Chapter 6

Asymptotic Methods

Introduction

Often an integral or solution of a differential equation is expressed in the form $f(x,n)$ or $g(x)$ where a closed (or simple) form for f or g is not obtainable. A classic example is the integral that represents probability given a normal distribution. This example is part of a select group used so often that tabular values are listed in many handbooks. If the problem under study does not fall into this category and information regarding the limiting behavior of a function is sought then we are in that branch of mathematics called asymptotic analysis. Briefly, for the functions f and g we see simpler functions $f_0(x,n)$ or $g_0(x)$ satisfying

$$f(x,n) = f_0(x,n)[1+o(1)]\ , \quad x \text{ or } n \to \infty$$

or

$$g(x) = g_0(x)[1+o(1)]\ , \quad x \to \infty\,.$$

The functions f_0 and g_0 are called asymptotic estimates of f and g, respectively.

The goal of this chapter is twofold. First, we are concerned with ascertaining f_0 and g_0 and, second, with bounding the error $\epsilon(x)$ where for example

$$g(x) = g_0(x)[1+\epsilon(x)]\,.$$

One method for bounding ϵ is to express ϵ in reciprocal powers of x. The result

$$\epsilon(x) = \frac{\epsilon_1}{x} + \frac{\epsilon_2}{x^2} + \cdots + \frac{\epsilon_{k-1}}{x^{k-1}} + \cdots$$

is called an asymptotic series. This series may not converge, and upon truncating the series after a finite number of terms we may again wish to bound the error, e.g. to bound $\eta_k(x)$ in

$$g(x) = g_0(x)\left[1 + \frac{\epsilon_1}{x} + \cdots + \frac{\epsilon_{k-1}}{x^{k-1}} + \eta_k(x)\right].$$

This chapter differs from most accessible texts in two ways. First, the chapter is introductory and elementary in that only real variable methods are used. Second, attention is restricted to those parts of asymptotic analysis that have a direct relation to the subject matter of the text. In particular, the methods presented here complement and extend the analysis in Chapters 5 and 7.

6.1 Integration by Parts

To fix ideas define the exponential integral

$$\mathrm{Ei}(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \tag{6.1.1}$$

where x is a large positive parameter. If one integration by parts is performed in the integral in (6.1.1) then

$$\mathrm{Ei}(x) = \frac{e^{-x}}{x} - \int_x^\infty \frac{e^{-t}}{t^2}\,dt \tag{6.1.2}$$

and the first term on the right-hand side of (6.1.2) is called an asymptotic estimate of the integral. Continuing integration by parts in (6.1.2) leads to the expression

$$\begin{aligned}\mathrm{Ei}(x) &= e^{-x}\left[\frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n}\right] + R_n(x) \\ &\equiv e^{-x}\left[\sum_{k=0}^{n-1}\frac{(-1)^k k!}{x^{k+1}}\right] + R_n(x)\end{aligned} \tag{6.1.3}$$

where

$$R_n(x) = (-1)^n n!\int_x^\infty \frac{e^{-t}}{t^{n+1}}\,dt\,. \tag{6.1.4}$$

Table 6.1.1: Ei(5) = .00114829

n	$\mathrm{Ei}^n(5)$	$\mathrm{Ei}(5) - \mathrm{Ei}^{(n)}(5)$
1	.00134758	-.000199
2	.00107807	+.000070
3	.00118588	-.000037
4	.00112119	+.000027
5	.00117294	-.000024
6	.00112119	+.000027
7	.00118329	-.000035
8	.00109635	+.000052
9	.00123545	-.000087
10	.00098508	+.000163
11	.00148583	-.000338

$\mathrm{Ei}^{(n)}(5) \equiv e^{-x}\left[\frac{1}{x} - \frac{1\,!}{x^2} + \frac{2\,!}{x^3} - \cdots + \frac{(n-1)\,!}{x^n}\right]$

Consider evaluating Ei(5) via the formula in (6.1.3). These results are displayed in Table 6.1.1.

A graph of the error (using the right-hand column in Table 6.1.1) has the oscillatory
appearance displayed in Figure 6.1.1.

A survey of the figure or the last column in Table 6.1.1 shows that as n increases up to 5 the error decreases, and as n increases past 5 there is a deterioration in the approximation. Indeed, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)\,!}{x^n} \tag{6.1.5}$$

diverges for every finite x but at $n = 5$ the error in approximating Ei(5) by five terms of (6.1.5) gives the value of the function to four digits (compare to quadratures). The series in (6.1.5) is called an asymptotic series (in the powers $1/x$) and such series often provide a good (if limited) approximation to the expression generating them (in this case the function $\mathrm{Ei}(x)$).

To see why $n = 5$ provided the best accuracy consider

$$(-1)^n R_n(x) \leq \frac{n\,!}{x^{n+1}} \int_x^\infty e^{-t} dt = \frac{n\,! \, c^{-x}}{x^{n+1}} \tag{6.1.6}$$

Figure 6.1.1:

valid, using (6.1.4), for all $0 < x \leq t < \infty$. Hence the error in truncating the asymptotic series is bounded in magnitude by the first deleted term and has the sign of the first deleted term (hence the oscillatory graph in Figure ??6.1.2).

The ratio of the magnitude of the $(n+1)^{\text{st}}$ term to the nth term is

$$\frac{n\,!\,/x^{n+1}}{(n-1)\,!\,/x^{n}} = \frac{n}{x}$$

which for fixed x is less than one until $n = [|x|] \equiv$ integer part of x and thereafter increases without bound ($n = x = 5$ in Table 6.1.1).

This example illustrates the typical behavior of asymptotic series: Usually only a limited accuracy is possible for each x. While it is possible, in this example, to pick the optimal value of n for given x, in practice the choice of the optimal n is more difficult if the error term is complicated.

The preceeding example is a special case of integrals in the form

$$g(x) = \int_x^\infty e^{\alpha t^m} f(t) dt \,, \tag{6.1.7}$$

with $\alpha = -1$, $m = 1$ and $f(t) = 1/t$. Two other important special cases are the Probability distribution

$$Erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \tag{6.1.8}$$

and the Fresnel integrals

$$C(x) + iS(x) = \int_x^\infty e^{it^2} dt \,. \tag{6.1.9}$$

One integration by parts in (6.1.8) gives

$$Erfc(x) = \frac{e^{-x^2}}{\sqrt{\pi}\, x} + \frac{1}{\pi} \int_x^\infty \frac{e^{-t^2}}{t^2} dt \,, \tag{6.1.10}$$

but, instead of continued integration by parts, consider the more general approach applied to (6.1.7). Define the change of variables

$$t = u^{1/m} \;, \quad u = v + x^m \tag{6.1.11}$$

so that (6.1.7) takes the form

$$\begin{aligned} g(x) &= \frac{1}{m} \int_{x^m}^\infty \frac{e^{\alpha u} f(u^{1/m})}{u^{(1-1/m)}} du \\ &= \frac{e^{\alpha x^m}}{m} \int_0^\infty \frac{e^{\alpha v} f((v + x^m)^{1/m})}{(v + x^m)^{(1-1/m)}} dv \,. \end{aligned} \tag{6.1.12}$$

Using (6.1.12) for (6.1.8) ($m = 2$, $\alpha = -1$ and $f(t) \equiv 1$) leads to the expression

$$\begin{aligned} Erfc(x) &= \frac{2e^{-x^2}}{\sqrt{\pi}\, 2} \int_0^\infty \frac{e^{-v}}{(v + x^2)^{1/2}} dv \\ &= \frac{e^{-x^2}}{\sqrt{\pi}\, x} \int_0^\infty \left(1 + \frac{v}{x^2}\right)^{-1/2} e^{-v} dv \\ &= \frac{e^{-x^2}}{\sqrt{\pi}\, x} \int_a^\infty \left\{ 1 + \sum_{k=1}^\infty \frac{(-1)^k (2k-1)\,!\,!}{(2k)\,!\,!\, x^{2k}} v^k \right\} e^{-v} dv \end{aligned} \tag{6.1.13}$$

$$\doteq \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left\{1+\sum_{k=1}^{\infty}\frac{(-1)^k(2k-1)!!}{(2k)!!\,x^{2k}}\,k!\right\}$$
$$= \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left\{1+\sum_{k=1}^{\infty}\frac{(-1)^k(2k-1)!!}{(2x^2)^k}\right\}$$

where $(2k-1)!! = (2k-1)(2k-3)\cdots 1$, and the integral

$$\int_0^{\infty} v^k e^{-v} dv = k!$$

have been used in line following the $\doteq$. Notice that the first term on the right-hand side of (6.1.13) is the same as that in (6.1.10). Indeed the series in (6.1.13) is the same as that obtained from continued integration by parts in (6.1.10) (Problem????6.1). As in the series (6.1.5) the extreme right of (6.1.13) diverges for every positive x. This, of course, is in violent contrast to the integral defining $Erfc(x)$ which converges for every x. It is not hard to find the "breakdown" in (6.1.13). The series expansion of $(1+v/x^2)^{-1/2}$ converges only in $|v/x^2| < 1$ (Problem ???6.2) but the series development was used over the entire range $v \in [0,\infty)$. It is however the case that the extreme right of (6.1.13) is an asymptotic expansion of $Erfc(x)$. The following powerful lemma justifies the expansion in (6.1.13).

Lemma 6.1.1 "Watson's" *For $|t| \le R$, let*

$$f(t) = \sum_0^{\infty} a_n t^n \tag{6.1.14}$$

and for $|t| > R$, let

$$|f(t)| \le C\, e^{bt}\,. \tag{6.1.15}$$

If $m > -1$, then

$$\left|\int_0^{\infty} e^{-xt} t^m f(t) dt \;-\; \sum_{k=0}^{n-1} \frac{a_k \Gamma(k+m+1)}{x^{k+m+1}}\right| \le M\,\frac{\Gamma(m+n+1)}{x^{m+n+1}} + C\, e^{-xR} \tag{6.1.16}$$

where C and M are constants independent of m, x and R.

The proof of this theorem can be found in a number of texts and is omitted here since we carry out a similar argument in the proof of the method of Laplace in 6.3.

Truncation of the series obtained from substitution of (6.1.14) into the integral in (6.1.16) yields

$$\int_0^{\infty} e^{-xt} t^m \sum_{k=0}^{n-1} q_k t^k \, dt \,,$$

a partial sum, $f(t)$ being replaced by the first n lines of its Taylor series. The factor t^m in the integrand in (6.1.16) serves to cope with singularities at $t = 0$ that may arise as a result of a transformation of the type (6.1.11).

The first term on the right of (6.1.16) is proportional to the first term neglected in replacing the series

$$\sum_0^{\infty} a_k \frac{\Gamma(m+k+1)}{x^{m+k+1}} \quad \text{by} \quad \sum_0^{n-1} a_k \frac{\Gamma(m+k+1)}{x^{m+k+1}} \,.$$

Watson's lemma is not a convenient tool for estimating the error of the asymptotic approximation; its chief use is that it justifies the familiar procedure of approximating

$$\int_0^{\infty} e^{-xt} t^m f(t) \, dt$$

by expanding f in powers of t and performing termwise integration. The examples 1 to 6 below illustrate the use and limitations of Watson's lemma.

