Sensitivity Analysis of Optimal Control Problems with Bang–Bang Controls

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Abstract—We study bang–bang control problems that depend on a parameter \( p \). For a fixed nominal parameter \( p_0 \), it is assumed that the bang–bang control has finitely many switching points and satisfies second order sufficient conditions (SSC). SSC are formulated and checked in terms of an associated finite-dimensional optimization problem w.r.t. the switching points and the free final time. We show that the nominal optimal bang–bang control can be locally embedded into a parametric family of optimal bang–bang controls where the switching points are differentialable function of the parameter. A well known sensitivity formula from optimization [7] is used to compute the parametric sensitivity derivatives of the switching points which also allows to determine the sensitivity derivatives of the optimal state trajectories.

I. INTRODUCTION

Sensitivity analysis for parametric optimal control problems has been studied extensively in the case that the control variable enters the system nonlinearly; cf., e.g., [5], [11], [12], [13]. In these papers, the basic assumption for sensitivity analysis is that the strict Legendre condition holds which precludes the application to bang–bang or singular controls. Here, we focus attention on optimal control problems with bang–bang controls. Recently, Agrachev et al. [1] have developed second–order sufficient conditions (SSC) for bang–bang controls which are stated in terms of an associated finite-dimensional nonlinear programming problem where the optimization variables are the switching points and the unknown initial state. A similar result on SSC including the case of a free terminal time can be derived using the approach in Maurer, Osmolovskii [14]. In nonlinear optimization, it is well known that sensitivity analysis and the property of parametric solution differentiability is closely connected to the verification of SSC [7], [2], [3], [4]. The purpose of this paper is to show that a similar picture arises for optimal bang–bang controls. The main results concerns the solution differentiability of switching points and optimal trajectories with respect to parameters in the system.

II. PARAMETRIC OPTIMAL CONTROL PROBLEMS WITH CONTROL APPEARING LINEARLY

We consider parametric optimal control problems that depend on a parameter \( p \in \mathbb{R}^q \) which is not among the optimization variables. Let \( x(t) \in \mathbb{R}^n \) denote the state variable and \( u(t) \in \mathbb{R} \) the control variable in the time interval \( t \in \Delta = [0, t_f] \) with a free final time \( t_f > 0 \). For simplicity of notation we restrict the discussion to a scalar control. The control variable is supposed to appear linearly in the control system. The following parametric optimal control problem will be denoted by \( OC(p) \) for \( p \in \mathbb{R}^q \) : determine a pair of functions \((x^p(\cdot), u^p(\cdot))\) and a final time \( t_f^p \) which minimize the cost functional

\[
J(x, u, t_f, p) := g(x(0), x(t_f), t_f, p)
\]

subject to the constraints on the interval \([0, t_f^p]\),

\[
\dot{x}(t) = f(t, x, u, p) = f_1(t, x, p) + F_1(t, x, p)u,
\]

\[
\varphi(x(0), x(t_f), t_f, p) = 0,
\]

\[
u_{\text{min}} \leq u(t) \leq u_{\text{max}}, \quad u_{\text{min}} < u_{\text{max}}.
\]

The functions \( g : \mathbb{R}^{2n+1} \times \mathbb{R}^q \to \mathbb{R}, F_1 : \mathbb{R}^{n+1} \times \mathbb{R}^q \to \mathbb{R}^n, \) and \( \varphi : \mathbb{R}^{2n+1} \times \mathbb{R}^q \to \mathbb{R}, 0 \leq r \leq 2n, \) are assumed to be twice continuously differentiable. For a fixed parameter \( p_0 \in \mathbb{R}^q \), the optimal control problem \( OC(p_0) \) is called the nominal problem. For simplicity of notation we choose \( p_0 = 0 \). Our aim is to study optimal solutions to the perturbed control problems \( OC(p) \) for parameters \( p \) in a neighborhood \( P \subset \mathbb{R}^q \) of the nominal parameter \( p_0 = 0 \).

