Hausaufgabe 8 (Abgabe bis Mittwoch, 4. Juni, 12 Uhr)

1. (From Evans, "PDEs") We say $v \in C^2(\Omega)$ is *subharmonic* if $-\Delta v \leq 0$ in Ω.
   (a) Prove for subharmonic v that $v(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} v \,\mathrm{d}y$ for all $B_r(x) \subset \Omega$. (1 pt)
   (b) Prove that therefore $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$. (1 pt)
   (c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic. (1 pt)
   (d) Prove $v := |\nabla u|^2$ is subharmonic, whenever u is harmonic. (1 pt)
2. (From Evans, "PDEs") Let Ω be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on Ω, such that
$$\max_{\overline{\Omega}} |u| \leq C(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f|)$$
whenever u is a smooth solution of
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega\,. \end{cases}$$
(Hint: $-\Delta(u + \frac{|x|^2}{2n}\lambda) \leq 0$ for $\lambda := \max_{\overline{\Omega}} |f|$.) (3 pt)
3. Derive the mean value formula for harmonic functions by integrating $u(y)\Delta\Phi(y-x)$ twice by parts on $B_r(x)$ for the fundamental solution Φ. (1 pt)
4. Let $\Delta u = 0$ for $r < 1$ with $u = g(\theta)$ on $r = 1$ (where r, θ are 2D polar coordinates). Derive Poisson's integral formula
$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)g(\phi)\,\mathrm{d}\phi}{1 + r^2 - 2r\cos(\theta - \phi)}\,.$$
(3 pt)
5. Find a Green's function for $\Delta u = f$ on $\{x \in \mathbb{R}^n \,|\, x_n > 0\}$ with $\partial u/\partial\nu = g$ on $\{x_n = 0\}$. (1 pt)
6. Find a Green's function for $\Delta u = f$ on $[0,1]$ with $u(0) = a$, $u(1) = b$. (1.5 pt)
7. (From Gilbarg & Trudinger, "Elliptic PDEs") Let $G^y(x)$ be the Green's function for the Dirichlet problem on a bounded domain Ω. Prove $G^y(x) = G^x(y) < 0$ for all $x, y \in \Omega$ with $x \neq y$. (1.5 pt)