Let \( x \in \mathbb{R}^n \) and \( r > 0 \). Henceforth we denote by \( B_r(x) \) the open ball of \( \mathbb{R}^n \) with center \( x \) and radius \( r \),

\[
B_r(x) = \{ y \in \mathbb{R}^n : |y - x| < r \},
\]

and by \( \partial B_r(x) \) the boundary of \( B_r(x) \),

\[
\partial B_r(x) = \{ y \in \mathbb{R}^n : |y - x| = r \}.
\]

If its center is the origin, the ball of radius \( r \) is denoted by \( B_r \) and, similarly, its spheric surface by \( \partial B_r \).

### 1. Coarea formula and polar coordinates

The following integration formula holds.

**Theorem 1.1.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be continuous and summable. Then, for each point \( x_0 \in \mathbb{R}^n \),

\[
\int_{\mathbb{R}^n} f(x) \, dx = \int_0^\infty \left( \int_{\partial B_r(x_0)} f(x) \, dS(x) \right) \, dr,
\]

where \( S \) denotes the surface measure on the boundary of \( B_r(x_0) \).

Theorem 1.1 can be proved passing in polar coordinates in \( \mathbb{R}^n \). Observe that the above theorem is a particular case of the following result.

**Theorem 1.2 (Coarea formula).** Let \( u : \mathbb{R}^n \to \mathbb{R} \) be Lipschitz and assume that for a.e. \( r \in \mathbb{R} \) the level set \( \{ x \in \mathbb{R}^n : u(x) = r \} \) is a smooth, \((n-1)\)-dimensional hypersurface in \( \mathbb{R}^n \). Suppose also \( f : \mathbb{R}^n \to \mathbb{R} \) to be continuous and summable. Then

\[
\int_{\mathbb{R}^n} f(x) |\nabla u(x)| \, dx = \int_{-\infty}^{+\infty} \left( \int_{\{u = r\}} f(x) \, dS(x) \right) \, dr.
\]

The Coarea Formula is a kind of “curvilinear” version of Fubini’s Theorem and allows to convert \( n \)-dimensional integrals into integrals over the level surfaces of a suitable function.

**Remark 1.3.** Theorem 1.1 follows from Theorem 1.2 by taking \( u(x) = |x - x_0| \).
2. Volume of the ball and measure of the spheric surface

In order to compute the volume $|B_r(x)|$ of the ball (0.1) and the measure $S(\partial B_r(x))$ of the spheric surface (0.2), let us introduce the so-called Gamma function. Let $t > 0$ and set

$$\Gamma(t) := \int_0^{+\infty} e^{-x}x^{t-1} \, dx.$$  \hspace{1cm} (0.5)

Let us first check that the definition of the Gamma function (0.5) is well-posed. Set $f(x) := e^{-x}x^{t-1}$; then, since $f(x) < x^{t-1}$ if $x > 0$ (being $e^{-x} < 1$) and $t-1 > -1$, we infer that $f$ is summable at 0. On the other hand, since $\lim_{x \to +\infty} x^{t+1}e^{-x} = 0$, there exists $M > 0$ such that $x^{t+1}e^{-x} < 1$ for all $x > M$ and, accordingly, $f(x) < 1/x^2$ for all $x > M$, which leads immediately to conclude the summability of $f$ at $+\infty$. Thus, $\Gamma(t) < \infty$ for all $t > 0$.

We see that

1. $\Gamma(1) = \int_0^{+\infty} e^{-x} \, dx = 1$
2. $\Gamma(t+1) = \int_0^{+\infty} x^t e^{-x} \, dx = t \Gamma(t)$ for all $t > 0$.

These two properties show that the Gamma function extends to $(0, \infty)$ the factorial of a number; indeed, $\forall n \in \mathbb{N}$

$$\Gamma(n+1) = n \Gamma(n) = n(n-1) \Gamma(n-1) = \cdots = n! \Gamma(1) = n!.$$

Another expression of the Gamma function is given by

$$\Gamma(t) = 2^{1-t} \int_0^{+\infty} e^{-\frac{y^2}{2}} y^{2t-1} \, dy,$$  \hspace{1cm} (0.6)

and it is obtained by applying the change of variables $x = y^2/2$ in (0.5).