III. BANG–BANG CONTROLS

Let \( p \in P \subset \mathbb{R}^q \) be an arbitrary parameter. A control process

\[
T^p = \{ (x^p(t), u^p(t)) | t \in [0, t_f^p], t_f^p > 0 \}
\]

is said to be admissible for problem \( OC(p) \), if \( x^p(\cdot) \) is absolutely continuous, \( u^p(\cdot) \) is measurable and essentially bounded and the pair of functions \((x^p(t), u^p(t))\) satisfies the constraints (2)–(4) on the interval \( \Delta = [0, t_f^p] \). The component \( x(t) \) will be called the state trajectory.

DEFINITION 3.1: An admissible control process \( T^p = \{ (x^p(t), u^p(t)) | t \in [0, t_f^p], t_f^p > 0 \} \) is said to be a strong (resp., a strict strong) minimum for \( OC(p) \) if there exists \( \varepsilon > 0 \) such that \( g(x(0), x(t_f), t_f, p) \geq g(x^p(0), x^p(t_f), t_f^p, p) \) (resp., \( g(x(0), x(t_f), t_f, p) > g(x^p(0), x^p(t_f), t_f^p, p) \)) holds for all admissible control processes \( T = \{ (x(t), u(t)) | t \in [0, t_f] \} \) (resp., for all admissible control processes \( T \) different from \( T^p \)) with \( |t_f^p - t_f^p| < \varepsilon \) and

\[
\max \{ |x(t) - x^p(t)|, t \in [0, t_f^p] \cap [0, t_f] \} < \varepsilon.
\]

First order necessary optimality conditions for a strong minimum for \( OC(p) \) are given by Pontryagin’s minimum
The Pontryagin or Hamiltonian function is
\[ H(t, x, u, \lambda, p) = \lambda f(t, x, u) + \lambda F_1(t, x, p)u, \]
where \( \lambda \in \mathbb{R}^n \) is a row vector while \( x, f, f_1, F_1 \) are column vectors. The factor of \( u \) in the Hamiltonian is called the switching function
\[ \sigma(t, x, \lambda, p) = \lambda F_1(t, x, p). \]

Furthermore, let us introduce the end–point Lagrange function
\[ l(x_0, x_f, t_f, p) = g(x_0, x_f, t_f, p) + \rho \sigma(x_0, x_f, t_f, p), \]
and a multiplier \( \rho \in \mathbb{R}^r \) (row vector).

In the sequel, partial derivatives of functions will be denoted by subscripts referring to the respective variables.

If \( T' = \{ (x^p(t), u^p(t)) | t \in [0, t_f], t_f > 0 \} \) provides a strong minimum for \( OC(p) \) then there exist a pair of functions \( (\lambda^p(t), \lambda^{lp}(t)) \neq 0 \) and a multiplier \( \rho^p \in \mathbb{R}^r \) that satisfy the following conditions for a.e. \( t \in [0, t_f] \):
\[
\begin{align*}
\dot{\lambda}^p(t) &= -H_x(t, x^p(t), u^p(t), \lambda^p(t), p), \\
\dot{\lambda}_0^p(t) &= -H_t(t, x^p(t), u^p(t), \lambda^p(t), p), \\
\lambda^p(0) &= -\frac{t}{2}x_0((x^p(0), x^p(0), t_f, p)), \\
\lambda^p(t_f) &= l_f(x^p(0), x^p(t_f), t_f, p), \\
\lambda_0^p(t_f) &= l_t(x^p(0), x^p(t_f), t_f, p), \\
H(t, x^p(t), u^p(t), \lambda^p(t), p) + \lambda_0^p(t) &= 0, \\
H(t, x^p(t), u^p(t), \lambda^p(t), p) &= \min_{u \in [u_{\min}, u_{\max}]} H(t, x^p(t), u, \lambda^p(t), p).
\end{align*}
\]

Henceforth, we shall use the notations \( f^p(t) = f(t, x^p(t), u^p(t), p), \sigma^p(t) = \sigma(t, x^p(t), \lambda^p(t), p) \), etc...

