show that $I : H_0^1(\Omega) \to H_0^1(\Omega)$ is a linear operator and it is compact, i. e. $I(H)$ is a compact subset of $H_0^1(\Omega)$ for any bounded subset H of $H_0^1(\Omega)$. (Hint: Use that the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and that the composition of a compact with a continuous operator is compact.)
- Choose $\sigma > 0$ such that $B_\sigma(u,u) = B(u,u) + \sigma\int_\Omega u^2 \,\mathrm{d}x$ is coercive. Show that we seek $\tilde{u} \in H_0^1(\Omega)$ with $B_\sigma(\tilde{u}, v) = (R(F) + \sigma Iu, v)_{H_0^1(\Omega)}$ and use the Lax–Milgram theorem to obtain an equation $(A - \sigma I)u = R(F)$.
- Using the previously proved uniqueness, derive the existence of a solution from the *Fredholm alternative*: If T is a compact operator, then $x - Tx = 0$ either has a nontrivial solution, or $I - T$ is invertible with bounded inverse.