Denoted by $Q_1 = [0, +\infty) \times [0, +\infty)$ the first quadrant of the plane, from (0.6) we deduce easily, applying first Fubini’s theorem and then passing in polar coordinates, that

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2 \left(\int_0^{+\infty} e^{-\frac{x^2}{2}} \, dx\right) \left(\int_0^{+\infty} e^{-\frac{y^2}{2}} \, dy\right) = 2 \iint_{Q_1} e^{-(x^2+y^2)/2} \, dx \, dy = 2 \int_0^{\pi/2} d\theta \int_0^{+\infty} g e^{-\frac{\varphi^2}{2}} \, d\varphi = \pi;$$
hence $\Gamma(1/2) = \sqrt{\pi}$.

Let $\omega_n$ and $\sigma_n$ denote the volume of the unit ball $B_1$ of $\mathbb{R}^n$ and the measure of the spheric surface of $B_1$, respectively; clearly

$$|B_r(x)| = r^n \omega_n, \quad S(\partial B_r(x)) = r^{n-1} \sigma_n. \quad (0.7)$$

**Theorem 2.1.** Let $n \geq 2$. Then, $\sigma_n = n \omega_n$.

**Proof.** Applying (0.3) with $f \equiv 1$, we get immediately

$$\omega_n = \int_{B_1} dx = \int_0^1 \left( \int_{\partial B_\varrho} dS \right) d\varrho = \int_0^1 S(\partial B_\varrho) d\varrho = \sigma_n \int_0^1 \varrho^{n-1} d\varrho = \frac{\sigma_n}{n}. \quad \square$$

Finally we give the expression of the volume $\omega_n$ of the unit ball, for all $n$.

**Theorem 2.2.** Let $n \geq 1$. Then

$$\omega_n = \frac{\pi^{n/2}}{(n/2) \Gamma(n/2)}. \quad (0.8)$$

**Proof.** Being $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$, (0.8) is verified for $n = 1$ and $n = 2$:

$$\omega_1 = \frac{\pi^{1/2}}{(1/2) \Gamma(1/2)} = 2, \quad \omega_2 = \frac{\pi}{1 \cdot \Gamma(1)} = \pi.$$  

Then, let us prove (0.8) for $n \geq 3$ by induction on $n$; suppose (0.8) is true for $n - 2$, with $n \geq 3$, and let us show that it is true for $n$.

Take $x \in B_1$; then we can write $x = (x', x'')$, with $x' = (x_1, x_2)$ and $x'' = (x_3, \ldots, x_n)$ such that

$$x' \in D_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

and

$$x'' \in (B_1)_{x'} = \{x'' \in \mathbb{R}^{n-2} : (x', x'') \in B_1\} = \{(x_3, \ldots, x_n) \in \mathbb{R}^{n-2} : x_3^2 + \cdots + x_n^2 < 1 - x_1^2 - x_2^2\}.$$
Thus, appealing to Fubini’s theorem and using the induction hypotesis, we have

\[ \omega_n = \int_{B_1} dx = \int_{D_1} dx' \int_{(B_1)'} dx'' \]

\[ = \int_{D_1} \omega_{n-2} \left(1 - x_1^2 - x_2^2\right)^{(n-2)/2} dx_1 dx_2 \]

\[ = \omega_{n-2} \int_0^{2\pi} d\theta \int_0^1 (1 - \rho)^{(n-2)/2} \rho d\rho \]

\[ = \frac{2\pi}{n} \omega_{n-2} = \frac{2\pi}{n} \frac{\pi^{(n-2)/2}}{\frac{n-2}{2} \Gamma\left(\frac{n-2}{2}\right)} \]

\[ = \frac{2\pi}{n} \frac{\pi^{(n-2)/2}}{\Gamma(n/2)} = \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)}. \]

\[ \square \]

References