SIAM J. CONTROL OPTIM.
Vol. 42, No. 6, pp. 2239–2263

SECOND ORDER SUFFICIENT CONDITIONS FOR TIME-OPTIMAL BANG-BANG CONTROL*

HELMUT MAURER† AND NIKOLAI P. OSMOLOVSKII‡

Abstract. We study second order sufficient optimality conditions (SSC) for optimal control problems with control appearing linearly. Specifically, time-optimal bang-bang controls will be investigated. In [N. P. Osmolovskii, *Sov. Phys. Dokl.*, 33 (1988), pp. 883–885; *Theory of Higher Order Conditions in Optimal Control*, Doctor of Sci. thesis, Moscow, 1988 (in Russian); *Russian J. Math. Phys.*, 2 (1995), pp. 487–516; *Russian J. Math. Phys.*, 5 (1997), pp. 373–388; *Proceedings of the Conference "Calculus of Variations and Optimal Control,"* Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 198–216; A. A. Milyutin and N. P. Osmolovskii, *Calculus of Variations and Optimal Control*, Transl. Math. Monogr. 180, AMS, Providence, RI, 1998], SSC have been developed in terms of the positive definiteness of a quadratic form on a critical cone or subspace. No systematical numerical methods for verifying SSC are to be found in these papers. In the present paper, we study explicit representations of the critical subspace. This leads to an easily implementable test for SSC in the case of a bang-bang control with one or two switching points. In general, we show that the quadratic form can be simplified by a transformation that uses a solution to a linear matrix differential equation. Particular conditions even allow us to convert the quadratic form to perfect squares. Three numerical examples demonstrate the numerical viability of the proposed tests for SSC.

Key words. optimal bang-bang control, second order sufficient conditions, Q-transformation to perfect squares, numerical verification, applications

AMS subject classifications. 49K15, 49K30, 65L10, 94C99

DOI. 10.1137/S0363012902402578

1. Introduction.

Second order sufficient optimality conditions (SSC) for optimal control problems subject to mixed control-state constraints have been studied by various authors; cf. Dunn [8, 9]; Malanowski [22]; Maurer and Pickenhain [30]; Maurer and Oberle [29]; Milyutin and Osmolovskii [31]; Osmolovskii [35, 36, 37, 38, 39, 40]; and Zeidan [48]. SSC amount to testing the positive definiteness of a certain quadratic form on the so-called critical cone or subspace. Provided that the strict Legendre–Clebsch condition holds, a well-known numerical recipe allows the conversion of the quadratic form to a perfect square. Namely, it suffices to check that an associated Riccati matrix differential equation has a bounded solution along the extremal trajectory. This test has been performed in a number of practical examples and has played a crucial role in sensitivity analysis of parametric control problems; cf., e.g., Augustin, Malanowski, and Maurer [2, 21, 22, 23, 24, 25, 27, 28]. Recently, the Riccati approach has been also extended to discontinuous controls (broken extremals) by Osmolovskii and Lempio [42].

The above mentioned tests for SSC are not applicable to optimal control problems with control appearing *linearly*. Bang-bang controls do belong to this class of problems. Though first and higher order *necessary* optimality conditions for bang-bang controls have been studied, e.g., in Bressan [3], Schättler [44], and Sussmann

*Received by the editors February 13, 2002; accepted for publication (in revised form) October 15, 2003; published electronically May 17, 2004.

http://www.siam.org/journals/sicon/42-6/40257.html

†Westfälische Wilhelms-Universität Münster, Institut für Numerische Mathematik, Einsteinstrasse 62, D–48149 Münster, Germany (maurer@math.uni-muenster.de). This author was supported by the Deutsche Forschungsgemeinschaft under grant MA 691/8–3.

‡Department of Applied Mathematics, Moscow State Civil Engineering University (MISI), Jaroslavskoe sh. 26, 129337 Moscow, Russia (nikolai@osmolovskii.msk.ru). This author was supported by the Russian Foundation for Fundamental Research under grant 00-15-96109.

[45, 46, 47], there is no systematic study of *sufficient* optimality conditions and their numerical verification. A general set of second order necessary and sufficient conditions for an extremal with a discontinuous control (cf. Osmolovskii [37]) can be derived from the theory of higher order conditions in Levitin, Milyutin, and Osmolovskii [20]. The main results for bang-bang controls which follow from these general conditions are given in Milyutin and Osmolovskii [31]. Some proofs missing in that book will appear in Osmolovskii [40]. Only recently, other authors have derived SSC for general bang-bang control problems with *fixed* final time (cf. Agrachev, Stefani, and Zezza [1]; Ledzewicz and Schättler [19]; and Noble and Schättler [33]).

In this paper, we shall consider the special class of *time-optimal* bang-bang controls with given initial and terminal state. To our knowledge, the paper of Sarychev [43] seems to be the only study on SSC for this class of problems. However, it is not clear how one might apply the SSC in this article to practical examples. Thus our aim is to derive SSC in a form that is also suitable for practical verification. The two main tools to achieve this goal will be (1) a detailed study of the critical subspace and (2) an adaptation of the above mentioned Riccati approach to bang-bang controls. The organization of the paper then is as follows. In section 2, Pontryagin's minimum principle and the bang-bang property are discussed. The accessory problem, respectively, the quadratic form and the critical subspace are introduced in section 3. SSC are given in a general form that is evaluated particularly for bang-bang controls with one or two switching points. Section 4 presents the Q-transformation whereby the quadratic form is simplified with the help of the solution Q of a linear differential equation. Positive definiteness conditions are given under which the quadratic form can be transformed into perfect squares. In section 5, we shall discuss three numerical examples that illustrate several numerical procedures for verifying positive definiteness of the corresponding quadratic forms.

2. Time-optimal bang-bang control problems.

2.1. Statement of the problem, strong minimum. We consider time-optimal control problems with control appearing linearly. Let $x(t) \in \mathbb{R}^{d(x)}$ denote the state variable and $u(t) \in \mathbb{R}^{d(u)}$ the control variable in the time interval $t \in \Delta = [0, T]$ with a nonfixed final time $T > 0$. For simplicity, the initial and terminal states are fixed in the following control problem:

$$\text{Minimize the final time} \quad T \tag{2.1}$$

subject to the constraints on the interval $\Delta = [0, T]$,

$$dx/dt = \dot{x} = f(t, x, u) = a(t, x) + B(t, x)u, \tag{2.2}$$

$$x(0) = x_0, \quad x(T) = x_1, \tag{2.3}$$

$$u(t) \in U, \quad (t, x(t)) \in \mathcal{Q}\,. \tag{2.4}$$

Here, x_0, x_1 are given points in $\mathbb{R}^{d(x)}$, $\mathcal{Q} \subset \mathbb{R}^{1+d(x)}$ is an open set, and $U \subset \mathbb{R}^{d(u)}$ is a convex polyhedron. The functions a, B are twice continuously differentiable on $\mathcal{Q}$ with B being a $d(x) \times d(u)$ matrix function. A trajectory or control process

$$\mathcal{T} = \{\, (x(t), u(t)) \mid t \in [0, T] \,\}$$

is said to be *admissible* if $x(\cdot)$ is absolutely continuous, $u(\cdot)$ is measurable and essentially bounded, and the pair of functions $(x(t), u(t))$ satisfies the constraints (2.2)–(2.4) on the interval $\Delta = [0,T]$. The component $x(t)$ will be called the *state trajectory*.

DEFINITION 2.1. *An admissible trajectory* $\mathcal{T}^0 = \{(x^0(t), u^0(t)) \mid t \in [0,T^0]\}$ *is said to be strongly (resp., strictly strongly) locally time-optimal if there exists* $\varepsilon > 0$ *such that* $T \geq T^0$ *(resp.,* $T > T^0$*) holds for all admissible* $\mathcal{T} = \{(x(t), u(t)) \mid t \in [0,T]\}$ *(resp., different from* $\mathcal{T}^0$*) with* $|T - T^0| < \varepsilon$ *and* $\max_{[0,T^0]\cap[0,T]} |x(t) - x^0(t)| < \varepsilon$.

2.2. Minimum principle. Let

$$\mathcal{T} = \{\, (x(t), u(t)) \mid t \in [0,T]\,\}$$

be a fixed admissible trajectory such that the control $u(\cdot)$ is a piecewise constant function on the interval $\Delta = [0,T]$ with *finitely many* points of discontinuity. In order to simplify notation we shall not use such symbols and indices as zero, hat, or asterisk to distinguish this trajectory from others. Denote by

$$\theta = \{t_1, \ldots, t_s\}, \quad 0 < t_1 < \cdots < t_s < T\,,$$

the finite set of all discontinuity points (jump points) of the control $u(t)$. Then $\dot{x}(t)$ is a piecewise continuous function whose discontinuity points belong to the set θ and, thus, $x(t)$ is a piecewise smooth function on Δ. Henceforth, we shall use the notation

$$[u]^k = u^{k+} - u^{k-}$$

to denote the jump of the function $u(t)$ at the point $t_k \in \theta$, where

$$u^{k-} = u(t_k - 0), \quad u^{k+} = u(t_k + 0)$$

are, respectively, the left-hand and the right-hand values of the control $u(t)$ at t_k. Similarly, we denote by $[\dot{x}]^k$ the jump of the function $\dot{x}(t)$ at the same point.

Let us formulate the first order necessary conditions of optimality for the trajectory $\mathcal{T}$, the *Pontryagin minimum principle*. To this end we introduce the *Pontryagin function* or Hamiltonian function

$$H(t,x,u,\psi) = \psi f(t,x,u) = \psi a(t,x) + \psi B(t,x)u, \tag{2.5}$$

where ψ is a row-vector of dimension $d(x)$, while x, u, f are column-vectors. In what follows, partial derivatives of the Pontryagin function and all other functions will be denoted by subscripts referring to the respective variables.

The factor of the control u in the Pontryagin function is called the *switching function*

$$\sigma(t,x,\psi) = \psi B(t,x).$$

Consider the pair of functions

$$\psi_0(\cdot) : \Delta \to \mathbb{R}^1, \quad \psi(\cdot) : \Delta \to \mathbb{R}^{d(x)},$$

which are continuous on Δ and continuously differentiable on each interval of the set $\Delta \setminus \theta$. Denote by M_0 the set of normed pairs of functions $(\psi_0(\cdot), \psi(\cdot))$ satisfying the conditions

$$\psi_0(T) \geq 0, \quad |\psi(0)| = 1, \tag{2.6}$$

$$\dot{\psi}(t) = -H_x(t, x(t), u(t), \psi(t)) \quad \forall\, t \in \Delta \setminus \theta, \tag{2.7}$$

$$\dot{\psi}_0(t) = -H_t(t, x(t), u(t), \psi(t)) \quad \forall\, t \in \Delta \setminus \theta, \tag{2.8}$$

$$\min_{u \in U} H(t, x(t), u, \psi(t)) = H(t, x(t), u(t), \psi(t)) \quad \forall\, t \in \Delta \setminus \theta, \tag{2.9}$$

$$H(t, x(t), u(t), \psi(t)) + \psi_0(t) = 0 \quad \forall\, t \in \Delta \setminus \theta\,. \tag{2.10}$$

Then the condition $M_0 \neq \emptyset$ is equivalent to the Pontryagin minimum principle. We assume that this condition is satisfied for the trajectory $\mathcal{T}$. We say in this case that $\mathcal{T}$ is an *extremal trajectory* for the problem. M_0 is a finite-dimensional *compact* set since in (2.6) the initial values $\psi(0)$ are assumed to belong to the unit ball of $\mathbb{R}^{d(x)}$. The case that there exists a multiplier $(\psi_0, \psi) \in M_0$ with $\psi_0(T) > 0$ will be called the *nondegenerate* or *normal* case.

