Optimization and Optimal Control in Banach Spaces

Problem sheet 1 - 2018-10-31

Exercise 1. Let $(C_k)_{k\in\mathbb{N}}$ be a sequence of convex subsets of H such that $C_k \subset C_{k+1}$ for all $k \in \mathbb{N}$ and set $C := \bigcup_{k\in\mathbb{N}} C_k$.

(i) Show that C is convex.

(ii) Find an example in which the sets $(C_n)_{n\in\mathbb{N}}$ are closed and C is not closed.

Exercise 2. Let $C \subset H$. $\overline{\operatorname{conv} C}$ is the closure of the convex hull of C. Define

$$\overline{\text{conv}}C := \text{Intersection of all closed, convex subsets } B \subset H \text{ such that } C \subset B$$

Show that $\overline{\operatorname{conv} C} = \overline{\text{conv}}C$.

Exercise 3. Let $H = \mathbb{R}^3$ and

$$C := \{(x_1, x_2, 0) \in H \mid x_1^2 + x_2^2 \leq 1\}.$$

(i) Determine the tangent and normal cones for points $x \in C$.

(ii) Compute the projection $P_C x$ for $x \in H$.

Exercise 4. Let $H = \mathbb{R}^n$ and

$$f(x) := \|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$$

for some $p \in (1, \infty)$. Determine the Fenchel–Legendre conjugate f^*.
Hint: Use Hölder's inequality, $\langle x, y\rangle \leq \|x\|_p \|y\|_q$ for $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and in particular the critical case where equality holds.