The minimum condition (13) yields the following control law
\[ u^p(t) = \begin{cases} 
0, & \text{if } \sigma^p(t) < 0, \\
\lambda_{\max}, & \text{if } \sigma^p(t) > 0, \\
\text{undetermined}, & \text{if } \sigma^p(t) = 0.
\end{cases} \]

In the sequel, the basic assumption is that the nominal switching function \( \sigma^0(t) \) has only finitely many zeroes \( t^0_k, k = 1, \ldots, s, \) with
\[ 0 = t^0_0 < t^0_1 < \ldots < t^0_k < \ldots < t^0_s < t^0_{s+1} = t_f. \]

Moreover, the following so–called strict bang–bang property is supposed to hold,
\[ \frac{d}{dt} \sigma^0(t^0_k) \neq 0, \quad k = 1, \ldots, s. \]

Thus, the nominal control \( u^0(t) \) is assumed to be a bang–bang control having exactly \( s \) switching points \( t^0_k, (k = 1, \ldots, s) \) in the interior of time interval \( [0, t^0_f] \),
\[ u^0(t) \equiv u^0_k \quad \text{for} \quad t^0_{k-1} < t < t^0_k, \]
\[ u^0_k \in \{u_{\min}, u_{\max}\}, \quad k = 1, \ldots, s + 1. \]

IV. SECOND ORDER SUFFICIENT CONDITIONS AND SENSITIVITY ANALYSIS FOR OPTIMAL BANG–BANG CONTROLS

SSC for the optimality of the control (18) have been developed in [1], [14]. We are going to show that the structure of the optimal bang–bang control remains stable under small perturbations \( p \) of \( p_0 \). To this end, we reformulate the control problem \( OC(p) \) as a finite–dimensional optimization problem. Consider the optimization vector
\[ z := (x_0(t_0, t_1), \ldots, t_s, t_f) \in \mathbb{R}^{n+s+1}, \quad \text{where} \]
\[ 0 = t_0 < t_1 < \ldots < t_k < \ldots < t_s < t_{s+1} = t_f, \]
which is taken from a neighborhood of the nominal vector \( z^0 = (x^0(0), t^0_1, \ldots, t^0_k) \). For any vector \( z \) in (19) we shall denote by \( u^z(t) \) the bang–bang control with
\[ u^z(t) \equiv u^z_k \quad \text{for} \quad t_{k-1} < t < t_k, \]
\[ (k = 1, \ldots, s + 1), \quad x(z) = x_0. \]

The initial condition \( x(0) = x_0 \) is given by the component \( x_0 \) of \( z \). Denote this solution by \( x(t; z, p) = x(t; x_0, t_1, \ldots, t_s, p) \).

We shall need the partial derivatives
\[ \frac{\partial x}{\partial x_0}(t, z^0, p_0) \quad \text{(an \( n \times n \) matrix),} \]
\[ \frac{\partial x}{\partial t_k}(t, z^0, p_0), \quad k = 1, \ldots, s + 1. \]

It follows from elementary properties of ODEs that these partial derivatives can be expressed via the fundamental transition matrix \( Y(t, \tau) \) of the linearized state equation. Let the \((n \times n)\)–matrix \( Y(t, \tau) \) with \( 0 \leq \tau \leq t \leq t_f \) be the solution to the matrix differential equation
\[ \frac{d}{dt} Y(t, \tau) = f^0(t) Y(t, \tau), \quad Y(0, \tau) = I_n \quad \text{(unity matrix)}. \]

Then the partial derivatives in (22) have the following explicit representations:
\[ \frac{\partial x}{\partial x_0}(t, z^0, p_0) = Y(t, 0), \quad \frac{\partial x}{\partial t_k}(t_f, z^0, p_0) = x^0(t_f), \]
\[ \frac{\partial x}{\partial t_k}(t, z^0, p_0) = Y(t, t^0_k)(\dot{x}(t^0_k) - \dot{x}(t^0_k +)), \quad t \geq t^0_k, \]
\[ \dot{x}(t^0_k) - \dot{x}(t^0_k +) = f^0(t^0_k) - f^0(t^0_{k+1}) = F^0(t_k)(u_k^0 - u_{k+1}^0), \]
where $f^0(t_k^0) - f^0(t_k^0+)$ denotes the jump at the switching point $t_k$. Using the solution $x(t, z, p)$ of the state equation we arrive at the following parametric optimization problem $OP(p)$ w.r.t. the optimization variable $z$ in (19):

$$\begin{align*}
\text{Minimize} & \quad G(z, p) := g(x_0, x(t_f; z, p), p) \\
\text{subject to} & \quad \Phi(z, p) := \varphi(x_0, x(t_f; z, p), p) = 0.
\end{align*}$$