Henceforth, it will be convenient to use the simple abbreviation (t) for all arguments $(t, x(t), u(t), \psi(t))$, e.g., $H(t) = H(t, x(t), u(t), \psi(t))$, $\sigma(t) = \sigma(t, x(t), \psi(t))$. The continuity of the pair of functions $(\psi_0(t), \psi(t))$ at the points $t_k \in \theta$ constitutes the Weierstrass–Erdmann necessary conditions for nonsmooth extremals. We formulate one more important condition of this type. Namely, for $(\psi_0, \psi) \in M_0$ and $t_k \in \theta$ consider the function

$$(\Delta_k H)(t) = H(t, x(t), u^{k+}, \psi(t)) - H(t, x(t), u^{k-}, \psi(t)) = \sigma(t, x(t), \psi(t))[u]^k.$$

This function has a derivative

$$D^k(H) := -\frac{d}{dt}(\Delta_k H)(t_k) = -\dot{\sigma}(t_k^{\pm})[u]^k,$$

where the values on the right-hand side are the same for the derivative $\dot{\sigma}(t_k^+)$ from the right and the derivative $\dot{\sigma}(t_k^-)$ from the left. In the case of a *scalar* control u, the total derivative $\sigma_t + \sigma_x \dot{x} + \sigma_\psi \dot{\psi}$ does not contain the control variable explicitly [17, 18] and, hence, the derivative of the switching function $\dot{\sigma}(t)$ is *continuous* at t_k. Then the minimum condition (2.9) immediately implies the following property.

PROPOSITION 2.2. *For each $(\psi_0, \psi) \in M_0$ the following conditions hold:*

$$D^k(H) = -\dot{\sigma}(t_k^{\pm})[u]^k \geq 0 \quad \textit{for } \ k = 1, \ldots, s. \tag{2.11}$$

2.3. Bang-bang control. The classical definition of a bang-bang control is that of a control which assumes values in the vertex set of the admissible polyhedron U in (2.4). We need a slightly more restrictive definition of a bang-bang control to obtain the sufficient conditions in Theorem 3.3. Let

$$\operatorname{Arg\,min}_{v \in U} \ \sigma(t)v$$

be the set of points $v \in U$ where the minimum of the linear function $\sigma(t)v$ is attained. For a given extremal trajectory $\mathcal{T} = \{\, (x(t), u(t)) \mid t \in \Delta \,\}$ with piecewise constant control $u(t)$ we shall say that $u(t)$ is a *bang-bang control* if there exists $(\psi_0, \psi) \in M_0$ such that

$$\operatorname{Arg\,min}_{v \in U} \sigma(t)v = [u(t-0), u(t+0)]\,, \tag{2.12}$$

where $[u(t-0), u(t+0)] = \{\alpha u(t-0) + (1-\alpha)u(t+0) \mid 0 \leq \alpha \leq 1\,\}$ denotes the line segment in $\mathbb{R}^{d(u)}$. Notice that $[u(t-0), u(t+0)]$ is a singleton $\{u(t)\}$ at each continuity point of the control $u(t)$ with $u(t)$ being a vertex of the polyhedron U. Only at the points $t_k \in \theta$ does the line segment $[u^{k-}, u^{k+}]$ coincide with an edge of the polyhedron.

If the control is *scalar*, $d(u) = 1$ and $U = [u_{min}, u_{max}]$, then the bang-bang property is equivalent to

$$\sigma(t, x(t), \psi(t)) \neq 0 \quad \forall\, t \in \Delta \setminus \theta,$$

which implies the following control law:

$$u(t) = \left\{ \begin{array}{lll} u_{min} & \text{if} & \sigma(t) > 0 \\ u_{max} & \text{if} & \sigma(t) < 0 \end{array} \right\} \quad \forall\, t \in \Delta \setminus \theta. \tag{2.13}$$

For vector-valued control inputs, condition (2.12) imposes further restrictions. For example, if U is the unit cube in $\mathbb{R}^{d(u)}$, condition (2.12) precludes simultaneous switching of the control components. However, this property holds for most examples; cf., e.g., the time-optimal control of a robot manipulator with $d(u) = 2$ in Chernousko, Akulenko, and Bolotnik [6]. Moreover, condition (2.12) will be indispensable in the sensitivity analysis of optimal bang-bang controls, a topic that we are currently investigating.

3. Critical subspace, quadratic form, and sufficient optimality conditions for bang-bang controls.

In order to formulate quadratic sufficient optimality conditions for a given extremal $\mathcal{T}$ with bang-bang control $u(\cdot)$ we shall introduce the space $\mathcal{Z}(\theta)$, the critical subspace $\mathcal{K} \subset \mathcal{Z}(\theta)$, and the quadratic form Ω defined in $\mathcal{Z}(\theta)$.

3.1. Critical subspace.

Denote by $P_\theta C^1(\Delta, R^n)$ the space of piecewise continuous functions

$$\bar{x}(\cdot) : \Delta \to \mathbb{R}^n$$

that are continuously differentiable on each interval of the set $\Delta \setminus \theta$. For each $\bar{x} \in P_\theta C^1(\Delta, R^n)$ and for $t_k \in \theta$ we use the abbreviation

$$[\bar{x}]^k = \bar{x}^{k+} - \bar{x}^{k-}, \quad \text{where} \quad \bar{x}^{k-} = \bar{x}(t_k - 0), \quad \bar{x}^{k+} = \bar{x}(t_k + 0).$$

Putting

$$\bar{z} = (\bar{T}, \xi, \bar{x}) \quad \text{with} \quad \bar{T} \in R^1, \quad \xi = (\xi_1, \ldots, \xi_s) \in R^s, \quad \bar{x} \in P_\theta C^1(\Delta, R^n),$$

we have

$$\bar{z} \in \mathcal{Z}(\theta) := \mathbb{R}^1 \times \mathbb{R}^s \times P_\theta C^1(\Delta, \mathbb{R}^n).$$

Denote by $\mathcal{K}$ the set of all $\bar{z} \in \mathcal{Z}(\theta)$ satisfying the following conditions:

$$\dot{\bar{x}}(t) = f_x(t, x(t), u(t))\bar{x}(t), \qquad [\bar{x}]^k = [\dot{x}]^k \xi_k, \quad k = 1, \ldots, s, \tag{3.1}$$

$$\bar{x}(0) = 0, \quad \bar{x}(T) + \dot{x}(T)\bar{T} = 0\,. \tag{3.2}$$

Then $\mathcal{K}$ is a subspace of the space $Z(\theta)$ which we call the *critical subspace*. Each element $\bar{z} \in \mathcal{K}$ is uniquely defined by the number $\bar{T}$ and the vector ξ. Consequently, the subspace $\mathcal{K}$ is *finite-dimensional*.

An explicit representation of the variations $\bar{x}(t)$ in (3.1) is obtained as follows. For each $k = 1, \ldots, s$, define the vector functions $y^k(t)$ as the solutions to the system

$$\dot{y} = f_x(t)y, \quad y(t_k) = [\dot{x}]^k, \quad t \in [t_k, T]. \tag{3.3}$$

For $t < t_k$ we put $y^k(t) = 0$ which yields the jump $[y^k]^k = [\dot{x}]^k$. It follows from the superposition principle for linear ODEs that

$$\bar{x}(t) = \sum_{k=1}^{s} y^k(t)\xi_k \tag{3.4}$$

from which we obtain the representation

$$\bar{x}(T) + \dot{x}(T)\bar{T} = \sum_{k=1}^{s} y^k(T)\xi_k + \dot{x}(T)\bar{T}\,. \tag{3.5}$$

Furthermore, denote by $x(t; t_1, \ldots, t_s)$ the solution of the state equation (2.2) using the optimal bang-bang control with switching points $t_1, \ldots, t_s$. It easily follows from elementary properties of ODEs that the partial derivatives of state trajectories w.r.t. the switching points are given by

$$\frac{\partial x}{\partial t_k}(t; t_1, \ldots, t_s) = -y^k(t) \quad \text{for } \ t \geq t_k, \ \ k = 1, \ldots, s. \tag{3.6}$$

This relation holds for all $t \in [0, T] \setminus \{t_k\}$, because for $t < t_k$ we have $\frac{\partial x}{\partial t_k}(t) = 0$ and $y^k(t) = 0$. Hence, (3.4) yields

$$\bar{x}(t) = -\sum_{k=1}^{s} \frac{\partial x}{\partial t_k}(t)\xi_k\,. \tag{3.7}$$

In the nondegenerate case $\psi_0(T) > 0$, the critical subspace simplifies as follows.

PROPOSITION 3.1. *If there exists $(\psi_0, \psi) \in M_0$ such that $\psi_0(T) > 0$, then $\bar{T} = 0$ holds for each $\bar{z} = (\bar{T}, \xi, \bar{x}) \in \mathcal{K}$.*

Proof. For arbitrary $(\psi_0, \psi) \in M_0$ and $\bar{z} = (\bar{T}, \xi, \bar{x}) \in \mathcal{K}$ we have

$$\frac{d}{dt}(\psi\bar{x}) = \dot{\psi}\bar{x} + \psi\dot{\bar{x}} = -\psi f_x(t)\bar{x} + \psi f_x(t)\bar{x} = 0,$$

and also

$$[\psi\bar{x}]^k = \psi(t_k)[\bar{x}]^k = \psi(t_k)[\dot{x}]^k\xi_k = [\psi\dot{x}]^k\xi_k = -[\psi_0]^k\xi_k = 0.$$

Consequently, $\psi(t)\bar{x}(t)$ is a constant function on $[0, T]$ which yields in view of (3.2)

$$0 = (\psi\bar{x})(0) = (\psi\bar{x})(T) = -\psi(T)\dot{x}(T)\bar{T} = \psi_0(T)\bar{T}.$$

Hence the inequality $\psi_0(T) > 0$ implies that $\bar{T} = 0$.

In section 3.2, we shall conclude from Theorem 3.3 that the property $\mathcal{K} = \{0\}$ essentially represents a first order sufficient condition. Since $\bar{x}(T) + \dot{x}(T)\bar{T} = 0$ by (3.2), the representations (3.4), (3.5), and Proposition 3.1 induce the following conditions for $\mathcal{K} = \{0\}$.

PROPOSITION 3.2. *Assume that one of the following conditions is satisfied:*

(a) *the $s+1$ vectors $y^k(T) = -\frac{\partial x}{\partial t_k}(T)$, $k = 1, \ldots, s$, $\dot{x}(T)$, are linearly independent,*

(b) *there exists $(\psi_0, \psi) \in M_0$ with $\psi_0(T) > 0$, and the s vectors $y^k(T) = -\frac{\partial x}{\partial t_k}(T)$, $k = 1, \ldots, s$, are linearly independent,*

(c) *there exists $(\psi_0, \psi) \in M_0$ with $\psi_0(T) > 0$, and the bang-bang control has one switching point, i.e., $s = 1$.*

Then the critical subspace is $\mathcal{K} = \{0\}$.

Now we discuss the case of two switching points, i.e., $s = 2$, to prepare the numerical example in section 5.2. Let us assume that $\psi_0(T) > 0$ and $[\dot{x}]^1 \neq 0$, $[\dot{x}]^2 \neq 0$.

By virtue of Proposition 3.1, we have $\bar{T} = 0$ and hence $\bar{x}(T) = 0$ for each element $\bar{z} \in \mathcal{K}$. Then the relations (3.2) and (3.4) yield

$$0 = \bar{x}(T) = y^1(T)\xi_1 + y^2(T)\xi_2\,. \tag{3.8}$$

The conditions $[\dot{x}]^1 \neq 0$ and $[\dot{x}]^2 \neq 0$ imply that $y^1(T) \neq 0$ and $y^2(T) \neq 0$, respectively. Futhermore, assume that $\mathcal{K} \neq \{0\}$. Then (3.8) shows that the nonzero vectors $y^1(T)$ and $y^2(T)$ are collinear, i.e.,

$$y^2(T) = \alpha y^1(T) \tag{3.9}$$

with some factor $\alpha \neq 0$. As a consequence, the relation $y^2(t) = \alpha y^1(t)$ is valid for all $t \in (t_2, T]$ since the functions $y^1(t)$ and $y^2(t)$ are continuous solutions to the system $\dot{y} = f_x(t)y$ in $(t_2, T]$. In particular, we have $y^2(t_2 + 0) = \alpha y^1(t_2)$ and thus

$$[\dot{x}]^2 = \alpha y^1(t_2) \tag{3.10}$$

which is equivalent to (3.9). In addition, the equalities (3.8) and (3.9) imply that

$$\xi_2 = -\frac{1}{\alpha}\xi_1. \tag{3.11}$$

We shall use these formulas in the next subsection.