Example 6.1.2 Put $t = v/x^2$ in the second line of (6.1.13) then the last line of (6.1.13) is the result of Watson's lemma applied to $f(t) = (1+t)^{-1/2}$.

Example 6.1.3 For the integral in (6.1.19), using (6.1.12) it follows that

$$\begin{aligned} C(x) + iS(x) &= \int_x^{\infty} e^{it^2} dt \\ &= \frac{1}{2} \int_{x^2}^{\infty} \frac{e^{iu}}{u^{1/2}} du \\ &= \frac{e^{ix^2}}{2} \int_0^{\infty} \frac{e^{iv}}{(v+x^2)^{1/2}} \, dv \,. \end{aligned}$$

Watson's lemma is not directly applicable, since the integrals

$$\int_0^{\infty} e^{iv} v^k \, dv$$

do not exist.

Example 6.1.4 The ratio of the magnitude of successive terms in the expression (6.1.16) using Watson's lemma is

$$r_k = \left| \frac{a_{k+1}}{a_k} \, \frac{k+m+1}{x} \right| .$$

If $a_{k+1}/a_k \to \rho$ as $k \to \infty$, where ρ is finite, $r_k \to \infty$ as $k \to \infty$ for each finite x. That is, the asymptotic series

$$\sum_0^\infty a_k \frac{\Gamma(m+k+1)}{x^{m+k+1}} \tag{6.1.17}$$

diverges for each finite x. The series (6.1.17) converges for all sufficiently large x if $\lim r_k < \infty$, $k \to \infty$. In this latter case the series expansion (6.1.15) converges to $f(t)$ for all finite values of t.

Example 6.1.5 The integral

$$I(y) = \int_0^\infty e^{yt - 1/3\, t^3} dt\ , \quad y \to 0$$

can be replaced by an asymptotic series in $x = 1/y^3$ by means of Watson's lemma. Setting

$$yt = (3v)^{1/3}\ , \quad y = x^{-1/3}$$

we get

$$\begin{aligned} I(y) = I(x^{-1/3}) &= 3^{-2/3} x^{1/3} \int_0^\infty e^{(3v)^{1/3} - xv} v^{-2/3} dv \\ &\sim 3^{-2/3} x^{1/3} \int_0^\infty \sum_{k=0}^\infty \frac{(3v)^{k/3}}{k\,!} e^{-xv} dv \\ &\sim \sum_{k=0}^\infty 3^{(k-2)/3} \, \frac{\Gamma\left(\frac{k+1}{3}\right)}{k\,!} y^k \end{aligned}$$

Here we have used the symbol $\sim$ to designate an asymptotic expansion that may not converge. This series converges for all finite y, and is equal to $I(y)$. (Why?)

Example 6.1.6

$$I(x) = \text{P.V.} \int_0^\infty \frac{3^{-2t}}{1-t^2}\, dt$$

Watson's lemma does not apply to this integral (since the proof of Watson's lemma does not take into account the possibility that the integral may be a principal-value integral). If we ignore the P.V., expand the denominator in powers of t and integrate term by term, we obtain

$$I(x) \sim \sum_{k=0}^\infty \frac{(2k)\,!}{x^{2k+1}}\, . \tag{6.1.18}$$

It can be shown that

$$I(x) = \int_0^\infty \frac{\sin xt}{1+t^2}\, dt\, .$$

If we perform repeated integration by parts in this last integral, then we do indeed obtain the asymptotic expansion in (6.1.18).

Example 6.1.7 If we apply integration by parts to the integral

$$I(x) = \int_0^\infty \frac{\cos xt}{1+t^2}\, dt$$

we obtain the asymptotic series

$$I(x) \sim \frac{0}{x} + \frac{0}{x^2} + \frac{0}{x^3} \,.$$

Indeed it may be shown that

$$I = \pi\, e^{-x}$$

which shows that, as $x \to \infty$, $I(x)$ converges to zero faster than any negative power of x. Therefore it does not have an asymptotic expansion in negative powers of x.

Example 6.1.8 Numerical integration can sometimes replace series expansions. For example, if we apply Gauss-Laguerre quadrature to the integral

$$\int_0^\infty e^{-xt} f(t) dt = \frac{1}{x} \int_0^\infty e^{-t} f\left(\frac{t}{x}\right) dt$$

we get

$$\int_0^\infty e^{-xt} f(t) dt = \frac{1}{x} \sum_{k=1}^{n} w_k f\left(\frac{t_k}{2}\right) + \frac{(n\,!)^2 f^{(2n)}(\xi)}{(2n)\,!\, x^{2n+1}} \tag{6.1.19}$$

for some $\xi = \xi(x) \in (0, \infty)$. The error term in (6.1.19) is obtained using the formula (5.6.27), we get

$$\left(\frac{d}{dt}\right)^{2n} f\left(\frac{t}{x}\right) = \frac{1}{x^{2n}} \left(\frac{d}{d(t/x)}\right)^{2n} f\left(\frac{t}{x}\right) \,.$$

Example 6.1.9 Watson's lemma can be readily adapted to integrals over a finite interval. In the case of the integral

$$I(x) = \int_0^T e^{-xt} t^m f(t) dt$$

we have two choices: (a) we can set $f(t) = 0$ for $t > T$ and replace $\int_0^T$ by $\int_0^\infty$, or (b) if $f(t)$ satisfies the conditions of Watson's lemma on the interval $(0, \infty)$, we can write

$$\int_0^T = \int_0^\infty - \int_T^\infty$$

and under the conditions of the lemma, in particular (6.1.15)

$$\begin{aligned}
\left|\int_T^\infty e^{-xt} t^m f(t) dt\right| &< C \int_T^\infty e^{-(x-b)\,t} t^m dt \\
&= C \int_0^\infty e^{-(x-b)\,(T+\tau)} (T+\tau)^m d\tau \\
&\leq C\, e^{-(x-b)\,T} \int_0^\infty e^{-(x-b)\,\tau} [(2T)^m + (2\tau)^m] d\tau \\
&= C\, e^{-(x-b)\,T} \left[\frac{(2T)^m}{x-b} + \frac{2^m \Gamma(m+1)}{(x-b)^{m+1}}\right]
\end{aligned}$$

Problems

6.1.1 Perform one more integration by parts in (6.1.10) to see that the method of integration by parts also leads to the second term on the extreme right of (6.1.13).

6.1.2 Show that

$$(1+x)^\alpha = \sum_{j=0}^{\infty} \frac{\alpha_j}{j\,!}\, x^j$$

where $\alpha_0 \equiv 1$, $\alpha_j = \alpha(\alpha-1)\cdots(\alpha-j+1)$. Note that if $\alpha = -1/2$ then

$$\alpha_j = \frac{(2j-1)\,!\,!}{2^j}.$$

6.1.3 Show that

$$I(k) \equiv \int_0^\infty v^k e^{-r}\, dr = k\,!\,.$$

Hint: Apply integration by parts to $I(k)$.

6.1.4 Do the term by term integration indicated in Example 5. Problem 6.2 with x replaced by $-x$ and $\alpha = -1$ is useful.

6.1.5 Perform two integrations by parts in

$$I(x) = \int_0^\infty \frac{\sin(xt)}{t^2+1}\, dt$$

to verify the first two terms of (6.1.18). In each step take the differentiated term to be the trigonometric function.

6.2 Definitions and Notations

In the previous section we have been using asymptotic series in powers of x^{-1} or x^{-2}. In this section we introduce the standard concept of a more general asymptotic series.

Definition 6.2.1 *Let f and ϕ be two functions defined on a domain R. We write*

$$f = 0(\phi)$$

if there exists a positive constant A such that

$$|f(x)| \leq A|\phi(x)|$$

for all $x \in R$. We also write $f = 0(\phi)$ as $x \to x_0$ if there is an open subset R_0 of R containing a point x_0 and a constant A, such that $|f(x)| \leq A|\phi(x)|$ for all $x \in R_0$.

Definition 6.2.2 *We write*

$$f = o(\phi)$$

as $x \to x_0$ if for any $\epsilon > 0$ there exists an open subset R_0 of R such that

$$|f(x)| \leq \epsilon|\phi(x)|$$

for all $x \in R_0$.

If $\phi(x) \neq 0$ on R, $x \neq x_0$, then $f = 0(\phi)$ means that $|f/\phi|$ is bounded, and $f = o(\phi)$ as $x \to x_0$ means that $|f/\phi|$ tends to zero as $x \to x_0$.

Definition 6.2.3 *A sequence of functions $\{\phi_k(x)\}_{k=1}^{\infty}$ is called an asymptotic sequence if there exists a neighborhood of x_0 where $\phi_k(x) \neq 0$ when $x \neq x_0$, $k = 0, 1, 2, \ldots$, and where*

$$\phi_{k+1} = 0(\phi_k)$$

as $x \to x_0$, $k = 0, 1, 2, \ldots$.

For example, the sequence $\{x^{-k}\}_{k=1}^{\infty}$ is an asymptotic sequence, with $x_0 = \infty$, and the neighborhood of x_0 being $\{x : x > 0\}$; similarly if $x_0 = -\infty$; the neighborhood of x_0 being $\{x : x < 0\}$.

Definition 6.2.4 *Let $\{\phi_k\}$ be an asymptotic sequence. The series $\sum_{k=0}^{\infty} a_k\phi_k$ is said to be an asymptotic expansion of f as $x \to x_0$ and we write*

$$f \sim \sum_{k=0}^{\infty} a_k\phi_k \ , \quad x \to x_0$$

if for every integer $n = 1, 2, 3, \ldots$,

$$f - \sum_{k=0}^{n-1} a_k \phi_k = 0(\phi_n)$$

as $x \to x_0$.

Example 6.2.5 In the asymptotic expansion of $Ei(x)$ in (6.1.3), the asymptotic sequence has

$$\phi_k(x) = e^{-x} x^{-(k+1)} \ , \quad k = 0, 1, 2, \ldots$$

and

$$a_k = (-1)^k k! \ , \quad k = 0, 1, 2, \ldots$$

Example 6.2.6 In the asymptotic expansion of $Erfc(x)$ in (6.1.13) the asymptotic sequence has

$$\phi_k(x) = e^{-x^2} x^{1-2k} \ , \quad k = 0, 1, 2, \ldots$$

and

$$\sqrt{\pi}\, a_0 = 1 \ , \ \sqrt{\pi}\, a_k = (-1)^k (2k-1)!! \ , \quad k = 1, 2, 3, \ldots .$$

6.3 The Method of Laplace

Integrals of the form

$$I(x) = \int_a^b e^{-xg(t)} f(t) dt \tag{6.3.1}$$

are called Laplace-type integrals, after Laplace, who developed an ingenious method of obtaining asymptotic approximation of these integrals. Let us briefly describe the method of obtaining a Laplace-type approximation of the integral (6.3.1).