The functions $G$ and $\Phi$ are twice continuously differentiable in view of (19). The Lagrange function for the parametric optimization problem $OP(p)$ becomes

$$\mathcal{L}(z, \rho, p) := G(z, p) + \rho \Phi(z, p), \quad \rho \in \mathbb{R}^r \text{ (row vector).}$$

The well known sensitivity result in Fiacco [7] translates as follows; cf. also [3], [4].

**Theorem 4.1:** (SSC and solution differentiability for the optimization problem $OP(p)$) Let $z^0 = (x_0^{(0)}, t_1^{(0)}, ..., t_s^{(0)}, t_f^{(0)})$ with $\rho^0 \in \mathbb{R}^r$ satisfy the SSC for the nominal optimization problem $OP(p_0)$, i.e., assume that the following conditions hold:

$$\begin{align*}
\mathcal{L}_z(z^0, \rho^0, p_0) = G_z(z^0, p_0) + \rho^0 \Phi_z(z^0, p_0) = 0, \\
\text{rank} \left( \Phi_z(z^0, p_0) \right) = r,
\end{align*}$$

$$\forall \; v \in \mathbb{R}^{n+s+1}, \quad v \neq 0, \quad \Phi_z(z^0, p_0)v = 0.$$  \hfill (29)

(29) Then there exists a neighborhood $P_0 \subset \mathbb{R}^q$ of $p_0$ such that for each $p \in P_0$ there are vectors $z^p = (x_0^p, t_1^p, ..., t_s^p, t_f^p)$ and $\rho^p \in \mathbb{R}^p$ which satisfy SSC for the parametric optimization problem $OP(p)$ and, hence, provide a strict minimum. Moreover, the function $p \rightarrow (z^p, \rho^p)$ is continuously differentiable in $P_0$.

The computation of the sensitivity derivatives proceeds via the following explicit formula

$$\begin{align*}
\left( \frac{dz^p}{dp} \right)^* dp \left( \frac{d\rho^p}{dp} \right)^* dp = - \left( \frac{\mathcal{L}_{zz}(z^p, \rho^p, p)}{\Phi_z(z(p), p)} \Phi_z(z(p), p)^* \right)^{-1} \\
\times \left( \frac{\mathcal{L}_{zp}(z^p, \rho^p, p)}{\Phi_p(z(p), p)} \Phi_p(z(p), p) \right),
\end{align*}$$

where the asterisk denotes the transpose; cf. [7], [3]. Note that the matrix on the right hand side, the so-called Kuhn–Tucker–matrix, is non–singular due to assumptions (30), (31) in Theorem 4.1.

On the basis of the solution $(z^p, \rho^p)$ to the optimization problem $OP(p)$ we are now going to construct a family of bang–bang controls and their associated adjoint variables that will give a strict strong minimum of the perturbed control problem $OC(p)$. As a first result we can express SSC for the nominal optimization problem $OC(p_0)$ in the following way.

**Theorem 4.2:** (SSC for the nominal control problem $OC(p_0)$) Suppose that the following conditions hold for the nominal control $u^0(t)$ defined in (18):

1. the vector $z^0 = (x_0^{(0)}, t_1^{(0)}, ..., t_s^{(0)}, t_f^{(0)})$ and the multiplier $\rho^0 \in \mathbb{R}^r$ satisfy the SSC for the nominal optimization problem $OP(p_0)$.
2. the strict bang–bang property $\frac{d}{dt} \sigma^0(t_k^0) \neq 0, \; k = 1, ..., s$, holds.

Then the pair $(u^0(\cdot), x^0(\cdot))$ provides a strict strong minimum for the nominal control problem $OC(p_0)$.