3.2. Quadratic form. In the sequel, second order partial derivatives will be denoted by double subscripts, e.g., $H_{xx} = D_x^2 H$. For $(\psi_0, \psi) \in M_0$ and $\bar{z} \in \mathcal{K}$ we define the functional

$$\begin{aligned}\Omega(\psi_0, \psi, \bar{z}) = \;& \sum_{k=1}^{s} (D^k(H)\xi_k^2 + 2[H_x]^k \bar{x}_{\text{av}}^k \xi_k) + \int_0^T \langle H_{xx}(t)\bar{x}(t), \bar{x}(t)\rangle\, dt \\ & -(\dot{\psi}_0(T) - \dot{\psi}(T)\dot{x}(T))\bar{T}^2\,,\end{aligned} \tag{3.12}$$

where

$$\bar{x}_{\text{av}}^k := \frac{1}{2}(\bar{x}^{k-} + \bar{x}^{k+}).$$

Note that the functional $\Omega(\psi_0, \psi, \bar{z})$ is linear in ψ_0 and ψ and quadratic in $\bar{z}$.

Now we introduce SSC for a bang-bang control which have been obtained by Osmolovskii; see [31, Part 2, chapter 3, section 12.4]. Some proofs missing in this book will appear in Osmolovskii [40].

THEOREM 3.3. *Let the following Condition $\mathcal{B}$ be fulfilled for the trajectory $\mathcal{T}$:*

(a) *$u(t)$ is a bang-bang control such that* (2.12) *holds;*

(b) *there exists $(\psi_0, \psi) \in M_0$ such that $D^k(H) > 0$ for $k = 1, \ldots, s$;*

(c) $\max_{(\psi_0,\psi)\in M_0} \Omega(\psi_0, \psi, \bar{z}) > 0 \quad \forall\, \bar{z} \in \mathcal{K} \setminus \{0\}$.

Then $\mathcal{T}$ is a strict strong minimum.

Remarks.

1. In this theorem, the sufficient Condition $\mathcal{B}$ is a natural strengthening of the corresponding necessary quadratic condition in the same problem; see [31, Part 2].
2. Condition (c) is automatically fulfilled if $\mathcal{K} = \{0\}$ holds (cf. Proposition 3.2), which gives a first order sufficient condition for a strong minimum.

3. If there exists $(\psi_0, \psi) \in M_0$ such that

$$\Omega(\psi_0, \psi, \bar{z}) > 0 \quad \forall \bar{z} \in \mathcal{K} \setminus \{0\},$$

then condition (c) is obviously fulfilled.

For boxes $U = \{u = (u_1, \ldots, u_{d(u)}) \in \mathbb{R}^{d(u)} \,|\, u_i^{\min} \le u_i \le u_i^{\max},\ i = 1, \ldots, d(u)\}$, condition (b) is equivalent to the property $\dot{\sigma}_i(t_k) \neq 0$ if t_k is a switching point of the ith control component $u_i(t)$. Note again that condition (2.12) precludes the simultaneous switching of two or more control components. A further remark concerns the case that the set M_0 of Pontryagin multipliers is not a singleton. This case has been illustrated in Osmolovskii [38, pp. 377–380] by the following time-optimal control problem for a linear system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = u, \quad |u| \le 1, \quad x(0) = a, \quad x(T) = b,$$

where $x = (x_1, x_2, x_3, x_4)$. It was shown in this paper that for some a and b there exists an extremal in this problem with two switching points of the control such that, under an appropriate normalization, the set M_0 is a segment. For this extremal, the maximum of the quadratic forms Ω over M_0 is positive on each nonzero element of the critical subspace and hence the sufficient conditions of Theorem 3.3 are satisfied.

3.3. Nondegenerate case. Let us assume the *nondegenerate* or *normal* case that there exists $(\psi_0, \psi) \in M_0$ such that the cost function multiplier $\psi_0(T)$ is positive. By virtue of Proposition 3.1 we have in this case that $\bar{T} = 0$ for all $\bar{z} \in \mathcal{K}$. Thus the critical subspace $\mathcal{K}$ is defined by the conditions

$$\dot{\bar{x}} = f_x(t)\bar{x}, \quad [\bar{x}]^k = [\dot{x}]^k \xi_k \quad (k = 1, \ldots, s), \quad \bar{x}(0) = 0, \quad \bar{x}(T) = 0. \tag{3.13}$$

In particular, these conditions imply $\bar{x}(t) \equiv 0$ on $[0, t_1)$ and $(t_s, T]$. Hence, we have $\bar{x}^{1-} = \bar{x}^{s+} = 0$ for all $\bar{z} \in \mathcal{K}$. Then the quadratic form (3.12) is equal to

$$\Omega(\psi, \bar{z}) = \sum_{k=1}^{s} (D^k(H)\xi_k^2 + 2[H_x]^k \bar{x}_{\mathrm{av}}^k \xi_k) + \int_{t_1}^{t_s} \langle H_{xx}(t)\bar{x}(t), \bar{x}(t)\rangle \, dt. \tag{3.14}$$

Just this case of a time-optimal (autonomous) control problem was studied by Sarychev [43]. He used a special transformation of the problem and obtained sufficient optimality condition for the transformed problem. It is not easy but possible to reformulate his results in terms of the original problem. The comparison of both types of conditions reveals that Sarychev used the same critical subspace, but his quadratic form is a lower bound for Ω. Namely, in his quadratic form the positive term $D^k(H)\xi_k^2$ has the factor $\frac{1}{4}$ instead of the factor 1 for the same term in Ω. Therefore, the sufficient Condition $\mathcal{B}$ is always fulfilled whenever Sarychev's condition is fulfilled. However, Osmolovskii has constructed an example of a control problem where the optimal solution satisfies Condition $\mathcal{B}$, but does not satisfy Sarychev's condition. Finally, Sarychev proved that his condition is sufficient for an L_1-minimum w.r.t. the control (which is a "Pontryagin minimum" [31] in this problem). In fact it could be proved that his condition is sufficient for a strong minimum.

3.4. Cases of one or two switching points of the control. From Theorem 3.3 and Proposition 3.2(c) we immediately deduce sufficient conditions for a bang-bang control with one switching point. The result will be used for the example in section 5.1 and is also applicable to the time-optimal control of an image converter discussed in Kim et al. [15].

THEOREM 3.4. *Let the following conditions be fulfilled for the trajectory $\mathcal{T}$:*

(a) *$u(t)$ is a bang-bang control with one switching point;*

(b) *there exists $(\psi_0, \psi) \in M_0$ such that $\psi_0(T) > 0$ and $D^1(H) > 0$.*

Then $\mathcal{T}$ is a strict strong minimum.

Now we turn our attention to the case of two switching points where $s = 2$. Assume the nondegenerate case $\psi_0(T) > 0$ and suppose that $[\dot{x}]^1 \neq 0$, $[\dot{x}]^2 \neq 0$ and $y^2(T) = \alpha y^1(T)$ as in (3.9). Otherwise, $\mathcal{K} = \{0\}$ holds and, hence, the first order sufficient condition for a strong minimum is satisfied. For any element $\bar{z} \in \mathcal{K}$ we have $\bar{T} = 0$, $\bar{x}^{1-} = 0$, $\bar{x}^{2+} = 0$. Consequently,

$$\bar{x}^1_{\rm av} = \frac{1}{2}[\bar{x}]^1 = \frac{1}{2}[\dot{x}]^1\xi_1, \quad \bar{x}^2_{\rm av} = \frac{1}{2}\bar{x}^{2-} = \frac{1}{2}y^1(t_2)\xi_1 = \frac{1}{2\alpha}[\dot{x}]^2\xi_1$$

in view of $\bar{x}(t) = y^1(t)\xi_1 + y^2(t)\xi_2$, $y^2(t_2 - 0) = 0$ and (3.10). Using these relations in the quadratic form (3.14) together with (3.11) and the conditions $y^2(t) = 0$ for all $t < t_2$, $[H_x]^k = -[\dot{\psi}]^k$, $k = 1, 2$, we compute the quadratic form for an element of the critical subspace as

$$\begin{aligned}
\Omega &= D^1(H)\xi_1^2 + D^2(H)\xi_2^2 - 2[\dot{\psi}]^1\bar{x}^1_{\rm av}\xi_1 - 2[\dot{\psi}]^2\bar{x}^2_{\rm av}\xi_2 + \int_{t_1}^{t_2}\langle H_{xx}\bar{x}, \bar{x}\rangle\, dt \\
&= D^1(H)\xi_1^2 + \tfrac{1}{\alpha^2}D^2(H)\xi_1^2 - [\dot{\psi}]^1[\dot{x}]^1\xi_1^2 + \tfrac{1}{\alpha^2}[\dot{\psi}]^2[\dot{x}]^2\xi_1^2 + \left(\int_{t_1}^{t_2}\langle H_{xx}y^1, y^1\rangle\, dt\right)\xi_1^2 \\
&= \rho\xi_1^2,
\end{aligned}$$

where

$$\rho := \left(D^1(H) - [\dot{\psi}]^1[\dot{x}]^1\right) + \frac{1}{\alpha^2}\left(D^2(H) + [\dot{\psi}]^2[\dot{x}]^2\right) + \int_{t_1}^{t_2}\langle H_{xx}y^1, y^1\rangle\, dt. \tag{3.15}$$

Thus, we obtain the following proposition.

PROPOSITION 3.5. *Assume that $\psi_0(T) > 0$, $s = 2$, $[\dot{x}]^1 \neq 0$, $[\dot{x}]^2 \neq 0$, and $y^2(T) = \alpha y^1(T)$ (which is equivalent to* (3.10)*) with some factor α. Then the condition of the positive definiteness of Ω on $\mathcal{K}$ is equivalent to the inequality $\rho > 0$, where ρ is defined by* (3.15)*.*

4. Sufficient conditions for positive definiteness of the quadratic form Ω on the critical subspace $\mathcal{K}$. In this section we consider the nondegenerate case in section 3.3 and assume

(i) $u(t)$ is a bang-bang control with $s > 1$ switching points;

(ii) there exists $(\psi_0, \psi) \in M_0$ such that $\psi_0(T) > 0$ and $D^k(H) > 0$, $k = 1, \dots, s$.

Under these assumptions the critical subspace $\mathcal{K}$ is defined by (3.13). Let $(\psi_0, \psi) \in M_0$ be a fixed element (possibly, different from that in assumption (ii)) and denote by $\Omega = \Omega(\psi_0, \psi, \cdot)$ the quadratic form for this element. Recall that Ω is given by (3.14). According to Theorem 3.3 *the positive definiteness of the quadratic form* (3.14) *on the subspace $\mathcal{K}$ in* (3.13) *is a sufficient condition for a strict strong minimum of the trajectory.* Now our aim is to find conditions that guarantee this property of positive definiteness. In what follows we shall use some ideas and results presented in Osmolovskii and Lempio [42], who have extended the Riccati approach in [4, 30, 22, 48] to broken extremals.

4.1. Q-transformation of Ω on $\mathcal{K}$. Let $Q(t)$ be a symmetric matrix on $[t_1, t_s]$ with piecewise continuous entries which are absolutely continuous on each interval of the set $[t_1, t_s] \setminus \theta$. Therefore, Q may have a jump at each point $t_k \in \theta$ including t_1, t_s, and thus the symmetric matrices Q^{1-} and Q^{s+} are also defined.

For $\bar z \in \mathcal{K}$ we obviously have

$$\int_{t_1}^{t_s} \frac{d}{dt}\langle Q\bar x, \bar x\rangle\, dt = \langle Q\bar x, \bar x\rangle\Big|_{t_1-0}^{t_s+0} - \sum_{k=1}^{s} [\langle Q\bar x, \bar x\rangle]^k ,$$

where $[\langle Q\bar x, \bar x\rangle]^k$ is the jump of the function $\langle Q\bar x, \bar x\rangle$ at the point $t_k \in \theta$. Using the conditions

$$\dot{\bar x} = f_x(t)\bar x, \quad \bar x^{1-} = \bar x^{s+} = 0,$$

we obtain

$$\sum_{k=1}^{s} [\langle Q\bar x, \bar x\rangle]^k + \int_{t_1}^{t_s} \langle (\dot Q + f_x^* Q + Q f_x)\bar x, \bar x\rangle\, dt = 0, \tag{4.1}$$

where the asterisk denotes transposition. Adding this zero-form to Ω we get

$$\Omega = \sum_{k=1}^{s}\Big(D^k(H)\xi_k^2 - 2[\dot\psi]^k \bar x_{\rm av}^k \xi_k + [\langle Q\bar x, \bar x\rangle]^k\Big) + \int_{t_1}^{t_s} \langle (H_{xx} + \dot Q + f_x^* Q + Q f_x)\bar x, \bar x\rangle\, dt. \tag{4.2}$$

We shall call this formula the *Q-transformation of Ω on $\mathcal{K}$*.