Assumption 6.3.1 *We suppose that the function* $\exp[-xg(t)]$ *has a unique maximum, located at a point* τ *in* $[a, b]$*. The following three possibilities occur most frequently (there are others that we omit):*

(i) τ *is an endpoint of* $[a, b]$*, e.g.* $\tau = a$ *and* $g'(\tau) \neq 0$.

(ii) τ *is an endpoint of* $[a, b]$*, e.g.* $\tau = a$ *and* $g'(\tau) = 0$, $g''(\tau) > 0$.

(iii) τ *is an interior point of* $[a, b]$, $g'(\tau) = 0$, $g''(\tau) > 0$.

Theorem 6.3.2 *Let one of the above possibilities be satisfied, let a and b be finite and let f be continuous on $[a, b]$.*

(i) If Assumption 6.3.1-(i) holds, and $\tau = a$ then

$$I(x) = \frac{e^{-xg(a)}}{xg'(a)} [f(a) + o(1)] , \quad x \to \infty \tag{6.3.2}$$

(ii) If Assumption 6.3.1-(ii) holds, and $\tau = a$ then

$$I(x) = \left[\frac{\pi}{2xg''(a)}\right]^{1/2} e^{-xg(a)}[f(a) + o(1)] , \quad x \to \infty \tag{6.3.3}$$

(iii) If Assumption 6.3.1-(iii) holds, then

$$I(x) = \left[\frac{2\pi}{xg''(\tau)}\right]^{1/2} e^{-xg(\tau)}[f(\tau) + o(1)] , \quad x \to \infty \tag{6.3.4}$$

We do not prove this theorem, since the arguments differ little from those of the proof of the Laplace expansion method that we carry out in the next section. We do however give a simple, intuitive explanation of the method of Laplace, for the special case

$$y(x) = \int_{-a}^{a} e^{-xt^2} f(t)dt . \tag{6.3.5}$$

The graph of $\exp(-xt^2)$ is shown in Figure 6.3.1.

As x increases, the neighborhood of $t = 0$ that contributes significantly to $y(x)$ in (6.3.5) becomes smaller and smaller.

Thus it makes no appreciable difference whether we replace $y(x)$ in (6.3.5) by
$\int_{-\epsilon}^{+\epsilon} e^{-xt^2} f(t)dt$, where ϵ is small but positive or by $\int_{-\infty}^{\infty} \exp(-xt^2)f(t)dt$ (in this latter case we set $f(t) = 0$ for $t > a$). In the region $-\epsilon \leq t \leq \epsilon$, $f(t) \cong f(0)$, and

$$y(x) \sim f(0) \int_{-\infty}^{\infty} e^{-xt^2} dt = \left(\frac{\pi}{x}\right)^{1/2} f(0) , \quad x \to \infty .$$

The following sequence of examples illustrates the uses of Laplace's theorem in each of the various cases ???(i)–(iii) cited above.

Figure 6.3.1: The graph of $exp(-xt^2)$.

Example 6.3.3 Consider the integral

$$I_k(x) = \int_0^\pi \cos(t) e^{-xt^k} dt \ , \quad k = 1 \text{ or } 2 \, .$$

The cases $k = 1$ or 2 are handled by Theorem 6.3.2-(i) or (ii) respectively and yield with $\tau = 0$

$$I_1(x) = \frac{1}{x} \left[1 + o(1)\right] \ , \quad x \to \infty \, ,$$

and

$$I_2(x) = \frac{1}{2} \left(\frac{\pi}{x}\right)^{1/2} \left[1 + o(1)\right] \ , \quad x \to \infty \, .$$

Example 6.3.4 The integral

$$I(x) = \int_{-a}^{a} e^{-x\cosh(t)} f(t) dt \ , \quad a > 0$$

is handled by case (iii) with $\tau = 0$. Equation (6.3.4) gives

$$I(x) = \left(\frac{2\pi}{x}\right)^{1/2} e^{-x} [f(0) + o(1)] \ , \quad x \to \infty \, .$$

Example 6.3.5 The integral

$$I(x, y) = \int_{-\infty}^{\infty} e^{-xt^2} \cos(yt) dt$$

has radically different behavior for large x (versus large y). Here we consider the former case. If $a > 1$ then

$$\begin{aligned} \left| \int_a^\infty e^{-xt^2} \cos yt \, dt \right| & \le \textstyle\int_a^\infty e^{-xt^2} dt \\ & \le \tfrac{1}{2} \textstyle\int_x^\infty e^{-u} du \\ & = \tfrac{1}{2} e^{-x} \quad . \end{aligned}$$

By Theorem 6.3.2-(iii)

$$\int_{-a}^{a} e^{-xt^2}\cos(yt)dt = \left(\frac{\pi}{x}\right)^{1/2}[1+o(1)]\ , \quad x\to\infty$$

so that

$$I(x,y) = \left(\frac{\pi}{x}\right)^{1/2}[1+o(1)] + \delta(x)$$

where

$$|\delta(x)| \le e^{-x}\,.$$

That is

$$I(x,y) = \left(\frac{\pi}{x}\right)^{1/2}[1+o(1)]\ , \quad x\to\infty\,.$$

It can shown that

$$I(x,y) = \left(\frac{\pi}{x}\right)^{1/2} e^{-y^2/4x}$$

so that the approximation $(\pi/x)^{1/2}$ is quite accurate for small $\big(y^2/(4x)\big)$.

Example 6.3.6 If n is an integer the n^{th} Bessel function may be defined by

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos(n\theta)d\theta\,.$$

It follows from Theorem 6.3.2-(ii) that

$$I_n(x) = \frac{e^x}{\sqrt{2\pi x}}[1+o(1)]\ , \quad x\to\infty\,.$$

Example 6.3.7 In the previous example set $u=\cos\theta$ and write

$$I_n(x) = \frac{1}{\pi}\int_{-1}^{1} e^{xu}\frac{T_n(u)}{(1-u^2)^{1/2}}\,du$$

where $T_n(u)=\cos\big(n\cos^{-1}(u)\big)$ is the n^{th} Chebyshev polynomial of degree n (Chapter 4). Now set $u=1-t$ to get

$$I_n(x) = \frac{1}{\pi}\int_0^2 \frac{e^{x(1-t)}T_n(1-t)}{t^{1/2}(2-t)^{1/2}}\,dt\,.$$

Using $(a=\tau=0)$ Theorem 6.3.2-(i) is not applicable since $f(t)=T_n(1-t)(2-t)^{-1/2}t^{-1/2}$ is not continuous on any subinterval of $[0,2]$ containing zero. However, replacing t by t^2 leads to

$$\begin{aligned} I_n(x) &= \frac{2}{\pi}\int_0^{\sqrt{2}} \frac{e^{x(1-t^2)}T_n(1-t^2)dt}{(2-t^2)^{1/2}} \\ &= \frac{e^x}{\sqrt{2\pi x}}[T_n(1)+o(1)] \\ &= \frac{e^x}{\sqrt{2\pi x}}[1+o(1)] \end{aligned}$$

where Theorem 6.3.2-(ii) has again been used.

Example 6.3.8 The Gamma function is defined by

$$\Gamma(x+1) = \int_0^\infty e^{-u} u^x du$$

and can be turned into a Laplace type integral by the substitution

$$u = x + 1 .$$

Thus

$$\begin{aligned}\Gamma(x+1) &= x^{x+1} e^{-x} \int_{-1}^\infty e^{-x[t-\ell n(t+1)]} dt \\ &= (2\pi x)^{1/2} \left(\frac{x}{e}\right)^x [1 + o(1)]\end{aligned}$$

where Theorem 6.3.2-(iii) with $\tau = 0$ was used.

In Problem 6.3.3 the complete asymptotic expansion of $I_0(x)$ was found by converting the integral to one for which Watson's lemma applies. This procedure is more general than that problem may have indicated.

Theorem 6.3.9 *In the integral*

$$I(x) = \int_a^b e^{-xg(t)} f(t) dt \tag{6.3.6}$$

let Assumption 6.3.1-(i) be satisfied. Let $g'(t) > 0$ on $[a,b]$ and define $t = h(w)$ through the equation

$$w = g(t) - g(a) \, , \, a \le t \le b \, . \tag{6.3.7}$$

If the function H satisfies

$$H(w) \equiv f(h(w))h'(w) = \sum_{k=0}^\infty \frac{H^{(k)}(0) w^k}{k\,!} \, , \quad |w| \le A \tag{6.3.8}$$

and

$$H(w) \le C\, e^{\beta w} \, , \, |w| > A \tag{6.3.9}$$

where

$$A = g(b) - g(a) \tag{6.3.10}$$

then

$$I(x) \sim e^{-xg(a)} \sum_{k=0}^\infty \frac{H^{(k)}(0)}{k\,!\, x^{k+1}} \, , \quad x \to \infty \, . \tag{6.3.11}$$

Theorem 6.3.10 *If Assumption 6.3.1-(ii) is satisfied and $g'(t) > 0$ on $(a,b]$, set*

$$w = [g(t) - g(a)]^{1/2} \tag{6.3.12}$$

and suppose that H satisfies (6.3.8) and

$$H(w) \le C\, e^{\beta w^2}\,,\ |w| > A \tag{6.3.13}$$

where

$$A = [g(b) - g(a)]^{1/2}\,. \tag{6.3.14}$$

In this case (6.3.6) satisfies

$$I(x) \sim \frac{e^{-xg(a)}}{2} \sum_{k=0}^{\infty} \frac{H^{(k)}(0)\Gamma(k+1/2)}{k\,!\,x^{(k+1/2)}}\,, \quad x \to \infty\,. \tag{6.3.15}$$

Theorem 6.3.11 *If Assumption 6.3.1-(iii) is satisfied and $g'(t) < 0$ on $a < t < \tau$, set*

$$w = \begin{cases} -[g(t) - g(\tau)]^{1/2} & , \ a \le t \le \tau \\ [g(t) - g(\tau)]^{1/2} & , \ \tau \le t \le b \end{cases}. \tag{6.3.16}$$

Assume that H satisfies (6.3.8) and (6.3.13) with

$$A = \min\{[g(b) - g(\tau)]^{1/2}\,,\ [g(a) - g(\tau)]^{1/2}\}\,. \tag{6.3.17}$$

In this case (6.3.6) satisfies

$$I(x) \sim e^{-xg(\tau)} \sum_{k=0}^{\infty} \frac{H^{(2k)}(0)\Gamma(k+1/2)}{(2k)\,!\,x^{k+1/2}}\,, \quad x \to \infty\,. \tag{6.3.18}$$

Proof: Since the proofs of each of the three theorems are similar we only prove the last one.

If the conditions of Theorem 6.3.11 are satisfied then

$$\begin{aligned} J(x) &\equiv \int_a^b e^{-x\left(g(t)-g(\tau)\right)} f(t)dt + \epsilon \\ &= \int_{-A}^{A} e^{-xw^2} H(w)dw + \epsilon \\ &= e^{-xg(\tau)} I(x) + \epsilon \end{aligned} \tag{6.3.19}$$

where

$$
\begin{aligned}
|\epsilon| &\leq 2C\int_A^\infty e^{-(x-\beta)w^2}dw \\
&= 2C\int_{(x-\beta)A^2}^\infty \frac{e^{-y}}{(x-\beta)^{1/2}}\, y^{-1/2}dy \qquad (6.3.20)\\
&\leq \frac{C\, e^{-(x-\beta)A^2}}{(x-\beta)^{1/2}}\,,\ (x-\beta)A^2 \geq 1\,.
\end{aligned}
$$

Thus $\epsilon = o(x^{-m})$ as $m \to \infty$ for all $m > 0$.