This theorem was proved in Agrachev et al. [1] for fixed final time $t_f$. It can be extended to the free final time case using the quadratic sufficient conditions and the representation of the critical cone derived in Maurer, Osmolovskii [14].

Now we focus attention on the parametric control problem $OC(p)$. Using the optimal solution $(z^p, \rho^p)$, we define the perturbed bang–bang control $u^p(t)$ by

$$\begin{align*}
u^p(t) \equiv u_k^0 \quad &\text{for } t_{k-1}^p < t < t_k^p \\
(k = 1, ..., s + 1, \quad t_{s+1}^p := t_f^p),
\end{align*}$$

where $u_k^0$ are the control values of the nominal control (18). Furthermore, consider the parametric state trajectory $x^p(t) := x(t, z^p, p)$ defined via (21). We intend to show that $(x^p(\cdot), u^p(\cdot))$ provide a strict strong minimum to the control problem $OC(p)$. For that purpose, let the adjoint functions $\lambda^p(t), \lambda^0_p(t)$ be solutions to the adjoints equations (8) and (9) with terminal conditions given by (11) and the second formula in (12),

$$\begin{align*}
\lambda^p(t_f^p) &= \ell_x(x^p(0), x^p(t_f^p), t_f^p, p), \\
\lambda^0_p(t_f^p) &= \ell_x(x^p(0), x^p(t_f^p), t_f^p, p).
\end{align*}$$

First, we have to validate the complete set of first order necessary conditions (8)–(14) for the optimal candidate $(x^p(\cdot), u^p(\cdot))$. Let $Y(t, \tau, p)$ be the transition matrix for the linearized system

$$\begin{align*}
\frac{d}{dt} Y(t, \tau, p) &= f^p(t) Y(t, \tau, p), \\
Y(\tau, \tau, p) &= I_n \text{ (unity matrix).}
\end{align*}$$

Then the solution of the adjoint equation (8), $\dot{\lambda}^p(t) = -\lambda^p(t) f^p(t)$, with terminal value (34) has the explicit representation

$$\lambda^p(t) = \lambda^p(t_f^p) Y(t_f^p, t, p), \quad 0 \leq t \leq t_f^p.$$  \hfill (36)

The first order Kuhn–Tucker condition for the optimization problem $OP(p)$ imply in view of (36) and the switching function (6),

$$\begin{align*}
0 &= \mathcal{L}_{t_k}(z^p, \rho^p, p) = G_{t_k}(z^p, p) + \rho^p \Phi_{t_k}(z^p, p) \\
&= \lambda^p(t_k^p) Y(t_k^p, t_{k+1}^p, p) (\dot{x}(t_k^p) - \dot{x}(t_{k+1}^p)) \\
&= \lambda^p(t_k^p) F_k^p(t_k^p)(u_k^0 - u_{k+1}^0) \\
&= \sigma^p(t_k^p)(u_k^0 - u_{k+1}^0),
\end{align*}$$

where $\lambda^p(t_k^p) = \lambda^p(t_f^p) Y(t_f^p, t_k^p, p) = \lambda^p(t_f^p)$.
where $\sigma^p(t)$ is the switching function defined in (6). In view of $u^p_k \neq u^p_{k+1}$, this relation immediately implies the switching conditions $\sigma^p(t^p_k) = 0$, $k = 1, \ldots, s$. The optimality condition for the optimal initial state yields
\[ 0 = L_{x_2}(x^p, p^p, p) = G_{x_2}(x^p, p) + \rho^p \Phi_{x_2}(x^p, p) \]
\[ = \lambda^p(t^p_f)Y(t^p_f, 0, p) + I_{x_2}(x^p(0), x^p(t^p_f), t^p_f, p) \]
\[ = \lambda^p(0) + I_{x_2}(x^p(0), x^p(t^p_f), t^p_f, p). \]

Hence, we have checked that the initial condition for $\lambda^p(0)$ in (10) holds. Analogously, one verifies that the adjoint function $H$ hence, we have checked that the initial condition for $\lambda^p(0)$ in (10) holds. Analogously, one verifies that the adjoint function $H$ satisfies the necessary conditions in (9), (12) and (13). It remains to check the minimum condition (14) and to show that strict bang–bang property (17) holds,
\[ \frac{d}{dt} \sigma^p(t^p_k) \neq 0, \quad k = 1, \ldots, s. \] (37)