In order to eliminate the integral term in Ω we assume that $Q(t)$ satisfies the following linear matrix differential equation:

$$\dot Q + f_x^* Q + Q f_x + H_{xx} = 0 \quad \text{on } [t_1, t_s] \setminus \theta. \tag{4.3}$$

It is interesting to note that the same equation is obtained from the modified Riccati equation in [30, equation (47)] when all control variables are on the boundary of the control constraints. Using (4.3) the quadratic form (4.2) reduces to

$$\Omega = \sum_{k=1}^{s} \omega_k , \quad \omega_k := D^k(H)\xi_k^2 - 2[\dot\psi]^k \bar x_{\rm av}^k \xi_k + [\langle Q\bar x, \bar x\rangle]^k . \tag{4.4}$$

Thus, we have proved the following statement.

PROPOSITION 4.1. *Let $Q(t)$ satisfy the linear differential equation* (4.3) *on $[t_1, t_s]\setminus \theta$. Then for each $\bar z \in \mathcal{K}$ the representation* (4.4) *holds.*

Now our goal is to derive conditions such that $\omega_k > 0$ holds on $\mathcal{K} \setminus \{0\}$ for $k = 1, \ldots, s$. We shall transform ω_k as in [42]. First we shall express it via the vector $(\xi_k, \bar x^{k-})$ and then via $(\xi_k, \bar x^{k+})$. To express ω_k as a quadratic form of $(\xi_k, \bar x^{k-})$, we use the formula

$$\bar x^{k+} = \bar x^{k-} + [\dot x]^k \xi_k, \tag{4.5}$$

which implies

$$\langle Q^{k+}\bar x^{k+}, \bar x^{k+}\rangle = \langle Q^{k+}\bar x^{k-}, \bar x^{k-}\rangle + 2\langle Q^{k+}[\dot x]^k, \bar x^{k-}\rangle \xi_k + \langle Q^{k+}[\dot x]^k, [\dot x]^k\rangle \xi_k^2.$$

Consequently,

$$[\langle Q\bar x, \bar x\rangle]^k = \langle [Q]^k \bar x^{k-}, \bar x^{k-}\rangle + 2\langle Q^{k+}[\dot x]^k, \bar x^{k-}\rangle \xi_k + \langle Q^{k+}[\dot x]^k, [\dot x]^k\rangle \xi_k^2.$$

Using this relation together with

$$\bar{x}_{\rm av}^k = \bar{x}^{k-} + \frac{1}{2}[\dot{x}]^k \xi_k$$

in the definition (4.4) of ω_k, we obtain

$$\begin{aligned} \omega_k \quad &= \quad \{D^k(H) + \Big(([\dot{x}]^k)^* Q^{k+} - [\dot{\psi}]^k\Big)[\dot{x}]^k\}\, \xi_k^2 \\ & \quad +2\Big(([\dot{x}]^k)^* Q^{k+} - [\dot{\psi}]^k\Big)\bar{x}^{k-}\xi_k + (\bar{x}^{k-})^*[Q]^k \bar{x}^{k-}. \end{aligned} \tag{4.6}$$

Here $[\dot{x}]^k$ and $\bar{x}^{k-}$ are column-vectors while $([\dot{x}]^k)^*$, $(\bar{x}^{k-})^*$, and $[\dot{\psi}]^k$ are row-vectors. Putting

$$q_{k+} = ([\dot{x}]^k)^* Q^{k+} - [\dot{\psi}]^k \tag{4.7}$$

we get

$$\omega_k = \big(D^k(H) + (q_{k+})[\dot{x}]^k\big)\, \xi_k^2 + 2(q_{k+})\bar{x}^{k-}\xi_k + (\bar{x}^{k-})^*[Q]^k \bar{x}^{k-}. \tag{4.8}$$

We immediately see from this representation that one way to enforce $\omega_k > 0$ is to impose the following conditions:

$$D^k(H) > 0, \quad q_{k+} = ([\dot{x}]^k)^* Q^{k+} - [\dot{\psi}]^k = 0, \quad [Q]^k \geq 0. \tag{4.9}$$

In practice, however, it might be difficult to check these conditions since it is necessary to satisfy the $d(x)$ equality constraints $q_{k+} = ([\dot{x}]^k)^* Q^{k+} - [\dot{\psi}]^k = 0$ together with the inequality constraints $[Q]^k \geq 0$. It is more convenient to express ω_k as a quadratic form in the variables $(\xi_k, \bar{x}^{k-})$ with the matrix

$$M_{k+} = \begin{pmatrix} D^k(H) + (q_{k+})[\dot{x}]^k & q_{k+} \\ (q_{k+})^* & [Q]^k \end{pmatrix}, \tag{4.10}$$

where q_{k+} is a row-vector and $(q_{k+})^*$ is a column-vector.

Similarly, using the relation

$$\bar{x}^{k-} = \bar{x}^{k+} - [\dot{x}]^k \xi_k,$$

we obtain

$$[\langle Q\bar{x}, \bar{x}\rangle]^k = \langle [Q]^k \bar{x}^{k+}, \bar{x}^{k+}\rangle + 2\langle Q^{k-}[\dot{x}]^k, \bar{x}^{k+}\rangle \xi^k - \langle Q^{k-}[\dot{x}]^k, [\dot{x}]^k\rangle \xi_k^2.$$

This formula together with the relation

$$\bar{x}_{\rm av}^k = \bar{x}^{k+} - \frac{1}{2}[\dot{x}]^k \xi_k$$

leads to the representation

$$\begin{aligned} \omega_k \quad &= \quad \{D^k(H) - \Big(([\dot{x}]^k)^* Q^{k-} - [\dot{\psi}]^k\Big)[\dot{x}]^k\}\, \xi_k^2 \\ & \quad +2\Big(([\dot{x}]^k)^* Q^{k-} - [\dot{\psi}]^k\Big)\bar{x}^{k+}\xi_k + (\bar{x}^{k+})^*[Q]^k \bar{x}^{k+}. \end{aligned} \tag{4.11}$$

Defining

$$q_{k-} = ([\dot{x}]^k)^* Q^{k-} - [\dot{\psi}]^k, \tag{4.12}$$

we get

$$\omega_k = \left(D^k(H) - (q_{k-})[\dot{x}]^k\right)\xi_k^2 + 2(q_{k-})\bar{x}^{k+}\xi_k + (\bar{x}^{k+})^*[Q]^k\bar{x}^{k+}. \tag{4.13}$$

Again, we see that $\omega_k > 0$ holds if we require the conditions

$$D^k(H) > 0, \quad q_{k-} = ([\dot{x}]^k)^*Q^{k-} - [\dot{\psi}]^k = 0, \quad [Q]^k \geq 0. \tag{4.14}$$

To find a more general condition for $\omega_k > 0$, we consider (4.13) as a quadratic form in the variables $(\xi_k, \bar{x}^{k+})$ with the matrix

$$M_{k-} = \begin{pmatrix} D^k(H) - (q_{k-})[\dot{x}]^k & q_{k-} \\ (q_{k-})^* & [Q]^k \end{pmatrix}. \tag{4.15}$$

Since the right-hand sides of equalities (4.8) and (4.13) are connected by the relation (4.5), the following statement obviously holds.

Proposition 4.2. *For each $k = 1, \ldots, s$, the positive (semi)definiteness of the matrix M_{k-} is equivalent to the positive (semi)definiteness of the matrix M_{k+}.*

Now we can prove the following theorem.

Theorem 4.3. *Let $Q(t)$ be a solution of the linear differential equation* (4.3) *on $[t_1, t_s] \setminus \theta$ which satisfies the following conditions:*

(a) *the matrix M_{k+} is positive semidefinite for each $k = 2, \ldots, s$;*

(b) $b_{k+} := D^k(H) + (q_{k+})[\dot{x}]^k > 0$ *for each $k = 1, \ldots, s-1$.*

Then Ω is positive on $\mathcal{K} \setminus \{0\}$.

Proof. Take an arbitrary element $\bar{z} = (\xi, \bar{x}) \in \mathcal{K}$. Let us show that $\Omega \geq 0$ for this element. Condition (a) implies that $\omega_k \geq 0$ for $k = 2, \ldots, s$. Condition (b) for $k = 1$ together with condition $\bar{x}^{1-} = 0$ implies that $\omega_1 \geq 0$. Consequently, $\Omega \geq 0$.

Assume that $\Omega = 0$. Then $\omega_k = 0$ for $k = 1, \ldots, s$. The conditions $\omega_1 = 0$, $\bar{x}^{1-} = 0$, and $b_{1+} > 0$ by formula (4.8) (with $k = 1$) yield $\xi_1 = 0$. Then $[\bar{x}]^1 = 0$ and hence $\bar{x}^{1+} = 0$. The last equality together with equation $\dot{\bar{x}} = f_x(t)\bar{x}$ shows that $\bar{x}(t) = 0$ in (t_1, t_2) and hence $\bar{x}^{2-} = 0$. Similarly, the conditions $\omega_2 = 0$, $\bar{x}^{2-} = 0$ and $b_{2+} > 0$ by formula (4.8) (with $k = 2$) imply that $\xi_2 = 0$ and $\bar{x}(t) = 0$ in (t_2, t_3). Therefore, $\bar{x}^{3-} = 0$, etc. Continuing this process we get $\bar{x} \equiv 0$ and $\xi_k = 0$ for $k = 1, \ldots, s-1$. Now using formula (4.4) for $\omega_s = 0$, as well as the conditions $D^s(H) > 0$ and $\bar{x} \equiv 0$, we obtain that $\xi_s = 0$. Consequently, we have $\bar{z} = 0$ which means that Ω is positive on $\mathcal{K} \setminus \{0\}$.

Similarly, using representation (4.13) for ω_k we can prove the following statement.

Theorem 4.4. *Let $Q(t)$ be a solution of the linear differential equation* (4.3) *on $[t_1, t_s] \setminus \theta$ which satisfies the following conditions:*

(a) *the matrix M_{k-} is positive semidefinite for each $k = 1, \ldots, s-1$,*

(b) $b_{k-} := D^k(H) - (q_{k-})[\dot{x}]^k > 0$ *for each $k = 2, \ldots, s$.*

Then Ω is positive on $\mathcal{K} \setminus \{0\}$.

4.2. *Q*-transformation of Ω to perfect squares. We shall formulate special jump conditions for the matrix Q at each point $t_k \in \theta$. This will make it possible to transform Ω to perfect squares and thus to prove its positive definiteness on $\mathcal{K}$.

Proposition 4.5 (see [42]). *Suppose that*

$$b_{k-} := D^k(H) - (q_{k-})[\dot{x}]^k > 0 \tag{4.16}$$

and that Q satisfies the jump condition at t_k

$$b_{k-}[Q]^k = (q_{k-})^*(q_{k-}), \tag{4.17}$$

where $(q_{k-})^*$ *is a column-vector while* q_{k-} *is a row-vector. Then* ω_k *can be written as the perfect square*

$$\omega_k = (b_{k-})^{-1}\left((b_{k-})\xi_k + (q_{k-})(\bar{x}^{k+})\right)^2 = (b_{k-})^{-1}\left(D^k(H)\xi_k + (q_{k-})(\bar{x}^{k-})\right)^2. \tag{4.18}$$

Proof. Using (4.13) and (4.17), we obtain

$$\begin{aligned}\omega_k &= (b_{k-})\xi_k^2 + 2(q_{k-})\bar{x}^{k+}\xi_k + (\bar{x}^{k+})^*[Q]^k\bar{x}^{k+}\\ &= (b_{k-})^{-1}\left((b_{k-})^2\xi_k^2 + 2(q_{k-})\bar{x}^{k+}(b_{k-})\xi_k + \left((q_{k-})\bar{x}^{k+}\right)^2\right)\\ &= (b_{k-})^{-1}\left((b_{k-})\xi_k + (q_{k-})(\bar{x}^{k+})\right)^2.\end{aligned}$$

Since

$$(b_{k-})\xi_k + (q_{k-})\bar{x}^{k+} = \left(D^k(H) - (q_{k-})[\dot{x}]^k\right)\xi_k + (q_{k-})\bar{x}^{k+}$$

$$= \quad D^k(H)\xi_k - (q_{k-})[\bar{x}]^k + (q_{k-})\bar{x}^{k+} = D^k(H)\xi_k + (q_{k-})\bar{x}^{k-},$$

we see that equality (4.18) holds.