Using (6.3.8) in (6.3.19) gives

$$
\begin{aligned}
\int_{-A}^{A} e^{-xw^2}H(w)dw &= \int_{-A}^{A} e^{-xw^2}\sum_{k=0}^{\infty}\frac{H^{(k)}(0)}{k\,!}\,w^k dw \\
&= \int_{-A}^{A} e^{-xw^2}\sum_{k=0}^{\infty}\frac{H^{(2k)}(0)}{(2k)\,!}\,w^{2k} dw\,.
\end{aligned}
\qquad (6.3.21)
$$

Putting $w^2 = v$ in (6.3.21) gives

$$
\begin{aligned}
\int_{-A}^{A} e^{-xw^2}H(w)dw &= \int_{0}^{A} e^{-xv}\sum_{k=0}^{\infty}\frac{H^{2k}(0)}{(2k)\,!}\,v^{k-1/2} dv \\
&\sim \sum_{k=0}^{\infty}\frac{H^{(2k)}(0)}{(2k)\,!}\,\frac{\Gamma(k+1/2)}{x^{k+1/2}}
\end{aligned}
\qquad (6.3.22)
$$

by Watson's lemma. Combining (6.3.20) and (6.3.22) in (6.3.19) gives (6.3.18).

Example 6.3.12 From (6.3.7)

$$
\frac{dw}{dt} = g'(t)\big|_{t=a} = g'(a)
$$

which combined with the change of variable $t = h(w)$ shows that the leading term of (6.3.11) is the same as (6.3.2).

Example 6.3.13 Integrals of the form

$$
I(n) = \int_a^b g(t)^n f(t)dt
$$

can sometimes be brought into the form of a Laplace integral via

$$-g(t) = \ell n\big(h(t)\big)\,.$$

Thus if h has exactly one maximum at $\tau \in [a,b]$ then

$$\begin{aligned} I(n) &= \int_a^b e^{-ng(t)} f(t)dt \\ &= \left[\frac{2\pi}{ng''(\tau)}\right]^{1/2} e^{-ng(\tau)}[f(\tau)+o(1)] \qquad (6.3.23) \\ &= \left[\frac{2\pi}{-nh''(\tau)}\right]^{1/2} [h(\tau)]^{n+1/2}[f(\tau)+o(1)]\,, \quad n\to\infty\,. \end{aligned}$$

Example 6.3.14 The Legendre polynomial $P_n(x)$ can be defined by

$$P_n(x) = \frac{1}{\pi}\int_0^\pi [x+(x^2-1)^{1/2}\cos t]^n dt\,, \quad x>1\,.$$

Use Example 8 with $f(t)=1$ and $h(t)=x+(x^2-1)^{1/2}\cos t$. Now h has a maximum at $t=0$ and $h''(0)=-(x^2-1)^{1/2}$ so from (6.3.23)

$$P_n(x) = \frac{1}{2\pi}\left[\frac{2\pi}{n(x^2-1)^{1/2}}\right]^{1/2} [x+(x^2-1)^{1/2}]^{n+1/2}[1+o(1)]\,, \quad n\to\infty\,.$$

Example 6.3.15 The function $h(w)$ in the previous theorems may be impossible to obtain in closed form. In this case the Gauss-Hermite quadrature proves helpful. With Assumption 6.3.1-(ii) and (6.3.6) we can write

$$I(x) = e^{-xg(\tau)}\int_a^b e^{-xg''(\tau)(t-\tau)^2/2} H(x,t)dt \qquad (6.3.24)$$

where

$$H(x,t) = \exp\{-x[g(t)-g(\tau)-(1/2)(t-\tau)^2 g''(\tau)]\} f(t)\,. \qquad (6.3.25)$$

In (6.3.24) set

$$t = \tau + \left[\frac{2}{xg''(\tau)}\right]^{1/2} y \equiv \tau + Cy$$

to find

$$\begin{aligned} I(x) &= C\,e^{-xg(\tau)}\int_{-\infty}^{\infty} e^{-y^2} H(x,\,\tau+Cy)dy \\ &\sim C\,e^{-xg(\tau)}\left[\sum_{k=1}^m w_k H(x,\,\tau+Cy_k)+\epsilon\right] \end{aligned} \qquad (6.3.26)$$

where the y_k and w_k are the nodes and weights of Gauss-Hermite quadrature. From (5.6.28), the error in (6.3.26) is

$$\epsilon = \frac{\sqrt{\pi}\;m!}{2^{2m}(2m)!}\left(\frac{\partial}{\partial y}\right)^{2m} H(x,\,\tau+Cy) \quad . \qquad (6.3.27)$$

We need only consider the exponential component of $H(x,y)$ in estimating the error ϵ and only the first term in the expansion

$$\begin{aligned} \exp\{ \quad & -x[g(t) - g(\tau) - (1/2)(t-\tau)^2 g''(\tau)]\} \\ & = \exp\{-x[g_3 c^3 y^3 + g_4 c^4 y^4 + \cdots]\}\,. \end{aligned} \tag{6.3.28}$$

However we have

$$\begin{aligned} \left(\frac{d}{dy}\right)^{2m} [\exp(\alpha y^3)] &= \alpha^{2m/3} \left(\frac{d}{dw}\right)^{2m} [\exp(w^3)] \\ &= 0(\alpha^{2m/3}) \\ &= 0(x^{-m/3}) \end{aligned}$$

so it follows that

$$\epsilon = 0(x^{-m/3})\,.$$

Problems

6.3.1 Write the integral in Example 2 as

$$I(x) = 2\int_0^a e^{-x\cosh(t)} f(t)dt\,.$$

Compare the result of Example 2 with the application of Theorem 6.3.2-(ii) to $I(x)$.

6.3.2 Apply Theorem 6.3.2 to the integral

$$I(x) = \int_{-\pi}^{\pi} e^{x\cos(t)} f(t)dt\,.$$

6.3.3 Use Watson's lemma to obtain the asymptotic expansion of

$$\begin{aligned} I_0(x) &= \frac{1}{\pi}\int_0^2 \frac{e^{x(1-t)}T_0(1-t)}{(2-t)^{1/2}t^{1/2}}\,dt \\ &= \frac{1}{\pi}\int_0^2 e^{-xt}t^{-1/2}\sum_{n=0}^{\infty}\frac{(1/2)_n t^n}{2^n n\,!}\,dt \\ &\sim \frac{e^x}{\sqrt{2\pi x}}\sum_{n=0}^{\infty}\frac{[(1/2)_n]^2}{2^n n\,!\,x^n}\,. \end{aligned}$$

See Example 8, Section 6.1.

6.3.4 Apply Theorem 6.3.2-(ii) to

$$I(x) = \int_{\pi}^{\infty} \cos(t) e^{-xt^2}\, dt\,.$$

6.3.5 The leading term in the result of Example 6 is called Stirling's formula, i.e.

$$n\,! \sim \sqrt{2\pi n}\,(n/e)^n\,.$$

It has been stated a number of times that the series

$$\sum_{n=0}^{\infty} \frac{n\,!}{x^{n+1}}$$

diverges for all positive x. Show this using Stirling's formula.

6.4 Darboux's Method

Darboux's method is particularly valuable in numerical work since it permits estimates of functions defined by a generating function and is thus of value for example in obtaining asymptotic estimates of orthogonal polynomials.

The concept the method depends on is best illustrated by a simple example. Consider the Taylor series development

$$\begin{aligned}\frac{1}{1-ax}\, e^x &= (1 + ax + a^2x^2 + \cdots)\left(1 + x + \frac{x^2}{2\,!} + \cdots\right) \\ &= 1 + (a+1)x + \left(a^2 + \frac{a}{1\,!} + \frac{1}{2}\right)x^2 + \left(a^3 + \frac{a^2}{1\,!} + \frac{a}{2\,!} + \frac{1}{3\,!}\right)x^3 + \cdots\end{aligned} \tag{6.4.1}$$

by straightforward multiplication of series. The coefficient of x^n can be written

$$a^n\left(1 + \frac{1}{1\,!\,a} + \frac{1}{2\,!\,a^2} + \cdots + \frac{1}{n\,!\,a^n}\right)$$

so that the expression in parenthesis is the n^{th} partial sum of $\exp(1/n)$. Hence the coefficient of x^n in (6.4.1) approaches $a^n \exp(1/n)$. In particular, for $a = 1$

$$F(x) = \frac{1}{1-x}\, e^x = \sum_{n=0}^{\infty} F_n x^n \tag{6.4.2}$$

where

$$F_n \to e \,\text{as}\, n \to \infty\,.$$

Hence, if $F(x)$ is the product of a function with a simple singularity at $x = 1$ and a function that has no singularity at the origin we surmise that the coefficients, F_n, of the Taylor series for $F(x)$ are proportional to $F(1)$ ($= e$ in (6.4.2)) are $n \to \infty$.

We want to obtain an asymptotic estimate of the Taylor coefficients F_n where

$$F(x) = \frac{1}{(1-x)^{\alpha}}\, f(x) = \sum_{n=0}^{\infty} F_n x^n\,. \tag{6.4.3}$$

If

$$f(x) = \sum_{n=0}^{\infty} f_n x^n\,, \quad |x| < 1 + \epsilon \tag{6.4.4}$$

where $f_0 \neq 0$ and $\epsilon > 0$ then (6.4.3) has a singularity at $x = 1$ as long as α is not a negative integer. We also assume that the Taylor series of f converges at $x = 1$, i.e. for $\mu > 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n\,!}\,(x-1)^n\,, \quad |x-1| < \mu\,. \tag{6.4.5}$$

Theorem 6.4.1 (Darboux) *Suppose that f in (6.4.3) satisfies (6.4.4) and (6.4.5). Then the coefficients F_n in (6.4.3) satisfy*

$$F_n = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\, f(1)[1 + 0(1/n)]\,, \quad n \to \infty\,. \tag{6.4.6}$$

Proof: Using (6.4.5) in (6.4.3) we have

$$F(x) = \frac{1}{(1-x)^{\alpha}}\, f(1) - (1-x)^{1-\alpha} f'(1) + \frac{(1-x)^{2-\alpha}}{2\,!}\, f''(1) - \cdots\,.$$

Expanding the factors $(1-x)^{n-\alpha}$ in powers of x ($|x| < 1$) we obtain

$$F(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left\{ \frac{(-1)^k f^{(k)}(1)}{k\,!}\,(\alpha - k + n - 1)_{(n)} \right\} x^n\,. \tag{6.4.7}$$

The exchange of order of summation is permitted since the series converges in the common region $\{x : |x| < 1 \text{ and } |x-1| < \mu\}$. Comparing (6.4.3) and (6.4.7) coupled with the uniqueness of power series expansions gives

$$F_n = \sum_{k=0}^{\infty} \frac{(-1)^k f^{(k)}(1)}{k\,!} (\alpha - k + n - 1)_{(n)} \,. \tag{6.4.8}$$

Since (Problem 6.4.3)

$$\frac{(\alpha - k - \ell + n - 1)_{(n)}}{(\alpha - k + n - 1)_{(n)}} = 0\left(\frac{1}{n^\ell}\right) \;, \quad \ell = 1, 2, \ldots \;, \quad n \to \infty$$

only the first term in the series (6.4.8) is significant, i.e.

$$\begin{aligned} F_n &= (\alpha + n - 1)_{(n)} f(1)[1 + 0(1/n)] \,, \quad n \to \infty \\ &= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} f(1)[1 + 0(1/n)] \,, \quad n \to \infty \\ &= \frac{n^{\alpha-1}}{\Gamma(\alpha)} f(1)[1 + 0(1/n)] \,, \quad n \to \infty \end{aligned} \tag{6.4.9}$$

where the last equality uses Stirling's formula.