This property is an immediate consequence of the fact that the function $p \rightarrow (x^p(0), x^p(t^p_f), t^p_f, \rho^p)$ is continuously differentiable and hence the terminal value $\lambda^p(t^p_f)$ is differentiable w.r.t. $p$. Then the neighborhood $P_0 \subset \mathbb{R}^q$ in Theorem 1 can be chosen sufficiently small as to ensure (37) and the property that $t^p_k, k = 1, \ldots, s$, are the only zeroes of the perturbed switching function $\sigma^p(t)$. Thus, we arrive at the main sensitivity result of this paper.

Theorem 4.3: (Solution differentiability for optimal bang–bang controls) Suppose that the nominal control $u^0(t)$ defined in (18) satisfies the assumptions in Theorem 4.2. Then there exists a neighborhood $P_0 \subset \mathbb{R}^q$ of $u_0$ such that with each $p \in P_0$ we can associate an initial value $x^0_p$, switching points $t^p_k, k = 1, \ldots, s$, and a final time $t^p_f$ such that the following holds true:
(a) the mapping $p \rightarrow (x^p_0, t^p_1, \ldots, t^p_s, t^p_f, \rho^p)$ is continuously differentiable.
(b) the control $u^p(t)$ given by (33) and the corresponding state trajectory $x^p(t) = x(t, z^p, p)$ defined via (21) provide a strict minimum to the perturbed control problem $OC(p)$.

This theorem generalizes the sensitivity result in Felgenhauer [6] for linear systems. For time–optimal bang–bang controls with fixed initial and final conditions, a similar result may be found in Kim [10].

Let us briefly sketch some computational aspects of sensitivity analysis. The numerical techniques in [2], [3], [4] can be adapted to bang–bang controls and allow to compute the vectors $z^p, \rho^p$ by direct optimization techniques. To facilitate computations, the arc scaling technique described in Kaya, Noakes [8], p.81, can be incorporated to determine the arc lengths $\xi := t_i - t_{i-1}$ ($i = 1, \ldots, s + 1$) of the bang-bang arcs. Then the parameter sensitivity derivatives $dt^p_k/d\rho$ ($k = 1, \ldots, s + 1$), $dx^p_k/d\rho$, $d\rho^p/d\rho$ are evaluated on the basis of formula (32). The sensitivity derivatives of the state trajectories and adjoint variables are determined by variational methods for ODEs. Under the assumption that there exists a parametrized family of extremals, methods for determining sensitivity derivatives may also be found in Kiefer, Schättler [9]. A systematic exposition of numerical methods for the sensitivity analysis of bang–bang controls will be given elsewhere. We conclude the paper with discussing the main ingredients of sensitivity analysis for a numerical example.

V. Example: Time–Optimal Control of the Van der Pol Oscillator with a Nonlinear Boundary Condition

In [14] we have studied the time–optimal control of a van der Pol oscillator with a nonlinear terminal condition. For example, we provide a numerical check of SSC using Theorems 4.1,4.2 and evaluate the sensitivity differentials of switching points on the basis of Theorem 4.3 and formula (32). The control problem is to minimize the endtime $t_f$ subject to the constraints
\[ \dot{x}_1(t) = x_2(t), \]
\[ \dot{x}_2(t) = -x_1(t) + x_2(t)(p - x_1^2(t)) + u(t), \]
where $p \in \mathbb{R}$ enters the dynamics (39) and has the nominal value $p_0 = 1$. For convenience, we shall drop the superscript $p$ for denoting the dependence on parameters. The Pontryagin or Hamiltonian function (5) is then
\[ H(x, u, \lambda, p) = \lambda_1 x_2 + \lambda_2 (-x_1 + x_2(p - x_1^2) + u). \] (43)