THEOREM 4.6. *Let* $Q(t)$ *satisfy the linear differential equation* (4.3) *on* $[t_1, t_s]\setminus\theta$. *Let condition* (4.16) *hold for each* $k = 1, \dots, s$ *and condition* (4.17) *hold for each* $k = 1, \dots, s-1$. *Then* Ω *is positive on* $\mathcal{K}\setminus\{0\}$.

Proof. By Proposition 4.5 and formulae (4.13), (4.15) the matrix M_{k-} is positive semidefinite for each $k = 1, \dots, s-1$, and hence both conditions (a) and (b) of Theorem 4.4 are fulfilled. Then by this theorem, Ω is positive on $\mathcal{K}\setminus\{0\}$.

Similar assertions hold for the jump conditions that use right-hand values of Q at each point $t_k \in \theta$.

PROPOSITION 4.7 (see [42]). *Suppose that*

$$b_{k+} := D^k(H) + (q_{k+})[\dot{x}]^k > 0 \tag{4.19}$$

and that Q *satisfies the jump condition at point* t_k

$$b_{k+}[Q]^k = (q_{k+})^*(q_{k+}). \tag{4.20}$$

Then

$$\omega_k = (b_{k+})^{-1}\left((b_{k+})\xi_k + (q_{k+})(\bar{x}^{k-})\right)^2 = (b_{k+})^{-1}\left(D^k(H)\xi_k + (q_{k+})(\bar{x}^{k+})\right)^2. \tag{4.21}$$

THEOREM 4.8. *Let* $Q(t)$ *satisfy the linear differential equation* (4.3) *on* $[t_1, t_s]\setminus\theta$. *Let condition* (4.19) *hold for each* $k = 1, \dots, s$ *and condition* (4.20) *hold for each* $k = 2, \dots, s$. *Then* Ω *is positive on* $\mathcal{K}\setminus\{0\}$.

4.3. Case of two switching points of the control. Let $s = 2$, i.e., $\theta = \{t_1, t_2\}$, and let $Q(t)$ be a symmetric matrix with absolutely continuous entries on $[t_1, t_2]$. Put

$$Q^k = Q(t_k), \quad q_k = ([\dot{x}]^k)^*Q^k - [\dot{\psi}]^k, \quad k = 1, 2.$$

THEOREM 4.9. *Let* $Q(t)$ *satisfy the linear differential equation* (4.3) *on* (t_1, t_2) *such that the following inequalities hold at* t_1, t_2*:*

$$D^1(H) + q_1[\dot{x}]^1 > 0, \quad D^2(H) - q_2[\dot{x}]^2 > 0. \tag{4.22}$$

Then Ω is positive on $\mathcal{K} \setminus \{0\}$.

Proof. In the case considered we have

$$Q^{1+} = Q^1, \quad q_{1+} = q_1, \quad Q^{2-} = Q^2, \quad q_{2-} = q_2$$

and

$$b_{1+} := D^1(H) + q_1[\dot{x}]^1 > 0, \quad b_{2-} := D^2(H) - q_2[\dot{x}]^2 > 0. \tag{4.23}$$

Define the jumps $[Q]^1$ and $[Q]^2$ by the conditions

$$b_{1+}[Q]^1 = (q_{1+})^*(q_{1+}), \quad b_{2-}[Q]^2 = (q_{2-})^*(q_{2-}). \tag{4.24}$$

Then $[Q]^1$ and $[Q]^2$ are symmetric matrices. Put

$$Q^{1-} = Q^{1+} - [Q]^1, \quad Q^{2+} = Q^{2-} + [Q]^2.$$

Then Q^{1-} and Q^{2+} are also symmetric matrices. Thus, we obtain a symmetric matrix $Q(t)$ satisfying (4.3) on (t_1, t_2), the inequalities (4.23), and the jump conditions (4.24). By Propositions 4.7 and 4.5, the terms ω_1 and ω_2 are nonnegative. In view of (4.4) we see that $\Omega = \omega_1 + \omega_2$ is nonnegative on $\mathcal{K}$. Suppose that $\Omega = 0$ for some $\bar{z} = (\xi, \bar{x}) \in \mathcal{K}$. Then $\omega_k = 0$ for $k = 1, 2$ and thus Propositions 4.7 and 4.5 give

$$b_{1+}\xi_1 + (q_{1+})\bar{x}^{1-} = 0, \quad b_{2-}\xi_2 + (q_{2-})\bar{x}^{2+} = 0.$$

But $\bar{x}^{1-} = 0$ and $\bar{x}^{2+} = 0$. Consequently, $\xi_1 = \xi_2 = 0$ and then conditions $\bar{x}^{1-} = 0$ and $[\bar{x}]^1 = 0$ imply that $\bar{x}^{1+} = 0$. The last equality together with equation $\dot{\bar{x}} = f_x(t)\bar{x}$ implies that $\bar{x}(t) = 0$ on (t_1, t_2). Thus $\bar{x} \equiv 0$ and then $\bar{z} = 0$. We have proved that Ω is positive on $\mathcal{K} \setminus \{0\}$.

4.4. Control system with a constant matrix B. In the case that $B(t, x) = B$ is a constant matrix, the adjoint equation has the form

$$\dot{\psi} = -\psi a_x,$$

which implies that

$$[\dot{\psi}]^k = 0, \quad k = 1, \ldots, s.$$

Therefore,

$$\begin{aligned} q_{k-} &= ([\dot{x}]^k)^* Q^{k-}, & q_{k+} &= ([\dot{x}]^k)^* Q^{k+}, \\ (q_{k-})^* q_{k-} &= Q^{k-}[\dot{x}]^k([\dot{x}]^k)^* Q^{k-}, & (q_{k+})^* q_{k+} &= Q^{k+}[\dot{x}]^k([\dot{x}]^k)^* Q^{k+}, \\ b_{k-} &= D^k(H) - ([\dot{x}]^k)^* Q^{k-}[\dot{x}]^k, & b_{k+} &= D^k(H) + ([\dot{x}]^k)^* Q^{k+}[\dot{x}]^k, \end{aligned}$$

where

$$D^k(H) = \dot{\psi}(t_k) B [u]^k, \quad k = 1, \ldots, s.$$

In case of two switching points with $s = 2$, the conditions (4.22) take the form

$$D^1(H) + \langle Q^1[\dot{x}]^1, [\dot{x}]^1) \rangle > 0\,, \quad D^2(H) - \langle Q^2[\dot{x}]^2, [\dot{x}]^2) \rangle > 0. \tag{4.25}$$

Now assume, in addition, that u is one-dimensional and that with $n = d(x)$

$$B = \beta e_n := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta \end{pmatrix}, \quad \beta > 0, \qquad U = [-c, c], \quad c > 0\,.$$

In this case we get

$$[\dot{x}]^k = B[u]^k = \beta e_n [u]^k, \quad k = 1, \ldots, s,$$

and thus

$$\langle Q^k [\dot{x}]^k, [\dot{x}]^k) \rangle = \beta^2 \langle Q^k e_n, e_n \rangle |[u]^k|^2 = 4\beta^2 c^2\, Q_{nn}(t_k),$$

where Q_{nn} is the element of matrix

$$Q = \begin{pmatrix} Q_{11} & \ldots & Q_{1n} \\ \vdots & \vdots & \vdots \\ Q_{n1} & \ldots & Q_{nn} \end{pmatrix}.$$

Moreover, in the last case we obviously have

$$D^k(H) = 2\beta c\, |\dot{\psi}_n(t_k)|, \quad k = 1, \ldots, s. \tag{4.26}$$

For $s = 2$ conditions (4.25) then yield the estimates

$$Q_{nn}(t_1) > -\frac{|\dot{\psi}_n(t_1)|}{2\beta c}\,, \qquad Q_{nn}(t_2) < \frac{|\dot{\psi}_n(t_2)|}{2\beta c}\,. \tag{4.27}$$

5. Numerical examples. In this section, we shall discuss three time-optimal control problems with fixed initial and final states $x(0) = x_0$ and $x(T) = x_1$. To solve these problems numerically, we need to reduce them to control problems with *fixed* final time. The procedure to achieve this goal is well known [11, 29] and consists of introducing a new time variable $\tau \in [0, 1]$ according to the transformation

$$t = \tau \cdot T\,, \quad \tau \in [0, 1]\,. \tag{5.1}$$

In what follows, we shall identify the function $y(\tau)$ with the function $y(\tau \cdot T)$ for all $y \in \{x, u, \psi\}$. This time transformation leads to the *augmented* state variable

$$\tilde{x} := \begin{pmatrix} x \\ T \end{pmatrix} \in \mathbb{R}^{d(x)+1}$$

for which we obtain the ODE and boundary conditions

$$\begin{aligned} &dx/d\tau = T \cdot f(\tau \cdot T, x(\tau), u(\tau)), \quad dT/d\tau = 0, \quad \tau \in [0, 1]\,, \\ &x(0) = x_0, \quad x(1) = x_1. \end{aligned} \tag{5.2}$$

In the same way, the adjoint equation (2.7) is rewritten as

$$d\psi/d\tau = -T \cdot H_x(\tau \cdot T, x(\tau), u(\tau), \psi(\tau)). \tag{5.3}$$

All examples in this section will treat *autonomous* problems for which we will be able to compute *nondegenerate* solutions with $\psi_0(T) > 0$ in (2.6). Then we may scale the equations such that $\psi_0(T) = 1$ holds. Furthermore, in the autonomous case it follows from (2.8) that $\psi_0(t) \equiv \psi_0(T) = 1$. Hence, (2.10) yields the following condition expressed in the new time variable τ:

$$\psi(\tau)\, f(x(\tau), u(\tau)) + 1 \equiv 0 \quad \forall\, \tau \in [0,1]\,. \tag{5.4}$$

Moreover, u can be expressed via x and ψ from the minimum principle (2.9),

$$\min_{u \in U} \psi(\tau)\, f(x(\tau), u) + 1 = 0 \quad \forall\, \tau \in [0,1]\,. \tag{5.5}$$

In the following examples, we shall use shooting methods (cf. Bulirsch [5] and Oberle and Grimm [34]) for solving the boundary value problem (5.2)–(5.5). Shooting methods are known to provide highly accurate solutions for which we shall carry out the second order test.

5.1. Time-optimal control of a Van der Pol oscillator. The following time-optimal control of a Van der Pol oscillator has been treated by several authors; cf., e.g., Kaya and Noakes [13, 14]. The state variables are the voltage $x_1(t) = U(t)$ at time $t \in [\,0, T\,]$ and $x_2(t) := \dot{x}_1(t)$. The control $u(t)$ is the voltage at the generator; cf. the tunneldiode oscillator in [29, Figure 5.1 in section 5].

