Partielle Differentialgleichungen I **Institut für Numerische und Angewandte Mathematik**
SS 2014 **Prof. Dr. Benedikt Wirth**
Hausaufgabe 7 (Abgabe bis Mittwoch, 28. Mai, 12 Uhr)

1. (From Gilbarg & Trudinger, "Elliptic PDEs") Let $u \in H^2(\Omega)$, $u = 0$ on $\partial\Omega \in C^1$. Prove the interpolation inequality: For every $\varepsilon > 0$,
$$\int_\Omega |Du|^2 \,\mathrm{d}x \leq \varepsilon \int_\Omega (\Delta u)^2 \,\mathrm{d}x + \frac{1}{4\varepsilon} \int_\Omega u^2 \,\mathrm{d}x\,.$$
(1 pt)

2. (From Evans, "PDEs") Let $\Omega \subset \mathbb{R}^n$ have smooth boundary. Prove that $C^{k,\alpha}(\overline{\Omega})$, $k \in \{0, 1, \ldots\}$, $\alpha \in (0, 1]$, is a Banach space. (4 pt)

3. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $p, q, r \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ derive the generalized Hölder inequality
$$\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$$
for $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ from the standard Hölder inequality. (1 pt)

   Analogously, derive
$$\|f_1 f_2 \cdots f_m\|_{L^r(\Omega)} \leq \|f_1\|_{L^{p_1}(\Omega)} \|f_2\|_{L^{p_2}(\Omega)} \cdots \|f_m\|_{L^{p_m}(\Omega)}$$
for $\frac{1}{p_1} + \ldots + \frac{1}{p_m} = \frac{1}{r}$. (1 pt)

4. (From Gilbarg & Trudinger, "Elliptic PDEs") Derive the maximum principle for harmonic functions u by considering the necessary conditions for a relative maximum. (Hint: First show that the function $v = u + \frac{\varepsilon}{4}|x|^2$ cannot have an internal local maximum.) (2 pt)

5. Let u_k be a monotonically increasing sequence of harmonic functions on the open domain Ω. Prove *Harnack's convergence theorem*: If $u_k(x_0)$ converges for an $x_0 \in \Omega$, then the sequence converges pointwise against a harmonic function. (3 pt)

6. Assume u is harmonic in $\Omega \subset \mathbb{R}^n$. Using the mean value formula for partial derivatives of u, prove that there is a constant C such that
$$|\nabla u(x)| \leq \frac{C}{r^{n+1}} \|u\|_{L^1(B_r(x))}$$
for each ball $B_r(x) \subset \Omega$. (2 pt)

7. (From Gilbarg & Trudinger, "Elliptic PDEs") Using the previous estimate, prove *Liouville's theorem*: A harmonic function defined over $\mathbb{R}^n$ and bounded is constant. (1 pt)