Optimization and Optimal Control in Banach Spaces

Problem sheet 4 - return on 2018-12-18

Exercise 1. Let $g \in C([0,1])$. For $n = 1, 2, \dots$ set

$$f_n \in C([0,1]), \qquad\qquad x \mapsto -g(x) \cdot \cos(2\pi\, n\, x).$$

Find the Γ-limit of $(f_n)_{n=1}^{\infty}$, in particular show that the lim-inf condition is satisfied and provide a recovery sequence for the lim-sup condition.

Exercise 2. We will study the duality between $C([0,1])$ and $\mathcal{M}([0,1])$. Recall that the total variation norm on $\mathcal{M}([0,1])$ is given by

$$\|\mu\|_{\mathcal{M}} := \sup_{\substack{f \in C([0,1]),\\ f \neq 0}} \frac{|\langle \mu, f \rangle|}{\|f\|_C}.$$

Let $(x_n)_n$ be a convergent sequence in $[0,1]$ with limit $x \in [0,1]$ with $x_n \neq x \ \forall\, n$.

(i) Show that $\delta_{x_n} \not\to \delta_x$, i.e. $\|\delta_{x_n} - \delta_x\|_{\mathcal{M}} \not\to 0$.

(ii) Show that $\delta_{x_n} \overset{*}{\rightharpoonup} \delta_x$.

Exercise 3. Let $X = \{0,1,2,3\}$ with the discrete topology. We can then identify $C(X) \simeq \mathcal{M}(X) \simeq \mathbb{R}^4$ with duality pairing $\langle \mu, f \rangle = \mu^\top f = \sum_{i=0}^{3} \mu_i f_i$ for $(\mu, f) \in \mathcal{M}(X) \times C(X)$. Similarly, we identify $C(X \times X) \simeq \mathcal{M}(X \times X) \simeq \mathbb{R}^{4\times 4}$.
Let the discrete cost function $c = (c_{i,j})_{i,j=0}^{3} \in C(X \times X)$ and marginals $\mu, \nu \in \mathcal{M}_+(X)$ be given by

$$c = \begin{pmatrix} 0 & 1 & 4 & 9 \\ 1 & 0 & 1 & 4 \\ 4 & 1 & 0 & 1 \\ 9 & 4 & 1 & 0 \end{pmatrix}, \qquad \mu = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \nu = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

In particular, $c_{i,j} = (i-j)^2$.

(i) Guess a minimizer π to the discrete transport problem $\min_{\pi \in \Pi(\mu,\nu)} \langle \pi, c \rangle$.

(ii) Prove that π is minimal by guessing corresponding dual maximizers $(\alpha, \beta) \in C(X)^2$ with $\langle \mu, \alpha \rangle + \langle \nu, \beta \rangle = \langle \pi, c \rangle$ and $\alpha_i + \beta_j \leq c_{i,j}$ for $i, j \in \{0,1,2,3\}$.

Hint: Use the primal-dual optimality conditions of optimal transport.