The control problem is to minimize the endtime T subject to the constraints

$$\dot{x}_1(t) = x_2(t)\,,\ \dot{x}_2(t) = -x_1(t) + x_2(t)(1 - x_1^2(t)) + u(t)\,, \tag{5.6}$$

$$x_1(0) = -0.4,\ x_2(0) = 0.6\,, \quad x_1(T) = 0.6,\ x_2(T) = 0.4\,, \tag{5.7}$$

$$|\,u(t)\,| \le 1 \quad \text{for} \ \ t \in [0, T\,]\,. \tag{5.8}$$

The Pontryagin function or Hamiltonian (2.5) becomes

$$H(x, u, \psi) = \psi_1 x_2 + \psi_2\,(-x_1 + x_2(1 - x_1^2) + u)\,. \tag{5.9}$$

The time transformation (5.1) yields the transformed state and adjoint equations (5.2), (5.3) in the time interval $\tau \in [0,1]$; for simplicity, the time argument τ will be omitted:

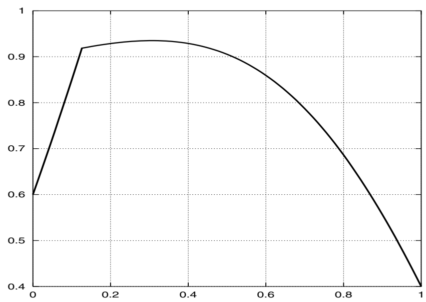
$$\begin{array}{rclrcl} dx_1/d\tau &=& T \cdot x_2\,, & dx_2/d\tau &=& T \cdot \big(-x_1 + x_2(1 - x_1^2) + u\big)\,, \\ d\psi_1/d\tau &=& T \cdot \psi_2(1 + 2x_1\,x_2)\,, & d\psi_2/d\tau &=& -T \cdot (\psi_1 + \psi_2(1 - x_1^2))\,, \\ dT/d\tau &=& 0\,. & & & \end{array} \tag{5.10}$$

The boundary conditions (5.7) and the condition (5.4) yield

$$\begin{array}{l} x_1(0) = -0.4,\ x_2(0) = 0.6, \quad x_1(1) = 0.6,\ x_2(1) = 0.4, \\ 0.4\psi_1(1) + \psi_2(1)(-0.344 + u(1)) + 1 = 0\,. \end{array} \tag{5.11}$$

The switching function $\sigma(x, \psi) = \psi_2$ determines the optimal control according to the control law (2.13),

$$u(\tau) = \left\{ \begin{array}{rl} 1 & \text{if} \quad \psi_2(\tau) < 0 \\ -1 & \text{if} \quad \psi_2(\tau) > 0 \end{array} \right\}. \tag{5.12}$$



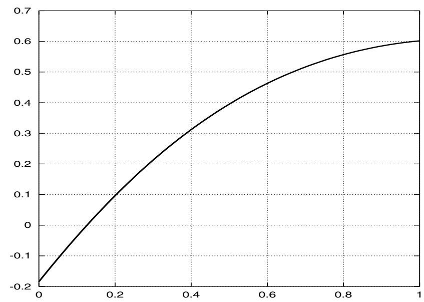


FIG. 5.1. *Van der Pol oscillator: state $x_2(\tau)$ and switching function $\sigma(\tau) = \psi_2(\tau)$, $\tau \in [0, 1]$.*

It can easily be seen that the *singular case*, where $\psi_2(\tau) \equiv 0$ holds in a time interval $[\tau_1, \tau_2]$, does not occur. In fact, $\psi_2(\tau) \equiv 0$ would imply $\psi_1(\tau) \equiv 0$ and thus $H[\tau] \equiv 0$ which would contradict the condition (5.4) in the autonomous case. Computations show that the optimal bang-bang control has the following structure with two bang-bang arcs and only one switching point τ_1:

$$u(\tau) = \left\{ \begin{array}{rll} 1 & \text{for} & 0 \le \tau \le \tau_1 \\ -1 & \text{for} & \tau_1 \le \tau \le 1 \end{array} \right\}. \tag{5.13}$$

Hence, we have to impose the switching condition

$$\sigma[\tau_1] = \psi_2(\tau_1) = 0 \tag{5.14}$$

to determine the switching point τ_1.

The task now is to solve the boundary value problem with the following components: the state and adjoint equations (5.10) using the optimal control structure (5.13), the boundary conditions (5.11) and the switching condition (5.14). Employing the code BNDSCO in [34] we obtain the state variables and adjoint variables displayed in Figure 5.1. The optimal final time, the switching point, and some selected values for the adjoint variables are

$$\begin{array}{lll} T = 1.25407473, & \tau_1 = 0.12624458, & t_1 = \tau_1 \cdot T = 0.1583201376, \\ \psi_1(0) = -1.08160561, & \psi_2(0) = -0.18436798, & \psi_1(\tau_1) = -1.08863205, \\ \psi_1(1) = -0.47781383, & \psi_2(1) = 0.60184112\,. & \end{array} \tag{5.15}$$

Since the bang-bang control has only one switching point, we are in the position to apply Theorem 3.4. To check the assumptions of this theorem it remains to verify the condition $D^1(H) = |\dot{\sigma}(t_1)[u]^1| > 0$. Indeed, in view of the adjoint equation (5.10) and the switching condition $\psi_2(\tau_1) = 0$ we find for the original time variable $t_1 = \tau_1 \cdot T$,

$$D^1(H) = |\dot{\sigma}(t_1)[u]^1| = 2|\psi_1(t_1)| = 2 \cdot 1.08863205 > 0\,.$$

Then Theorem 3.4 asserts that the computed solution is a strict strong minimum.

Let us briefly discuss the optimal solution for the following boundary values (cf. Kaya and Noakes [14]) different from those in (5.7),

$$x_1(0) = x_2(0) = 1, \quad x_1(T) = x_2(T) = 0\,. \tag{5.16}$$

The optimal bang-bang control has two bang-bang arcs with one switching point τ_1. However, the control structure is reversed as compared to the one in (5.13):

$$u(\tau) = \left\{ \begin{array}{rll} -1 & \text{for} & 0 \le \tau \le \tau_1 \\ 1 & \text{for} & \tau_1 \le \tau \le 1 \end{array} \right\}. \tag{5.17}$$

We get the following numerical results,

$$\begin{array}{lll} T = 3.09520234, & \tau_1 = 0.23358852, & t_1 = \tau_1 \cdot T = 0.72300373, \\ \psi_1(0) = 0.94728449, & \psi_2(0) = 0.97364224, & \psi_1(\tau_1) = 1.70467637, \\ \psi_1(1) = 0.19669125, & \psi_2(1) = -1\,, & \end{array}$$

for which we obtain

$$D^1(H) = |\dot{\sigma}(t_1)[u]^1| = 2|\psi_1(t_1)| = 2 \cdot 1.70467637 > 0\,.$$

Theorem 3.4 shows again that the computed solution is a strict strong minimum.

5.2. Time-optimal control of the Rayleigh problem. The Rayleigh problem is concerned with the same electric circuit as treated in the previous section. However, the state variables are different since now the state variable $x_1(t) = I(t)$ denotes the electric current; cf. the dynamical model in [12, 27, 28, 29].

The control problem is to minimize the endtime T subject to

$$\dot{x}_1(t) = x_2(t)\,, \quad \dot{x}_2(t) = -x_1(t) + x_2(t)(1.4 - 0.14x_2(t)^2) + 4u(t)\,, \tag{5.18}$$

$$x_1(0) = x_2(0) = -5\,, \quad x_1(T) = x_2(T) = 0\,, \tag{5.19}$$

$$|\,u(t)\,| \le 1 \quad \text{for } \; t \in [0, T\,]\,. \tag{5.20}$$

The Pontryagin function (2.5) for this problem is

$$H(x, u, \psi) = \psi_1 x_2 + \psi_2\,(-x_1 + x_2(1.4 - 0.14x_2^2) + 4u). \tag{5.21}$$

The time transformation (5.1) and the transformed state and adjoint equations (5.2), (5.3) in the time interval $\tau \in [0, 1]$ lead to the following equations; again, the time argument τ will be omitted:

$$\begin{array}{rclrcl} dx_1/d\tau & = & T \cdot x_2\,, & dx_2/d\tau & = & T \cdot (-x_1 + x_2(1.4 - 0.14x_2^2) + 4u)\,, \\ d\psi_1/d\tau & = & T \cdot \psi_2\,, & d\psi_2/d\tau & = & -T \cdot (\psi_1 + \psi_2(1.4 - 0.42x_2^2))\,, \\ dT/d\tau & = & 0\,. & & & \end{array} \tag{5.22}$$

The boundary conditions (5.19) and the condition (5.4) yield, in view of (5.21),

$$x_1(0) = x_2(0) = -5, \quad x_1(1) = x_2(1) = 0, \quad 4\,\psi_2(1)\,u(1) + 1 = 0. \tag{5.23}$$

The switching function $\sigma(x, \psi) = 4\psi_2$ determines the optimal control via the minimum condition (2.13):

$$u(\tau) = \left\{ \begin{array}{rll} 1 & \text{if} & \psi_2(\tau) < 0 \\ -1 & \text{if} & \psi_2(\tau) > 0 \end{array} \right\}. \tag{5.24}$$

Again, the *singular case* with $\psi_2(\tau) \equiv 0$ holding in a time interval $[\tau_1, \tau_2]$ can be eliminated. Hence, the optimal control is bang-bang. In view of the special terminal

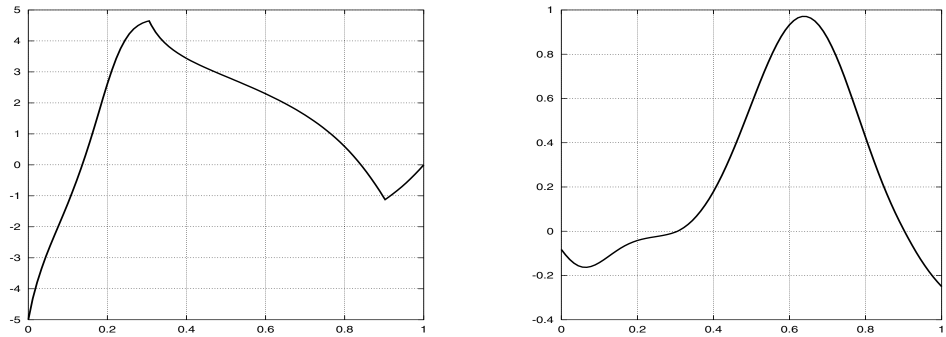


FIG. 5.2. *Rayleigh problem: state $x_2(\tau)$ and switching function $\psi_2(\tau) = \sigma(\tau)/4$, $\tau \in [0,1]$.*

conditions for the state, a simple reasoning reveals that the optimal control cannot be composed of only two bang-bang arcs. Computations show that the optimal control comprises the following three bang-bang arcs:

$$u(\tau) = \left\{ \begin{array}{rll} 1 & \text{for} & 0 \le \tau \le \tau_1 \\ -1 & \text{for} & \tau_1 \le \tau \le \tau_2 \\ 1 & \text{for} & \tau_2 \le \tau \le 1 \end{array} \right\}. \tag{5.25}$$

This control structure yields the two switching conditions

$$\psi_2(\tau_1) = 0\,, \quad \psi_2(\tau_2) = 0\,. \tag{5.26}$$

Thus we have to solve the multipoint boundary value problem consisting of the state and adjoint equations (5.22) with the optimal control structure (5.25), the boundary conditions (5.23), and the switching conditions (5.26).

The code BNDSCO in [34] yields the final time, the switching points, and some selected values for the adjoint variables as follows:

$$\begin{array}{rclrcl} T & = & 3.66817339\,, & & & \\ \tau_1 & = & 0.30546718\,, & \tau_2 & = & 0.90236928\,, \\ t_1 & = & \tau_1 \cdot T = 1.12050658\,, & t_2 & = & \tau_2 \cdot T = 3.31004698\,, \\ \psi_1(0) & = & -0.12234128\,, & \psi_2(0) & = & -0.08265161\,, \\ \psi_1(\tau_1) & = & -0.21521225\,, & \psi_1(\tau_2) & = & 0.89199176\,, \\ \psi_1(1) & = & 0.84276186\,, & \psi_2(1) & = & -0.25\,. \end{array} \tag{5.27}$$

Figure 5.2 displays the state variable $x_2(\tau)$ and the switching function $\psi_2(\tau)$ which match precisely the control laws (5.24) and (5.25).