Example 6.4.2 The Bernouli numbers B_n are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n\,!} \,. \tag{6.4.10}$$

Recall Problem ??5.4.2 with $x = 0$. Adding $t/2$ to the left-hand side of (6.4.10) gives

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t}{2}\left(\frac{e^t + 1}{e^t - 1}\right)$$

which is an even function of t. Hence

$$B_1 = -1/2 \;,\; B_{2n+1} = 0 \;,\; n = 1, 2, \ldots$$

and

$$\frac{t}{2}\left(\frac{e^t + 1}{e^t - 1}\right) = \sum_{n=0}^{\infty} B_{2n} \frac{t^{2n}}{(2n)\,!} \,. \tag{6.4.11}$$

The left-hand side of (6.4.11) has a singularity at $t = 2\pi i$ so by writing $t = 2\pi i x$ in (6.4.11) the singularity is transformed to $x = 1$;

$$\begin{aligned} \pi i x \left(\frac{e^{2\pi i x} + 1}{e^{2\pi i x} - 1}\right) &= \sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{(2\pi)^{2n}}{2n\,!} x^{2n} \\ &= \frac{\pi x \cos(\pi x)}{\sin(\pi x)} \,. \end{aligned} \tag{6.4.12}$$

Now $x/\big(\sin(\pi x)\big)$ is singular at $x = \pm 1$ and we eliminate these singularities by defining

$$f(x) = \pi x(x^2 - 1)\,\frac{\cos(\pi x)}{\sin(\pi x)}\,.$$

Now f is an even function of x $f(1) = f(-1) = 2$ so the generating function (6.4.12) for the B_{2n} can be written

$$\frac{f(x)}{1 - x^2} = \sum_{n=0}^{\infty} (-1)^n\,\frac{B_{2n}(2\pi)^{2n}}{(2n)\,!}\,x^{2n}\,. \tag{6.4.13}$$

Replacing x^2 by y in (6.4.13) and using Darboux's theorem gives

$$(-1)^{n+1} B_{2n}\frac{(2\pi)^{2n}}{(2n)\,!} = f(1)[1 + 0(1/n)]\ , \quad n \to \infty\,.$$

That is

$$B_{2n} = (-1)^{n+1} 2\frac{(2n)\,!}{(2\pi)^{2n}}\,[1 + 0(1/n)]\ , \quad n \to \infty\,.$$

Example 6.4.3 The Chebyshev polynomials are generated by

$$f(t) = \frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x) t^n\,.$$

Now $1 - 2tx + t^2$ vanishes at

$$t = x \pm (x^2 - 1)^{1/2}\ , \quad x > 1$$

so if we put $\beta = x + (x^2 - 1)^{1/2} > 1$ then

$$1/\beta = x - (x^2 - 1)^{1/2}$$

and

$$x = \frac{1}{2}\,(\beta + 1/\beta)\,.$$

Hence

$$\begin{aligned} f(t) &= \frac{1 - 1/2t(\beta + \beta^{-1})}{\beta - \beta^{-1}}\left[\frac{1}{\beta^{-1} - t} - \frac{1}{\beta - t}\right] \\ &\equiv F_1(t) + F_2(t) \end{aligned} \tag{6.4.14}$$

where

$$F_1(t) = \left(\frac{\beta^2 - 1/2\beta t(\beta^2 + 1)}{\beta^2 - 1}\right)\left(\frac{1}{1 - \beta t}\right) \tag{6.4.15}$$

and

$$F_2(t) = -\left(\frac{\beta - 1/2t(\beta^2 + 1)}{\beta(\beta^2 - 1)}\right)\left(\frac{1}{1 - \beta^{-1} t}\right)\,. \tag{6.4.16}$$

In Problem 6.4.4 you are asked to show that if

$$F(t) = \frac{1}{(1-\gamma t)^{\alpha}} f(t) = \sum_{n=0}^{\infty} F_n t^n \tag{6.4.17}$$

then the conclusion of Darboux's theorem (6.4.6) reads

$$F_n = \frac{\gamma^n n^{\alpha-1}}{\Gamma(\alpha)} f(\gamma^{-1})[1 + 0(1/n)]\,. \tag{6.4.18}$$

Putting $F_j(t) = \sum_{n=0}^{\infty} F_{jn} t^n$ $(j = 1, 2)$ we have by (6.4.18), using $\gamma = \beta$ in F_1 and $\gamma = \beta^{-1}$ in F_2,

$$F_{1n} = \frac{1}{2}\beta^n[1 + 0(1/n)]\ , \quad n \to \infty$$

and

$$F_{2n} = \frac{1}{2}\beta^{-n}[1 + 0(1/n)]\ , \quad n \to \infty\,.$$

Hence

$$\begin{aligned} T_n(x) &= F_{1n} + F_{2n} \\ &= \frac{1}{2}(\beta^n + \beta^{-n})[1 + 0(1/n)] \\ &= \frac{1}{2}\beta^n[1 + 0(1/n)] \\ &= \frac{1}{2}[x + (x^2-1)^{1/2}]^n[1 + 0(1/n)]\,. \end{aligned}$$

Example 6.4.4 The Legendre polynomials $P_n(\cos\theta)$, $0 < \theta < \pi$, can be defined by the generating function

$$F(t) = \frac{1}{(1 - 2t\cos\theta + t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(\cos\theta) t^n\,.$$

Rewriting

$$F(t) = \frac{1}{(1 - te^{i\theta})^{1/2}(1 - te^{-i\theta})^{1/2}}$$

then P_n can be asymptotically estimated as $n \to \infty$ using (6.4.17) and (6.4.18). In the notation of (6.4.17) with $\beta = e^{i\theta}$ and $\alpha = 1/2$ we have

$$\begin{aligned} P_n(\cos\theta) &= \tfrac{n^{-1/2}}{\Gamma(1/2)}\left[\tfrac{e^{in\theta}}{(1-e^{2i\theta})^{1/2}} + \tfrac{e^{-in\theta}}{(1-e^{-2i\theta})^{1/2}}\right][1 + 0(1/n)] \\ &= \tfrac{1}{(\pi n)^{1/2}}[u + \hat{u}][1 + 0(1/n)] \end{aligned} \tag{6.4.19}$$

where, with $i^{1/2} = e^{\pi/4i}$,

$$u = \frac{e^{in\theta}}{(1 - e^{2i\theta})^{1/2}} = \frac{e^{in\theta + i\theta/2}}{\left[\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) 2i\right]^{1/2}} = \frac{e^{i[(n+1/2)\theta - \pi/4]}}{\sqrt{2}\,\sin(\theta)} \tag{6.4.20}$$

and $\hat{u}$ is obtained by replacing i by $-i$ in (6.4.20). Hence

$$\begin{aligned}
P_n(\cos\theta) &= \frac{1}{(\pi n)^{1/2}}\left\{\frac{e^{i[(n+1/2)\theta-\pi/4]}}{\sqrt{2}\sin^{1/2}\theta}+\frac{e^{-i[(n+1/2)\theta-\pi/4]}}{\sqrt{2}\sin^{1/2}\theta}\right\}[1+0(1/n)] \\
&= \left(\frac{2}{\pi n\sin\theta}\right)^{1/2}\cos[(n+1/2)\theta-\pi/4][1+0(1/n)] \\
&= \left(\frac{2}{\pi n\sin\theta}\right)^{1/2}\cos[(n+1/2)\theta-\pi/4]+0\left(\frac{1}{n^{3/2}}\right).
\end{aligned}$$

Problems

6.4.1 Using the series development

$$\frac{1}{(1-ax)^2} = 1+2ax+3a^2x^2+4a^3x^3+\cdots \tag{6.4.21}$$

find an estimate of F_n in

$$F(x) = \frac{1}{(1-ax)^2}\,e^x = \sum_{n=0}^{\infty} F_n x^n$$

by the procedure outlined in (6.4.1). What is your result for $a=1$? Check your result via Darboux's theorem.

6.4.2 Obtain the series in (6.4.21). Hint:

$$\frac{d}{dx}\left(\frac{1}{1-ax}\right) = \frac{a}{(1-ax)^2} \qquad \text{and (6.4.1).}$$

6.4.3 Recall that

$$(\gamma)_{(n)} = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}.$$

Show that

$$\frac{(\gamma-\ell)_{(n)}}{(\gamma)_{(n)}} = 0\left(\frac{1}{n^\ell}\right), \quad \ell=1,2,\, n\to\infty.$$

6.4.4 Obtain from (6.4.17) the formula (6.4.18). Put $\gamma t = x$ in (6.4.17) so that

$$\frac{1}{(1-\gamma t)^\alpha}f(t) = \frac{1}{(1-x)^\alpha}f(x/\gamma) = \sum_{n=0}^{\infty}F_n x^n.$$

Now consider (6.4.3) and (6.4.6).