Note that this is an autonomous problem for which we shall evaluate the necessary conditions in normal form. Hence, the adjoint variable $\lambda_0$ in (9) and (12) becomes $\lambda_0(t) = 1$. The adjoint equations (8) and boundary conditions (11) are
\[ \dot{\lambda}_1 = \lambda_2 (1 + 2x_1 x_2), \quad \lambda_1(t_f) = 2\rho x_1(t_f), \]
\[ \dot{\lambda}_2 = -\lambda_1 - \lambda_2 (p - x_1^2), \quad \lambda_2(t_f) = 2\rho x_2(t_f), \] (44)

where $\rho \in \mathbb{R}$. The boundary condition (13) associated with the free final time $t_1$ yields
\[ 0 = 1 + \lambda_1(t_f) x_2(t_f) + \lambda_2(t_f) (-x_1(t_f) + x_2(t_f)(p - x_1(t_f)^2) + u(t_f)). \] (45)

The switching function is $\sigma(t) = H_u(t) = \lambda_2(t)$. For the nominal parameter $p_0 = 1$ we have found in [14] the optimal bang-bang control with one switching point $t_1$, $u(t) = \begin{cases} -1 & \text{for } 0 \leq t \leq t_1 \\ 1 & \text{for } t_1 \leq t \leq t_f \end{cases}. \] (46)

This implies the switching condition $\sigma(t_1) = \lambda_2(t_1) = 0$. (47)
Rather than optimizing \( t_1 \) and \( t_f \) directly, we determine the arclengths \( \xi_1 := t_1, \xi_2 := t_f - t_1 \) and solve the scaled problem; cf. problem formulation (PM2) in [8], p. 81:

\[
\begin{align*}
\min & \quad t_f = \xi_1 + \xi_2 \\
\text{s.t.} & \quad \dot{x}_1 = \zeta \cdot x_2, \quad \dot{x}_2 = \zeta \cdot ( -x_1 + x_2 (p - x_1^2) + u ),
\end{align*}
\]  

(48)

where

\[
\zeta(t) \equiv 2\xi_1 \text{ for } t \in [0, 1/2], \quad \zeta(t) \equiv 2\xi_2 \text{ for } t \in [1/2, 1].
\]

For the nominal parameter \( p_0 = 1 \), the routine NUDOCCCS of Büskens [2] yields the following set of selected values for the switching point, final time and state and adjoint variables:

\[
\begin{align*}
t_1 &= 0.7139356, & t_f &= 2.864192, \\
\lambda_1(0) &= 0.9890682, & \lambda_2(0) &= 0.9945782, \\
x_1(t_1) &= 1.143759, & x_2(t_1) &= -0.568784, \\
\lambda_1(t_1) &= 1.758128, & \lambda_2(t_1) &= 0, \\
x_1(t_f) &= 0.06985245, & x_2(t_f) &= -0.187405, \\
\lambda_1(t_f) &= 0.4581826, & \lambda_2(t_f) &= -1.229244, \\
\rho &= 3.279646.
\end{align*}
\]

(49)

Using the scaled problem (48) it can be shown that the associated parametric sensitivity derivatives

\[
\frac{\partial x_i}{\partial p}(t, p_0), \quad i = 1, 2,
\]

are continuous functions on \([0, 1]\). Figure 2 displays the sensitivity derivatives on the original time intervals \([0, t_1]\) and \([t_1, t_f]\).

![Fig. 2. Parametric sensitivity derivatives of the state variables.](image)

VI. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

Parametric bang-bang control problems with a scalar control have been considered. Under the assumption that the bang-bang control has finitely many switching points, the bang-bang control problem can be reformulated as a finite-dimensional optimization problem. Then second order sufficient conditions (SSC) for the bang-bang control are stated in terms of SSC for the resulting optimization problem [1], [14]. A classical sensitivity result for optimization problems [7] yields the desired result of solution differentiability of the switching points w.r.t. parameters. This allows for the numerical verification of SSC and the computation of parametric sensitivity derivatives using the techniques in [2], [3], [4].

B. Future Works

Future work concerns the sensitivity analysis of multi-dimensional bang–bang controls. We shall work out a detailed exposition of the numerical techniques for computing parametric sensitivity derivatives of the switching points and state trajectories. Applications will be given in the area of robot control, control of chemical batch reactions and drug design of modeling HIV.

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VIII. REFERENCES