We are going to show now in two different ways that the computed control provides a strict strong minimum. First, we compute the quantities $D^k(H) = -\dot{\sigma}(t_k)[u]^k$, $k = 1,2$, where $-\dot{\sigma}(t_k) = -4\dot{\psi}_2(t_k) = 4\psi_1(t_k)$ holds in view of the adjoint equation in (5.22) evaluated in the original time variable $t \in [0,T]$. Inserting the values from (5.27) we find

$$D^1(H) = 8 \cdot 0.21521225 = 1.7269800 > 0\,,\; D^2(H) = 8 \cdot 0.89199176 = 7.1359341 > 0\,.$$

The variational system $\dot{y} = f_x(t)y$ with $y = (y_1, y_2)$ in (3.3) reads explicitly

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = -y_1 + (1.4 - 0.42x_2^2)y_2\,.$$

The initial values for the variations $y^1(t)$, $y^2(t)$ w.r.t. the switching points t_1, t_2 are

$$y^1(t_1) = [\dot{x}]^1 = \begin{pmatrix} 0 \\ -8 \end{pmatrix}, \quad y^2(t_2) = [\dot{x}]^2 = \begin{pmatrix} 0 \\ 8 \end{pmatrix}.$$

At the second switching point t_2 we find $y^1(t_2) = (0, 2.517130)$. In view of the initial value $y^2(t_2) = (0, 8)$, this already implies that the vectors $y^1(T)$ and $y^2(T)$ are *linearly dependent*. Explicitly, we get $y^1(T) = (1.084614, 3.656286)$, $y^2(T) = (3.447153, 11.620490)$ which gives $y^2(T) = \alpha y^1(T)$ with $\alpha = 3.17823$ in relation (3.9). Thus, condition (b) in Proposition 3.2 asserting the zero critical subspace is not satisfied here. Here, the critical subspace is a one-dimensional subspace and the test for optimality proceeds via Proposition 3.5 by verifying that the number ρ in (3.15) is positive. Using the above variational vectors we compute

$$\int_{t_1}^{t_2} \langle H_{xx}(t) y^1(t), y^1(t) \rangle \, dt = -0.84 \int_{t_1}^{t_2} x_2(t) \psi_2(t) (y_2^1(t))^2 dt \; = \; -0.97063758.$$

Finally, observing the relations $[\dot{\psi}]^1 = [\dot{\psi}]^2 = 0$ and inserting the computed values of $D^1(H)$, $D^2(H)$ and α we obtain

$$\rho = 1.726980 + 0.706448 - 0.970638 = 1.462790 > 0.$$

Hence, we have shown that the solution described by (5.27) is a strict strong minimum.

An alternative proof of optimality proceeds via Theorem 4.9. Consider the symmetric 2×2 matrix

$$Q(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{12}(t) & Q_{22}(t) \end{pmatrix}.$$

The linear equation (4.3), $\dot{Q} = -Qf_x - f_x^* Q - H_{xx}$, in the original time variable $t \in [t_1, t_2]$ leads to the following three ODEs:

$$\begin{aligned} \dot{Q}_{11} &= 2\, Q_{12}, \\ \dot{Q}_{12} &= -Q_{11} - Q_{12}(1.4 - 0.42 x_2^2) + Q_{22}, \\ \dot{Q}_{22} &= -2\,(Q_{12} + Q_{22}(1.4 - 0.42 x_2^2)) + 0.84 \psi_2 x_2. \end{aligned} \tag{5.28}$$

We have to find a solution $Q(t)$ that satisfies the estimates (4.22), respectively, (4.27) at the switching points t_1 and t_2,

$$Q_{22}(t_1) > -\frac{|\psi_1(t_1)|}{8} = -0.026901531\,, \quad Q_{22}(t_2) < \frac{|\psi_1(t_2)|}{8} = 0.11149897\,. \tag{5.29}$$

These conditions hold if we choose, e.g., the following initial values at the switching point t_1,

$$Q_{11}(t_1) = 0, \quad Q_{12}(t_1) = 0, \quad Q_{22}(t_1) = -0.02,$$

which produce the value $Q_{22}(t_2) = -0.048826568$ at the second switching point. Then Theorem 4.9 assures us that the computed solution (5.27) provides a strict strong minimum.

5.3. Time-optimal control of a nuclear reactor. Hassan, Ghonaimy, and Abdel Malek [10] have presented a model for the time-optimal control of a nuclear reactor. A detailed solution has been given in Maurer [26]. Now our aim is to verify second order conditions for this specific solution. The model comprises the state variables x_1, neutron density; x_2, delayed neutron concentration; and x_3, reactivity. The control problem is to minimize the final time T subject to

$$\begin{array}{lll} \dot{x}_1(t) = k_1(x_3(t)-1)x_1(t) + k_2x_2(t)\,, & x_1(0) = n_0, & x_1(T) = n_f, \\ \dot{x}_2(t) = k_1x_1(t) - k_2x_2(t)\,, & x_2(0) = n_0k_1/k_2, & x_2(T) = n_fk_1/k_2, \\ \dot{x}_3(t) = u(t)\,, & x_3(0) = 0, & x_3(T) = 0, \\ |\,u(t)\,| \le 0.2 & \text{for } \ t \in [0,T\,]\,. & \end{array} \tag{5.30}$$

The constants are $k_1 = 5.0$, $k_2 = 0.1$, $n_0 = 0.04$, $n_f = 0.06$. The Pontryagin function or Hamiltonian (2.5) becomes

$$H(x,u,\psi) = \psi_1(k_1(x_3-1)x_1 + k_2x_2) + \psi_2(k_1x_1 - k_2x_2) + \psi_3 u\,. \tag{5.31}$$

The time transformation (5.1) and the scaled equations (5.2)–(5.4) yield the following state and adjoint equations and boundary conditions:

$$\begin{array}{lcllll} dx_1/d\tau & = & T\cdot(k_1(x_3-1)x_1 + k_2x_2)\,, & x_1(0) = 0.04, & x_1(1) = 0.06, \\ dx_2/d\tau & = & T\cdot(k_1x_1 - k_2x_2)\,, & x_2(0) = 2, & x_2(1) = 3, \\ dx_3/d\tau & = & T\cdot u(\tau)\,, & x_3(0) = 0, & x_3(1) = 0, \\ d\psi_1/d\tau & = & -T\cdot(\psi_1k_1(x_3-1) + \psi_2k_1)\,, & & \\ d\psi_2/d\tau & = & T\cdot k_2(\psi_2 - \psi_1)\,, & & \\ d\psi_3/d\tau & = & -T\psi_1k_1x_1\,, & \psi_3(0) = -5\,, & \psi_3(1) = -5\,. \end{array} \tag{5.32}$$

The switching function $\sigma(x,\psi) = \psi_3(t)$ determines the optimal control via $u(t) = -0.2\,sign(\psi_3(t))$. The optimal control computed in [26] is composed of three bang-bang arcs,

$$u(\tau) = \left\{ \begin{array}{rll} 0.2 & \text{for} & 0 \le \tau \le \tau_1 \\ -0.2 & \text{for} & \tau_1 \le \tau \le \tau_2 \\ 0.2 & \text{for} & \tau_2 \le \tau \le 1 \end{array} \right\}, \tag{5.33}$$

which imply the two further switching conditions

$$\psi_3(\tau_1) = 0, \quad \psi_3(\tau_2) = 0\,. \tag{5.34}$$

The earlier computations in [26] are confirmed by the code BNDSCO in [34] which yields the following solution of the boundary value problem (5.32)–(5.34):

$$\begin{array}{rclrcl} T & = & 7.04780685, & & & \\ \tau_1 & = & 0.47987830\,, & t_1 & = & \tau_1\cdot T = 3.38208957\,, \\ \tau_2 & = & 0.97987830\,, & t_2 & = & \tau_2\cdot T = 6.90599299\,, \\ \psi_1(0) & = & -2.97015515\,, & \psi_2(0) & = & -2.84546900\,, \\ \psi_1(\tau_1) & = & -5.22557130\,, & \psi_2(\tau_1) & = & -2.22864972\,, \\ x_1(\tau_1) & = & 0.11014294\,, & x_1(\tau_2) & = & 0.06078025\,, \\ \psi_1(\tau_2) & = & 78.6539693\,, & \psi_2(\tau_2) & = & -3.53032114\,, \\ \psi_1(1) & = & 165.786058\,, & \psi_2(1) & = & -5.25230261\,. \end{array} \tag{5.35}$$

The state variable $x_3(\tau)$ and the switching function $\sigma(\tau) = \psi_3(\tau)$ are displayed in Figure 5.3.

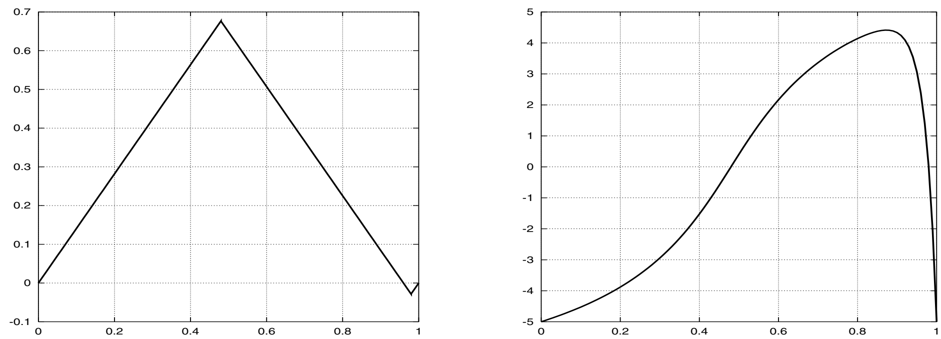


FIG. 5.3. *Nuclear reactor: state $x_3(\tau)$ and switching function $\sigma(\tau) = \psi_3(\tau)$.*

As in the foregoing example, we can show in two different ways that the computed control provides a strict strong minimum. The quantities

$$D^k(H) = -\dot{\sigma}(t_k)[u]^k = 0.4\,|\dot{\psi}_3(t_k)| = 0.4\,|\psi_1(t_k)k_1x_1(t_k)|, \quad k = 1, 2,$$

are computed on the basis of solution data in (5.35) as

$$D^1(H) = 1.15111957 \, > 0, \quad D^2(H) = 9.56121580 \, > 0\,.$$

Evaluating the variational system (3.3), $\dot{y} = f_x(t)y$ with $y = (y_1, y_2, y_3)$, we get

$$\dot{y}_1 = k_1(x_3 - 1)y_1 + k_2y_2 + k_1x_1y_3, \quad \dot{y}_2 = k_1y_1 - k_2y_2, \quad \dot{y}_3 = 0.$$

The initial values for the variations $y^1(t)$, $y^2(t)$ w.r.t. t_1, t_2 are

$$y^1(t_1) = [\dot{x}]^1 = \begin{pmatrix} 0 \\ 0 \\ -0.4 \end{pmatrix}, \quad y^2(t_2) = [\dot{x}]^2 = \begin{pmatrix} 0 \\ 0 \\ 0.4 \end{pmatrix}.$$

This leads to the following variational vectors at the terminal time T:

$$y^1(T) = \begin{pmatrix} -0.04508835 \\ -1.0424039 \\ -0.4 \end{pmatrix}, \quad y^2(T) = \begin{pmatrix} 0.012216498 \\ 0.0048217532 \\ 0.4 \end{pmatrix}, \quad \dot{x}(T) = \begin{pmatrix} 0 \\ 0 \\ 0.2 \end{pmatrix},$$

which obviously are *linearly independent.* Thus, either condition (a) or (b) in Proposition 3.2 implies that the critical cone is $\mathcal{K} = \{0\}$. Hence, Theorem 3.3 asserts that the solution candidate characterized by (5.35) provides indeed a strict strong minimum.

Alternatively, it is instructive to use also the test of optimality in Theorem 4.9. Since $d(x) = 3$ we consider the symmetric 3×3 matrix $Q(t) = (Q_{ik})_{1 \le i,k \le 3}$. By evaluating the linear equation (4.3) one immediately recognizes that the equations for Q_{11}, Q_{12}, Q_{22} are homogeneous in these variables and can thus be satisfied by $Q_{11}(t) = Q_{12}(t) = Q_{22}(t) \equiv 0$. The remaining three equations then simplify to

$$\begin{aligned} \dot{Q}_{13} &= -Q_{13}k_1(x_3 - 1) - Q_{23}k_1 - \psi_1 k_1, \\ \dot{Q}_{23} &= -Q_{13}k_2 + Q_{23}k_2, \\ \dot{Q}_{33} &= -2Q_{13}k_1x_1. \end{aligned} \tag{5.36}$$

Our task is to find a solution to these ODEs which satisfies the estimates (4.22) or (4.27) at the switching points t_1 and t_2. Since

$$\frac{|\dot{\psi}_3|}{2\beta c} = \frac{k_1|\psi_1 x_1|}{0.4} = 12.5|\psi_1 x_1|,$$

conditions (4.27) require that the following estimates be satisfied:

$$\begin{array}{lcrcr} Q_{33}(t_1) & > & -12.5\,|\psi_1(t_1)x_1(t_1)| & = & -7.1944973, \\ Q_{33}(t_2) & < & 12.5\,|\psi_1(t_2)x_1(t_2)| & = & 59.788260. \end{array} \tag{5.37}$$

The strategy for finding appropriate initial values at the point t_1 is the following: we fix the initial values

$$Q_{13}(t_1) = 0, \quad Q_{33}(t_1) = 0,$$

and determine $Q_{23}(t_1)$ in such a way that the inequality $Q_{33}(t_2) < 59.788260$ holds. We found that the initial value $Q_{23}(t_1) = 4.23$ produced the value $Q_{33}(t_2) = -96.953435$. Hence, the inequalities (5.37) hold and Theorem 4.9 asserts that the computed solution is a strict strong minimum.