6.5 Asymptotic Methods in Differential Equations

This section is concerned with the approximation of the solution of the differential equation

$$\frac{d^2y}{dx^2} + p(x)\,\frac{dy}{dx} + q(x)y = 0 \tag{6.5.1}$$

for large x. Two fairly typical equations are

$$y'' + \left(2 + \frac{1}{x}\right) y' + \frac{y}{x} = 0 \tag{6.5.2}$$

and

$$y'' + \left(1 + \frac{1}{x^2}\right) y = 0 \tag{6.5.3}$$

For large x, we may suspect that the solutions of these equations may, for a first approximation, have the same form as the solutions of

$$\begin{aligned} y'' + 2y' &= 0 \quad (y = A + Be^{-2x}) \\ y'' + y &= 0 \quad (y = A\sin x + B\cos x) \end{aligned}$$

A very simple method for the solution of such equations has been known for more than a century and the solutions so obtained in descending powers of x are called Stokes-type solutions. If $p(x)$ and $q(x)$ in (6.5.1) are bounded, $x \to \infty$, the equation can be written as

$$y'' + \left(a_0 + \frac{a_1}{x} + \cdots\right) y' + \left(b_0 + \frac{b_1}{x} + \cdots\right) y = 0\,. \tag{6.5.4}$$

Assuming that the solutions of the differential equation have the form

$$\begin{aligned} y_1(x) &= e^{\lambda_1 x} x^{\sigma_1} \left[A_0 + \tfrac{A_1}{x} + \tfrac{A_2}{x^2} + \cdots\right] \\ y_2(x) &= e^{\lambda_2 x} x^{\sigma_2} \left[B_0 + \tfrac{B_1}{x} + \tfrac{B_2}{x^2} + \cdots\right] \end{aligned} \tag{6.5.5}$$

the parameters λ_1, λ_2 and σ_1, σ_2 are determined by

$$\begin{aligned} \lambda_1 + \lambda_2 &= -a_0 \qquad \lambda_1\lambda_2 = b_0 \\ \sigma_1 + \sigma_2 &= -a_1 \qquad \lambda_1\sigma_2 + \lambda_2\sigma_1 = b_1 \end{aligned} \tag{6.5.6}$$

The first two of these equations imply that λ_1 and λ_2 are different unless $b_0 = 1/4\, a_0^2$; we exclude this case temporarily. Except in the excluded case, σ_1 and σ_2 are determined, and the remaining coefficients A_i and B_i of (6.5.5) can be determined
sequentially by familiar methods.

Sometimes, this method suffices for the problem in hand and it has the advantage that there are no formulas to remember.

Example 6.5.1 The integral

$$y(x) = \frac{e^{-x}}{\pi} \int_{-1}^{1} e^{-xu}(1-u^2)^{-1/2}du = e^{-x} I_0(x)$$

satisfies the differential equation (6.5.2) where $I_0(x)$ is the modified Bessel function that satisfies $xI_0''(x) + I_0'(x) - xI_0(x) = 0$. Using (6.5.6) with $a_0 = 2$, $a_1 = 1$, $b_0 = 0$ and $b_1 = 1$ we find

$$\begin{aligned} \lambda_1 &= 0 \quad \lambda_2 = -2 \\ \sigma_1 &= \sigma_2 = -1/2 \end{aligned}$$

for (6.5.5). Hence, the solutions of (6.5.2) have the form

$$y_1(x) = x^{-1/2}\left[A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots\right]$$

and

$$y_2(x) = e^{-2x}x^{-1/2}\left[B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \cdots\right].$$

It is easy to obtain the series solution

$$\begin{aligned} y(x) &= A_0 x^{-1/2}\left[1 + \frac{1}{8x} + \frac{9}{128x^2} + \cdots\right] \\ &\quad + B_0 e^{-2x} x^{-1/2}\left[1 - \frac{1}{8x} + \frac{9}{128x^2} - \cdots\right] \end{aligned}$$

The determination of the constants A_0 and B_0 is always the difficult point in asymptotic solutions of differential equations. Usually the leading terms must be obtained by alternative methods.

Example 6.5.2 For equation (6.5.3) the Stokes procedure gives

$$\lambda_1 + \lambda_2 = 0 \qquad \lambda_1\lambda_2 = 1$$

so that $\lambda_1 = i$ and $\lambda_2 = -i$ and

$$\sigma_1 + \sigma_2 = 0 \qquad i(\sigma_2 - \sigma_1) = 0$$

so that $\sigma_1 = \sigma_2 = 0$. From (6.5.5) the solutions of (6.5.3) have the form

$$y_1(x) = e^{ix}\left[A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots\right] \tag{6.5.7}$$

and

$$y_2(x) = e^{-ix}\left[B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \cdots\right] \tag{6.5.8}$$

Example 6.5.3 The excluded case of equal roots in the analysis of (6.5.4) ($\lambda_1 = \lambda_2$) presents no grave difficulties. In this case we the factor $\exp(\lambda x)$ from the solution and assume that a_0 and b_0 are zero in (6.5.4). The solution is then of the form

$$y_\pm(x) = e^{\pm\mu x^{1/2}} x^\sigma \left(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots\right) \tag{6.5.9}$$

where

$$2(\sigma - 1/4) = -a_1 \qquad \frac{1}{4}\mu^2 = b_1 \neq 0\,. \tag{6.5.10}$$

For a number of reasons, mathematical interest has turned towards the study of differential equations simpler in form than (6.5.1). The first of these reasons is that a change of dependent variable yields a differential equation with no dy/dx term. If we write

$$y = u \exp\left[-\frac{1}{2}\int^x p(t)dt\right], \tag{6.5.11}$$

then (6.5.1) becomes

$$\frac{d^2u}{dx^2} + \left(q - \frac{1}{2}p' - \frac{1}{4}p^2\right)u = 0\,. \tag{6.5.12}$$

Hence, we can without loss of generality suppose that the middle term of (6.5.1) absent and in the remainder of this section we are concerned with equations of the form

$$y'' + g(x)y = 0\,. \tag{6.5.13}$$

A second reason for concentration on equations of the type (6.5.13) rather than (6.5.1) is that error analysis of (6.5.13) is much simpler; another reason is that excellent methods are available for (6.5.13).

Even for (6.5.13), it is necessary to imply a simplifying condition, in order to reach a tractable analysis: *In the remainder of this chapter, it is assumed that $g(x)$ is one-signed on the interval where the solution is to be determined.* When g has one sign in one part of the interval and a different sign in another part or when $g(x)$ oscillates between positive and negative values on the interval, the analysis is considerably more difficult and is beyond the scope of this book.

Problems

6.5.1 Show that $y(x) = B_0 + A_0 e^{-2x}$ is the solution of $y'' + 2y' = 0$ as follows: With $p(x) = 2$ we see that $a_0 = 2$ and $a_j = 0$ for all $j \geq 1$. Similarly

$b_j = 0$ $(j \geq 0)$ since $g(x) = 0$. The relations in (6.5.5) then give $\lambda_1 = -2$, $\lambda_2 = 0$, $\sigma_1 = \sigma_2 = 0$ so that

$$y_1(x) = e^{-2x}\left(A_0 + \frac{A_1}{x} + \cdots\right)$$

and

$$y_2(x) = B_0 + \frac{B_1}{x} + \cdots .$$

Substitute each of these in the differential equation to find $A_j = 0$ $(j \geq 1)$ and $B_j = 0$ $(j \geq 1)$.

6.5.2 Find the first few A_j in (6.5.7) as follows: Direct differentiation of $y_1(x)$ gives

$$y_1''(x) + y_1(x) = e^{ix}\left[2i\left(\frac{-A+1}{x^2} - \frac{2A_2}{x^3} - \cdots\right) + \left(\frac{2A_1}{x^3} + \frac{3!\,A_2}{x^4} + \cdots\right)\right]$$

and

$$\frac{1}{x^2}\,y_1(x) = e^{ix}\left(\frac{A_0}{x^2} + \frac{A_1}{x^3} + \cdots\right) .$$

Adding these two lines and equating coefficients of like powers gives the A_j $(j \geq 1)$ in terms of A_0 $(A_1 = A_0/2i,\ A_2 = 3/8\,A_0)$. Conclude that

$$y_1(x) = A_0\left\{\sin x\left[1 - \frac{3}{8x^2} + \cdots\right] - \cos x\left[\frac{1}{2x} - \frac{3}{4x^3} + \cdots\right]\right\} .$$

6.5.3 For the differential equation in (6.5.2) show that the equation (6.5.12) takes the form

$$u''(x) + \left[-1 + \frac{1}{4x^2}\right]u(x) = 0\,.$$

As in the line following (6.5.3) we expect (for large x) the solutions of the u equation are approximated by the solutions of $u''(x) - u(x) = 0$; i.e. $u_1(x) = A_0e^x$ and $u_2(x) = B_0e^{-x}$. Use this in conjunction with the change of variable in (6.5.11) to find the leading terms of the solutions $y_1(x)$ and $y_2(x)$ displayed in Example 6.5.1.

6.6 WKB Methods

When $g(x)$ in (6.5.13) is one-signed throughout the interval of interest a change of independent variable followed by a change of dependent variable brings (6.5.1) to one of the forms

$$y''(x) - [1 + f(x)]y(x) = 0 \tag{6.6.1}$$

or

$$y''(x) + [1 + f(x)]y(x) = 0 \tag{6.6.2}$$

where $f(x) \to 0$, $x \to \infty$. The first form (6.6.1) is reached when $g(x)$ in (6.5.13) is negative and the second (6.6.2) when $g(x)$ is positive. For the moment, we suppose that the requisite changes of variable have been made (the connection with (6.5.13) is established later).

Consider, for example, (6.6.1) which is analogous to the simple equation

$$y'' - y = 0\,.$$

This equation has the solutions

$$y = \exp(\pm x) = \exp\left[\pm \int^x 1\,dt\right].$$

Using this analogy, we see that a solution of (6.6.1) has the form

$$y = \exp\left[\int^x (1 + \eta(t))dt\right] \tag{6.6.3}$$

so that

$$y' = [1 + \eta(x)]y$$

and

$$y'' = [(1 + \eta)^2 + \eta']y\,.$$

Equation (6.6.1) is satisfied if

$$\eta' + (1 + \eta)^2 = 1 + f(x)\,. \tag{6.6.4}$$

If we assume that η can be expressed as the series

$$\eta = \eta_1 + \eta_2 + \eta_3 + \cdots$$

then (6.6.4) becomes

$$1 + 2\eta_1 + (2\eta_2 + \eta_1^2) + (2\eta_3 + 2\eta_1\eta_2) + \cdots = 1 + f - \eta_1' - \eta_2' - \eta_3' - \cdots$$

and these equations are soluble for the η_i if they are arranged in the form

$$\begin{aligned} 2\eta_1 &= f \\ 2\eta_2 &= -\eta_1^2 - \eta_1' \\ 2\eta_3 &= -2\eta_1\eta_2 - \eta_2' \,. \end{aligned}$$

From these, we obtain in sequence

$$\begin{aligned} \eta_1 &= \frac{1}{2} f \\ \eta_2 &= -\frac{1}{4} f' - \frac{1}{8} f^2 \\ \eta_3 &= \frac{1}{4} f f' + \frac{1}{8} f'' + \frac{1}{16} f^3 \,. \end{aligned}$$

If the derivatives of f approach zero, $x \to \infty$, a few terms of this series may yield a satisfactory asymptotic approximation. Since each of the terms η_i is to be integrated in the expression (6.6.3), it is not possible to effect the integration explicitly until the form of $f(x)$ is given.

Clearly, (6.6.1) has a second solution of the form

$$y = \exp\left[\int^x (-1 + \eta(t))dt\right]$$

but it is simpler to obtain the form of this solution by the method of Example 6.6.1 below.

Example 6.6.1 A solution of the equation

$$y'' = a^2[1 + f(x)]y$$

is of the form

$$y = \exp\left[\int^x \big(a + \eta(t)\big)dt\right]$$

where

$$\eta = \eta_1 + \eta_2 + \eta_3 + \cdots \,.$$

Verify that

$$\begin{aligned} \eta_1 &= \frac{af}{2} \qquad \eta_2 = -\frac{f'}{4} - \frac{af^2}{8} \\ \eta_3 &= \frac{ff'}{4} + \frac{f''}{8a} + \frac{af^3}{16} \,. \end{aligned}$$

The second solution of equations (6.6.1) can now be obtained on replacing a by -1. Similarly, the two (complex) solutions of (6.6.2) are obtained on replacing a by i and $-i$.

Example 6.6.2 The equation

$$y'' = a^2 \left[1 + \frac{b}{x^2}\right] y$$

is a special case of (6.6.1) or (6.6.2), but is of some generality. It can be shown that the Stokes-type solution of this equation is

$$\begin{aligned} y &= A_0 e^{ax} \left[1 - \frac{ab}{2x} - \frac{b(2-a^2b)}{8x^2} - \frac{b(2-a^2b)(6-a^2b)}{48ax^3} - \cdots\right] \\ &\quad + B_0 e^{-ax} \left[1 + \frac{ab}{2x} - \frac{b(2-a^2b)}{8x^2} + \frac{b(2-a^2b)(6-a^2b)}{48ax^3} - \cdots\right] . \end{aligned}$$

Example 6.6.3 Verify from Example 6.6.1 that

$$y'' = \left(1 + \frac{1}{x^2}\right) y$$

has the solution

$$y = \exp\left[x - \frac{1}{2x} - \frac{1}{4x^2} - \cdots\right]$$

and verify from Example 2 that the solution can be written as

$$y = e^x \left[1 - \frac{1}{2x} - \frac{1}{8x^2} - \cdots\right] .$$

Verify that these solutions are consistent up to the terms in $1/x^2$ (Problem 6.6.4).

Example 6.6.4 Verify that

$$y_1(x) = \exp\left(x - \frac{1}{2x}\right)$$

satisfies the equation

$$y_1'' = \left(1 + \frac{1}{x^2}\right) y_1 + r(x)$$

where

$$r(x) = -\left(\frac{1}{x^3} - \frac{1}{4x^4}\right) y_1 .$$

The device used for the approximate solution of (6.6.1) and (6.6.2) can be used with advantage for the more general equation (6.5.13). For a reason that appears presently, we replace (6.5.13) by

$$y'' = \frac{g(x)}{h^2} y \, , \quad g(x) > 0 \, , \quad a \le x \le b \tag{6.6.5}$$

where we assume that $g(x)$ is positive throughout the interval where the solution is to be determined and h is a parameter that we may choose freely. In particular, putting $h = \pm 1$, we obtain the equation

$$y'' = g(x) y$$

in which the multiplier of y on the right is a positive function; again, putting $h = \pm i$, we obtain the equation

$$y'' = -g(x)y$$

in which the multiplier of y on the right is a negative function. For the moment, it is convenient to regard h as a small positive parameter.

We seek a solution of (6.1.5) of the form

$$y = \exp\left[\frac{1}{h}\int^x (\mu_0 + h\mu_1 + h^2\mu_2^2 + \cdots)dt\right] \tag{6.6.6}$$

where

$$\mu_i = \mu_i(t) \quad , \quad i = 0, 1, 2, \ldots$$

is a differentiable function of t. Proceeding as before, we find the equation

$$\frac{1}{h^2}\left[\mu_0 + h\mu_1 + h^2\mu_2 + \cdots\right]^2 + \frac{1}{h}\left[\mu_0' + h\mu_1' + h^2\mu_2' + \cdots\right] = \frac{g(x)}{h^2}$$

where the primes denote differentiation with respect to x. Equating the coefficients of powers of h on both sides of this equation, we arrive at the sequence of equations

$$\begin{aligned} \mu_0^2 &= g \\ 2\mu_0\mu_1 + \mu_0' &= 0 \\ 2\mu_0\mu_2 + \mu_1^2 + \mu_1' &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \end{aligned} \tag{6.6.7}$$

The first three equations yield

$$\begin{aligned} \mu_0 &= \pm g^{1/2} \qquad \mu_1 = -\frac{1}{2}\frac{\mu_0'}{\mu_0} \\ \mu_2 &= \frac{1}{4\mu_0}\left[\frac{\mu_0''}{\mu_0} - \frac{3}{2}\frac{(\mu_0')^2}{\mu_0^2}\right] \ . \end{aligned} \tag{6.6.8}$$

If we denote by y_1 the approximation obtained by retaining the terms in μ_0 and μ_1 in (6.6.6), then

$$y_1(x) \quad = \exp\left[\tfrac{1}{h}\int^x \left\{\mu_0(t) - \tfrac{h}{2}\tfrac{\mu_0'(t)}{\mu_0(t)}\right\} dt\right]$$

$$= |\mu_0|^{-1/2} \exp\left[\tfrac{1}{h}\int^x \mu_0(t)dt\right] \tag{6.6.9}$$

where $\mu_0 = \pm g^{1/2}$. The function $y_1(x)$ is called the WKB approximation. We surmise that the solution $y_1(x)$ is related to the true solution $y(x)$ by the relation

$$y(x) = y_1(x)\left[1 + 0\Big(\exp\Big[h\int^x \eta_2(t)dt\Big]\Big)\right].$$

This surmise cannot be confirmed by the present analysis since the convergence of the series μ_i cannot be demonstrated. We can however establish that $y_1(x)$ satisfies a differential equation whose coefficients are close to the original equation. For

$$\begin{aligned} y_1(x) &= \exp\left[\frac{1}{h}\int^x \left\{\mu_0 - \frac{h}{2}\frac{\mu_0'}{\mu_0}\right\} dt\right] \\ \frac{d^2y_1}{dx^2} &= \left[\frac{\mu_0^2}{h^2} - 2\mu_2\right] y_1 \\ &= \left[\frac{g(x)}{h^2} - 2\mu_2\right] y_1 . \end{aligned} \tag{6.6.10}$$

For h small, we observe that the second term in brackets above is independent of h and is relatively small, $h \to 0$. For $h = 1$ we require

$$\frac{2\mu_2}{\mu_0^2} = \frac{2\mu_2}{g} \ll 1 .$$

It is important to realize that (6.6.9) embraces two solutions since

$$\mu_0 = \pm g^{1/2}$$

and we know from the general theory that a second-order differential equation possesses two independent solutions.

Example 6.6.5 When the function g in (6.6.5) given by

$$g(x) = 1 - \frac{1}{x^2+1}$$

we see from (6.6.8) that

$$\mu_0 = \pm\tfrac{x}{(1+x^2)^{1/2}}$$
$$\mu_2 = \mp\tfrac{3}{4(1+x^2)^2}\left[1 \pm \tfrac{1}{2x^2}\right].$$

For $x \geq 10$ then

$$|\mu_2| < 7\cdot 5\times 10^{-5}.$$

Example 6.6.6 When

$$\begin{aligned} g(x) &= e^{2x} \qquad\qquad \mu_0 = \pm e^x \\ \frac{|\mu_2|}{g(x)} &= \frac{1}{8}e^{-2x} \\ &< 6\times 10^{-6} \text{ for } x \geq 5 \end{aligned}$$

Example 6.6.7 Taking $h = \pm i$ then the WKB approximation of the equation

$$y'' = -g(x)y \quad , \quad g(x) > 0$$

is

$$y_1(x) = |\mu_0|^{-1/2}\exp\left[\mp i\int^x \mu_0(t)dt\right].$$

Example 6.6.8 It is sometimes desirable to express the WKB approximation explicitly in terms of g. Equation (6.6.9) can be expressed as

$$y_1(x) = |g(x)|^{-1/4}\exp\left[\frac{1}{h}\int^x g^{1/2}(t)dt\right] \tag{6.6.11}$$

and

$$\mu_2 = \frac{1}{8}\left[\frac{g''}{g} - \frac{5}{4}\,\frac{(g')^2}{g^2}\right]. \tag{6.6.12}$$

The equation (6.6.10) becomes

$$\frac{d^2y_1}{dx^2} = \left[\frac{g(x)}{h^2} - \frac{1}{4}\left(\frac{g''}{g} - \frac{5}{4}\,\frac{(g')^2}{g^2}\right)\right]y_1 \tag{6.6.13}$$

but the earlier notation seems preferable.

To begin the analysis implied after (6.6.9) we consider

$$y'' \pm [1 + f(x)]y = 0$$

and subsequently extend the analysis to the more general equations

$$y'' \pm g(x)y = 0 \quad , \quad g(x) > 0\,.$$

It is convenient to consider positive and negative signs separately, so in this direction we state two separate but related lemmas.

Lemma 6.6.9 *If $y(x)$ satisfies the differential equation*

$$y'' + [1 + f(x)]y = 0 \quad , \quad a \leq x \leq b \tag{6.6.14}$$

and if we can find a $w(x)$ that satisfies the equation

$$w'' + [1 + f(x)]w = r(x) \quad , \quad a \leq x \leq b \tag{6.6.15}$$

where

$$\int_a^b |f(t)|dt < \infty \quad , \quad \int_a^b |r(t)|dt < \infty \,, \tag{6.6.16}$$

then there exists a solution $y(x)$ of (6.6.14) such that

$$|y(x) - w(x)| \leq \exp\left[\int_x^b |f(t)|dt\right] \int_x^b |r(t)|dt \,. \tag{6.6.17}$$

Proof: On subtracting equation (6.6.15) from (6.6.14) and writing

$$\epsilon = y - w \tag{6.6.18}$$

we find that ϵ satisfies the differential equation

$$\epsilon'' + [1 + f(x)]\epsilon = -r(x) \,. \tag{6.6.19}$$

It can be verified that $\epsilon(x)$ defined by (6.6.19) satisfies the integral equation

$$\epsilon(x) = \int_x^b \sin(x - t)[f(t)\epsilon(t) + r(t)]dt \,, \quad a \leq x < b \tag{6.6.20}$$

and it is much easier to obtain a bound on $\epsilon(x)$ from the integral equation. It can also be verified that

$$\epsilon(x) \to 0 \quad , \quad \epsilon'(x) \to 0 \,, \quad x \to b$$

and from these properties it is clear that $\epsilon(x)$ defined by (6.6.20) implies $y(x) \to w(x)$, $y'(x) \to w'(x)$, $x \to b$. Since

$$|\sin(x - t)| \leq 1$$

it follows that

$$|\epsilon(x)| \leq \int_a^b [\,|f(t)|\,|\epsilon(t)| + |r(t)|\,]\,dt \,.$$

Let us define $E(x)$ by

$$E(x) = \int_x^b [\,|f(t)|E(t) + |r(t)|\,]\,dt\,.$$

Differentiating $E(x)$ with respect to x, we find that $E(x)$ satisfies the first-order equation

$$E'(x) = -|f(x)|E(x) - r(x) \quad , \quad E(b) = 0\,.$$

Writing

$$F(x) = \int_x^b |f(t)|dt$$

the solution of this differential equation is

$$E(x) = \int_x^b \exp[F(x) - F(t)]|r(t)|dt$$

and it is now clear that $E(x)$ is positive in $a \leq x \leq b$ and that

$$|\epsilon(x)| \leq E(x)\,.$$

The expression for the error can be simplified slightly; since $F(x)$ is positive and decreasing in $a \leq x \leq b$, we have

$$F(x) - F(t) \leq F(x)\,.$$

Hence finally

$$|\epsilon(x)| \leq \exp(F(x)) \int_x^b |r(t)|dt$$

where

$$F(x) = \int_x^b |f(t)|dt\,. \tag{6.6.21}$$

This is the inequality (6.6.17).