6. Conclusion. We have considered time-optimal bang-bang control problems with finitely many switching points. SSC for such problems amount to the requirement that a certain quadratic form be positive on a finite-dimensional critical subspace. An explicit representation of the critical subspace has been derived in terms of the variations of the state trajectories w.r.t. the switching points. For bang-bang controls with one or two switching points, this approach results in a rather straightforward test of SSC. To treat the general case, we have shown that the so-called Q-transformation allows us to convert the quadratic form to another quadratic form which might be better suited for practical verification. The resulting numerical test then consists in determining a solution of a linear matrix differential equation which satisfies additional jump conditions at the switching points. The viability of the presented tests has been demonstrated by three numerical examples. Further examples with applications of bang-bang control to the design of lasers may be found in the dissertation of Kim [16].

Though the techniques have been developed in this paper only for time-optimal bang-bang controls with fixed terminal conditions, the basic ideas apply as well to arbitrary bang-bang control problems with general cost functionals and boundary conditions. Results for this general approach will be presented in a future paper that will also highlight a more detailed analysis of the boundary conditions.

During the revision of this paper we became aware of the work of Agrachev, Stefani, and Zezza [1], where a different approach to SSC for bang-bang controls is presented for problems with *fixed* final time. Agrachev and his coauthors reduce the bang-bang control problem to a finite-dimensional optimization problem w.r.t. the switching times and show that it suffices to test SSC for this optimization problem. Currently, we are implementing this approach and are in the process of comparing it with the numerical methods given in the present paper. Recently, we have been able to show that the SSC given in Theorem 3.3 are equivalent to the SSC in Agrachev, Stefani, and Zezza [1] in the case when the set M_0 of Lagrange multipliers is a singleton which is not assumed in Theorem 3.3. The SSC developed in this paper and in [1] will pave the way to a theoretical and computational sensitivity analysis for bang-bang control problems which is similar in spirit to that developed in [2, 21, 22, 23, 24, 25, 27, 28].

Acknowledgments. We are grateful to the referees for helpful remarks and suggestions to improve the paper. We are also indebted to Dirk Augustin for providing us the bang-bang control for the Rayleigh problem in section 5.2.

REFERENCES

[1] A. A. Agrachev, G. Stefani, and P. L. Zezza, *Strong optimality for a bang-bang trajectory*, SIAM J. Control Optim., 41 (2002), pp. 991–1014.

[2] D. Augustin and H. Maurer, *Computational sensitivity analysis for state constrained optimal control problems*, Ann. Oper. Res., 101 (2001), pp. 75–99.

[3] A. Bressan, *A high order test for optimality of bang-bang controls*, SIAM J. Control Optim., 23 (1985), pp. 38–48.

[4] A. E. Bryson and Y. C. Ho, *Applied Optimal Control*, Revised printing, Hemisphere Publishing, Washington, D.C., 1975.

[5] R. Bulirsch, *Die Mehrzielmethode zur numerischen Lösung von nichtlinearen Randwertproblemen und Aufgaben der optimalen Steuerung*, Report of the Carl-Cranz Gesellschaft, Oberpfaffenhofen, Germany, 1971.

[6] F. L. Chernousko, L. D. Akulenko, and N. N. Bolotnik, *Time-optimal control for robotic manipulators*, Optimal Control Appl. Methods, 10 (1989), pp. 293–311.

[7] A. L. Dontchev, W. W. Hager, A. B. Poore, and B. Yang, *Optimality, stability, and convergence in nonlinear control*, Applied Math. Optim., 31 (1995), pp. 297–326.

[8] J. C. Dunn, *Second-order optimality conditions in sets of L^∞ functions with range in a polyhedron*, SIAM J. Control Optim., 33 (1995), pp. 1603–1635.

[9] J. C. Dunn, *On L^2 sufficient conditions and the gradient projection method for optimal control problems*, SIAM J. Control Optim., 34 (1996), pp. 1270–1290.

[10] M. A. Hassan, M. A. R. Ghonaimy, and N. R. Abdel Malek, *Computational solution of the nuclear reactor minimum time start-up problem with state constraints*, in 2nd IFAC Symposium on Multivariable Technical Control Systems, Düsseldorf, Germany, Oct. 11–13, North–Holland, Amsterdam, 1971.

[11] M. Hestenes, *Calculus of Variations and Optimal Control Theory*, John Wiley, New York, 1966.

[12] D. H. Jacobson and D. Q. Mayne, *Differential Dynamic Programming*, American Elsevier, New York, 1970.

[13] Y. C. Kaya and J. L. Noakes, *Computations and time-optimal controls*, Optimal Control Appl. Methods, 17 (1996), pp. 171–185.

[14] Y. C. Kaya and J. L. Noakes, *Computational method for time-optimal switching control*, J. Optim. Theory Appl., 117 (2003), pp. 69–92.

[15] J.-H. R. Kim, H. Maurer, Yu. A. Astrov, M. Bode, and H.-G. Purwins, *High-speed switch-on of a semiconductor gas discharge image converter using optimal control methods*, J. Comput. Phys., 170 (2001), pp. 395–414.

[16] J.-H. R. Kim, *Optimierungsmethoden und Sensitivitätsanalyse für optimale bang-bang Steuerungen mit Anwendungen in der Nichtlinearen Optik*, Dissertation, Institut für Numerische Mathematik, Universität Münster, Münster, Germany, 2002.

[17] H. W. Knobloch, *Higher Order Necessary Conditions in Optimal Control Theory*, Lecture Notes in Comput. Sci. 34, Springer-Verlag, Berlin, 1981.

[18] A. J. Krener, *The high order maximum principle and its application to singular externals*, SIAM J. Control Optim., 15 (1977), pp. 256–293.

[19] U. Ledzewicz and H. Schättler, *Optimal bang-bang controls for a two-compartment model in cancer chemotherapy*, J. Optim. Theory Appl., 114 (2002), pp. 609–637.

[20] E. S. Levitin, A. A. Milyutin, and N. P. Osmolovskii, *Conditions of high order for a local minimum in problems with constraints*, Uspeki Mat. Nauk, 33 (1978), pp. 85–148 (in Russian); Russian Math. Surveys, 33 (1978), pp. 97–168 (in English).

[21] K. Malanowski, *Stability and sensitivity of solutions to nonlinear optimal control problems*, Appl. Math. Optim., 32 (1995), pp. 111–141.

[22] K. Malanowski, *Stability and sensitivity analysis for optimal control problems with control-state constraints*, Dissertationes Math. (Rozprawy Mat.), 394, (2001), pp. 1–51.

[23] K. Malanowski and H. Maurer, *Sensitivity analysis for parametric control problems with control-state constraints*, Comput. Optim. Appl., 5 (1996), pp. 253–283.

[24] K. Malanowski and H. Maurer, *Sensitivity analysis for state constrained optimal control problems*, Discrete Contin. Dynam. Systems, 4 (1998), pp. 241–272.

[25] K. Malanowski and H. Maurer, *Sensitivity analysis for optimal control problems subject to higher order state constraints. Optimization with data perturbations*, II, Ann. Oper. Res., 101 (2001), pp. 43–73.

[26] H. Maurer, *On optimal control problems with bounded state variables and control appearing linearly*, SIAM J. Control Optim., 15 (1977), pp. 345–362.

[27] H. Maurer and D. Augustin, *Second order sufficient conditions and sensitivity analysis for the controlled Rayleigh problem*, in Parametric Optimization and Related Topics, IV, J. Guddat et al., eds., Lang, Frankfurt, 1997, pp. 245–259.

[28] H. Maurer and D. Augustin, *Sensitivity analysis and real-time control of parametric optimal control problems using boundary value methods*, in On-line Optimization of Large Scale Systems, M. Grötschel et al., eds., Springer-Verlag, Berlin, 2001, pp. 17–55.

[29] H. Maurer and H. J. Oberle, *Second order sufficient conditions for optimal control problems with free final time: The Riccati approach*, SIAM J. Control Optim., 41 (2002), pp. 380–403.

[30] H. Maurer and S. Pickenhain, *Second-order sufficient conditions for optimal control problems with mixed control-state constraints*, J. Optim. Theory Appl., 86 (1995), pp. 649–667.

[31] A. A. Milyutin and N. P. Osmolovskii, *Calculus of Variations and Optimal Control*, Transl. Math. Monogr. 180, AMS, Providence, RI, 1998.

[32] L. W. Neustadt, *Optimization: A Theory of Necessary Conditions*, Princeton University Press, Princeton, NJ, 1976.

[33] J. Noble and H. Schättler, *Sufficient conditions for relative minima of broken extremals in optimal control theory*, J. Math. Anal. Appl., 269 (2002), pp. 98–128.

[34] H. J. Oberle and W. Grimm, *BNDSCO — A Program for the Numerical Solution of Optimal Control Problems*, Internal report 515–89/22, Institute for Flight Systems Dynamics, DLR, Oberpfaffenhofen, Germany, 1989.

[35] N. P. Osmolovskii, *High-order necessary and sufficient conditions for Pontryagin and bounded-strong minima in the optimal control problems*, Dokl. Akad. Nauk SSSR, Ser. Cybernetics and Control Theory, 303 (1988), pp. 1052–1056 (in Russian); Sov. Phys. Dokl., 33 (1988), pp. 883–885 (in English).

[36] N. P. Osmolovskii. *Theory of Higher Order Conditions in Optimal Control*, Doctor of Sci. thesis, Moscow, Russia, 1988 (in Russian).

[37] N. P. Osmolovskii, *Quadratic conditions for nonsingular extremals in optimal control (a theoretical treatment)*, Russian J. Math. Phys., 2 (1995), pp. 487–516.

[38] N. P. Osmolovskii, *Quadratic conditions for nonsingular extremals in optimal control (examples)*, Russian J. Math. Phys., 5 (1998), pp. 373–388.

[39] N. P. Osmolovskii, *Second-order conditions for broken extremals*, in Calculus of Variations and Optimal Control, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 198–216.

[40] N. P. Osmolovskii, *Quadratic extremality conditions for broken extremals in the general problem of calculus of variations*, J. Math. Sci., submitted.

[41] N. P. Osmolovskii and F. Lempio, *Jacobi conditions and the Riccati equation for a broken extremal*, J. Math. Sci., 100 (2000), pp. 2572–2592.

[42] N. P. Osmolovskii and F. Lempio, *Transformation of quadratic forms to perfect squares for broken extremals*, Set-Valued Anal., 10 (2002), pp. 209-232.

[43] A. Sarychev, *First- and second-order sufficient optimality conditions for bang-bang controls*, SIAM J. Control Optim., 35 (1997), pp. 315–340.

[44] H. Schättler, *On the local structure of time-optimal bang-bang trajectories in* $\mathbb{R}^3$, SIAM J. Control Optim., 26 (1988), pp. 186–204.

[45] H. J. Sussmann, *A bang-bang theorem with bounds on the number of switchings*, SIAM J. Control Optim., 17 (1979), pp. 629–651.

[46] H. J. Sussmann, *The structure of time-optimal trajectories for single-input systems in the plane: The* C^∞ *nonsingular case*, SIAM J. Control Optim., 25 (1987), pp. 433–465.

[47] H. J. Sussmann, *The structure of time-optimal trajectories for single-input systems in the plane: The general real analytic case*, SIAM J. Control Optim., 25 (1987), pp. 868–904.

[48] V. Zeidan, *The Riccati equation for optimal control problems with mixed state-control constraints: Necessity and sufficiency*, SIAM J. Control Optim., 32 (1994), pp. 1297–1321.