The existence of a bounded solution $\epsilon(x)$ implies the existence of a bounded solution

$$y(x) = w(x) + \epsilon(x)$$

of (6.6.14). Notice that (6.6.15) has two independent solutions and correspondingly there are two solutions of (6.6.14).

The corresponding lemma for the equation

$$y'' = [1 + f(x)]y \quad , \quad a \leq x \leq b \tag{6.6.22}$$

may be formulated thus:

Lemma 6.6.10 *If $w_1(x)$ and $w_2(x)$ satisfy the differential equations*

$$\begin{aligned} \frac{d^2 w_1}{dx^2} &= [1+f(x)]w_1 + r_2 e^{x} \quad , \quad a \le x \le b \\ \frac{d^2 w_2}{dx^2} &= [1+f(x)]w_2 + r_2 e^{-x} \quad , \quad a \le x \le b \end{aligned} \tag{6.6.23}$$

and if

$$\int_a^b |f(t)|dt < \infty \quad , \quad \int_a^b |r_i(t)|dt < \infty \quad , \quad i = 1, 2 \, .$$

Then, in the notation,

$$\begin{aligned} w_1 &= e^{x} v_1 & \qquad w_2 &= e^{-x} v_2 \\ y_1 &= e^{x} z_1 & \qquad y_2 &= e^{-x} z_2 \end{aligned}$$

There exist solutions y_1 and y_2 of (6.6.22) such that

$$|z_1(x) - v_1(x)| \le \frac{1}{2} \exp\Big[\int_a^x |f(t)|dt\Big] \Big(\int_a^x |r_1(t)|dt\Big)$$

and

$$|z_2(x) - v_2(x)| \le \frac{1}{2} \exp\Big[\int_x^b |f(t)|dt\Big] \Big(\int_x^b |r_2(t)|dt\Big) \, .$$

The proof of this lemma is similar to the proof of the preceding lemma and may be omitted; the principal steps of the proof are suggested in the problems that follow.

Problems

6.6.1 If w_1 satisfies the first of (6.6.23) and if

$$\epsilon_1 = y_1 - w_1$$

show that ϵ_1 satisfies the differential equation

$$\epsilon_1'' = [1 + f(x)]\epsilon_1 - r_1 e^{x}$$

and the integral equation

$$\epsilon_1 = \int_b^x \sinh(x-t) \left[f(t)\epsilon_1(t) - r_1(t)e^{t} \right] dt \, .$$

Writing $\epsilon_1 = \eta_1 e^x$, show that η_1 satisfies the integral equation

$$\eta_1(x) = \int_a^x e^{-(x-t)} \sinh(x-t) \, [f(t)\eta_1 - r_1(t)] \, dt \, .$$

Note that

$$|e^{-(x-t)} \sinh(x-t)| \le \frac{1}{2} \, , \quad t \le x$$

and that the lower limit a may be taken to be $-\infty$.

6.6.2 If w_2 satisfies the second of (6.6.23) and if

$$\epsilon_2 = y_2 - w_2$$

show that ϵ_2 satisfies the equations

$$\begin{aligned} \epsilon_2'' &= [1 + f(x)]\epsilon_2 - r_2 e^{-x} \\ \epsilon_2 &= \int_x^b \sinh(t-x) \left[f(t)\epsilon_2(t) - r_2(t) e^{-t} \right] dt \, . \end{aligned}$$

Writing $\epsilon_2 = \eta_2 e^{-x}$, show that

$$\eta_2(x) = \int_x^b e^{-(t-x)} \sinh(t-x) \, [f(t)\eta_2 - r_2(t)] \, dt \, .$$

The limit b may be taken to be $+\infty$.

6.6.3 If

$$y'' = \left[1 + \frac{1}{x^m} \right] y$$

where m is an integer ≥ 2, show that the coefficients of the asymptotic series

$$y = e^x \sum_{k=0}^{\infty} \frac{A_k}{x^k}$$

are determined by the equations

$$\begin{aligned} 0 &= \frac{-2(m-1)A_{m-1} - A_0}{x^m} - 2 \sum_{k=0}^{\infty} \frac{(m+k)A_{m+k}}{x^{m+k+1}} \\ &\quad + \sum_{k=0}^{\infty} \frac{(m+k-1)(m+k)A_{m+k+1}}{x^{m+k+1}} - \sum_{k=0}^{\infty} \frac{A_{m+k}}{x^{m+k}} \end{aligned}$$

and that the non-zero coefficients are determined by

$$\begin{aligned}
A_{m-1} &= -\frac{1}{2(m-1)}\,A_0 \\
A_{m+k} &= \frac{1}{2}\,(m+k-1)A_{m+k-1}\ , \quad k=0,1,2,\ldots,m-3 \\
A_{2m+k-2} &= \frac{1}{2}\,(2m+k-3)A_{2m+k-3} \\
&\quad -\frac{1}{2(2m+k-3)(2m+k-2)}\,A_{m+k-1}\ , \quad k=0,1,2,\ldots
\end{aligned}$$

Further, apply the preceding to the differential equation with $m=4$ to show that the relative error of the approximation

$$y_1(x) = A_0 e^x \left[1-\frac{1}{6x^3}\right]$$

is less than

$$\frac{1}{2}\,A_0\left\{\frac{1}{24x^4}+\frac{1}{6x^6}\right\}\exp\left(\frac{1}{6x^3}\right)\ .$$

6.6.4 Referring to Example 3 verify the steps

$$\begin{aligned}
y(x) &= \exp\left[x-\frac{1}{2x}-\frac{1}{4x^2}-\cdots\right] \\
&= e^x\left[\left(1-\frac{1}{2x}+\frac{1}{2(4x)^2}\ \cdots\right)\left(1-\frac{1}{4x^2}+\frac{1}{2(4x^2)^2}\ \cdots\right)\cdots\right] \\
&= e^x\left[1-\frac{1}{2x}-\frac{1}{8x^2}-\cdots\right]
\end{aligned}$$

so that the consistency of the last line of that example is established.

6.7 Liouville's Transformation

It remains to bring the equations

$$\begin{aligned}
y''+g(x)y &= 0 \qquad\qquad g(x)>0\ , \quad a\le x\le b \\
y''-g(x)y &= 0 \qquad\qquad g(x)>0\ , \quad a\le x\le b
\end{aligned} \tag{6.7.1}$$

within the scope of the error-bound analysis of the preceding section; we consider first the first equation of (6.7.1).

It is convenient to retain the earlier notation so that μ_0 denotes either the positive or negative square root of $g(x)$ and to write

$$\mu = +g^{1/2}(x)\,. \tag{6.7.2}$$

The change of independent variable

$$\xi = \int_a^x \mu(t)dt \tag{6.7.3}$$

defines a one-to-one map of the interval $a < x < b$ onto the interval $0 < \xi < c$ where

$$c = \int_a^b \mu(t)dt\,. \tag{6.7.4}$$

From (6.7.3)

$$\frac{d\xi}{dx} = \mu(x)$$

and

$$\begin{aligned} \frac{dy}{dx} &= \mu(x)\,\frac{dy}{d\xi} \\ \frac{d^2y}{dx^2} &= \mu^2\,\frac{d^2y}{d\xi^2} + \frac{d\mu}{dx}\,\frac{dy}{d\xi}\,. \end{aligned}$$

The first of (6.7.1) becomes, on division by μ^2,

$$\frac{d^2y}{d\xi^2} + \frac{1}{\mu^2}\,\frac{d\mu}{dx}\,\frac{dy}{d\xi} + y = 0\,. \tag{6.7.5}$$

The middle term in the above equation can be removed by the change of dependent variable (6.5.11). It is convenient to write temporarily

$$\begin{aligned} \mu(x) &= \lambda(\xi) \\ \frac{d\mu}{dx} &= \lambda\,\frac{d\lambda}{d\xi}\;,\quad \frac{1}{\mu^2}\,\frac{d\mu}{dx} = \frac{1}{\lambda}\,\frac{d\lambda}{d\xi}\,. \end{aligned} \tag{6.7.6}$$

We can now make the change of dependent variable

$$\begin{aligned} y &= u\exp\left[-\frac{1}{2}\int^{\xi}\frac{1}{\lambda(t)}\,\frac{d\lambda(t)}{dt}\,dt\right] \\ &= \lambda^{-1/2}(\xi)u \end{aligned}$$

and, from (6.5.12), the equation in u is

$$\frac{d^2u}{d\xi^2} + \left[1 - \frac{1}{2}\left\{\frac{1}{\lambda}\,\frac{d^2\lambda}{d\xi^2} - \frac{1}{2\lambda^2}\left(\frac{d\lambda}{d\xi}\right)^2\right\}\right]u = 0 \tag{6.7.7}$$

which is of the form (6.6.2). Hence the error-bound (6.6.17) may be used for the approximate solutions of (6.7.7) and hence of the first of (6.7.1) since y is related to u by known transformations. Writing

$$\mu(t) = \lambda(s)$$

and

$$\frac{ds}{dt} = \mu(t)$$

the expression $F(\xi)$ in (6.6.21) can be expressed in terms of t and x. We find

$$\begin{aligned} F(\xi) &= \frac{1}{2}\int_x^b \frac{1}{\mu}\left[\frac{1}{\mu}\,\frac{d^2\mu(t)}{dt^2} - \frac{3}{2\mu^2}\left(\frac{d\mu(t)}{dt}\right)^2\right]dt \\ &= 2\int_x^b \frac{|\mu_2(t)|}{\mu(t)}\,dt \end{aligned} \tag{6.7.8}$$

in the notation of (6.6.8) if μ_2 as there defined is replaced by μ.

Example 6.7.1 Let us rewrite (6.7.7) in the form

$$\frac{d^2u}{d\xi^2} + [1 + h(\xi)]u = 0 \tag{6.7.9}$$

where

$$h(\xi) = -\frac{1}{2}\left[\frac{1}{\lambda}\,\frac{d^2\lambda}{d\xi^2} - \frac{1}{2\lambda^2}\left(\frac{d\lambda}{d\xi}\right)^2\right].$$

The function

$$u_1 = \exp(\pm i\xi)$$

satisfies the equation

$$\frac{d^2u_1}{d\xi^2} + [1 + h(\xi)]u_1 = \rho(\xi) \tag{6.7.10}$$

where

$$\rho(\xi) = h(\xi)u_1$$

in the notation of (6.6.8) if μ_2 as there defined is replaced by μ. Also

$$\int_\xi^c |\rho(s)|ds = 2\int_x^b \frac{|\mu_2(t)|}{\mu(t)}\,dt$$

where the left-hand side is the integral of Problem 6.6.1.

Further Reading

N. Bleistein and R.A. Handelsman. *Asymptotic Expansions of Integrals*. Dover Publications, Inc., New York, 1986.

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