Acoustic Mammography in the Time Domain

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1 Introduction

The purpose of this note is to study a reconstruction algorithm for an acoustic breast scanner as described in [2]. Its essential features are that it is two-dimensional, and that it has a ring geometry. The essential features of our reconstruction algorithm are that it is in time domain and that it makes use of the Kaczmarz method. Algorithms of this type have been described for a variety of imaging problems in chapter 7 of [3]. They are versions of the widely used adjoint methods. In the context of breast screening frequency domain versions of this algorithm were described and tested in [1] and [5].

2 The mathematical model

We consider a plane domain Ω with boundary Γ. Ideally Ω is a circle, but its exact shape is not relevant as long as it is sufficiently regular. On Γ acoustic transducers (typically 256) are placed which can act as sources and as receivers. The object to be imaged is situated in Ω. It is defined by its speed of sound $c(x), x \in \Omega$. We assume c to be the ambient sound speed c_0 outside a compact subset of Ω. We put $c^2(x) = c_0^2/(1 + f(x))$. Hence $f = 0$ outside a compact subset of Ω. The pressure $u(x, t)$ at $x \in \Omega$ at time $t, 0 < t < T$ is assumed to satisfy

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2(x)\left(\Delta u(x,t) + q(t)p(x-s)\right), \tag{1}$$

$$x \in \Omega,\ 0 < t < T. \tag{2}$$

and $u = 0,\ t < 0$. Here, $s \in \Gamma$ is a transducer that acts as source, q is the source wavelet, and p is the intensity profile of the source. For a finite number of sources (typically 256), the function $g_s(r,t) = u(r,t)$ is measured at $r \in \Gamma$ by means of the transducers that act as receivers. From all these measurements the function c, i.e. the function f, has to be reconstructed. We consider g_s as a function of f, writing $g_s = R_s(f)$ with the nonlinear operator $R_s : L_2(\Omega) \longrightarrow L_2(\Gamma \times (0,T))$. Then the reconstruction problem amounts to solving the nonlinear system $R_s(f) = g_s$ for all sources s.

3 The Kaczmarz method

The Kaczmarz method is an iterative procedure for solving systems such as $R_s(f) = g_s$ for some set of sources s. The update is given by

$$f \leftarrow f - \alpha (R_s'(f))^* (R_s(f) - g_s). \tag{3}$$

This is done for all sources s. Once all sources have been used the process is started again. One pass through all the sources is called a sweep.

The derivative R_s' is easily computed in the following way; see chapter 7 of [3]: In order to evaluate $(R_s'(f))(h)$ for some $h \in L_2(\Omega)$ solve

$$c^{-2}\frac{\partial^2 w}{\partial t^2} = \Delta w - \frac{h}{c_0^2}\frac{\partial^2 u}{\partial t^2}, \tag{4}$$

$$x \in \Omega,\ 0 < t < T \tag{5}$$

with $w = 0,\ t < 0$ and with u the solution of (1,2). Then $R_s'(f) = w|_\Gamma$.

Computing the adjoint operator $R_s'(f) : L_2(\Gamma \times (0,T)) \longrightarrow L_2(\Omega)$ is more tricky. We follow the procedure in [6]. For functions w, z in $R^2 \times (0,T)$ that decay sufficiently fast as $|x|$ tends to infinity we have by integration by parts

$$\int_0^T \int_{R^2} (c^{-2}\frac{\partial^2 w}{\partial t^2} - \Delta w) z dx dt = \int_0^T \int_{R^2} w (c^{-2}\frac{\partial^2 z}{\partial t^2} - \Delta z) dx dt + \left[\int_{R^2} c^{-2}(\frac{\partial w}{\partial t} z - w\frac{\partial z}{\partial t}) dx\right]_0^T.$$

Let g_Γ be the distribution defined for $g \in L_2(\Gamma \times (0,T))$ by

$$\int_0^T \int_{R^2} g_\Gamma \phi dx dt = \int_0^T \int_\Gamma g \phi dx dt. \tag{6}$$

Choosing w as in (4,5) and z as the solution of the final value problem

$$\frac{\partial^2 z}{\partial t^2} = c^2(\Delta z + g_\Gamma) \tag{7}$$

with $z = 0,\ t > T$ we obtain

$$-\int_0^T \int_{R^2} \frac{h}{c_0^2} \frac{\partial^2 u}{\partial t^2} z dx dt = \int_0^T \int_{R^2} w g_\Gamma dx dt = \int_\Gamma (R_s'(f)h) g dx.$$

This means that for any $g \in L_2(\Gamma \times (0,T))$

$$(R_s'(f))^\star g = -\frac{1}{c_0^2} \int_0^T z \frac{\partial^2 u}{\partial t^2} dt.$$

Thus each step of the Kaczmarz method requires the solution of one initial value problem (1,2) and one final value problem (7), and this has to be done for each source. Thus the numerical effort is considerable but manageable.

4 Finding an initial approximation

The Kazmarz method starts out from an initial approximation f_0 that has to be chosen. It has been shown heuristically in [7] that a condition for f_0 that implies convergence is

$$|\int (f - f_0) ds| \leq \lambda \tag{8}$$

where the integration is along the geodesics of the background medium f_0 and λ is the largest wavelength in the source wavelet q.

With the typical wavelengths in medical imaging being in the order of mm, (8) is difficult to satisfy. A simple trick helps: One does a first reconstruction with a wavelet that has a wavelength big enough to satisfy (8) for $f_0 = 0$, say. It turns out that the result of this first reconstruction is a low pass filtered version of f. This low pass filtered version is used as the initial approximation f_0. Often this initial approximation satisfies (8) and hence leads to a convergent process.

Another possibility is to extract travel time information from the scattering data g. This assumes that the paths are essentially straight. Let $T_{r,s}$ be the travel time from source s to receiver r. $T_{r,s}$ can be easily found by searching for the maximum of $g_s(r,t)$ as a function of t. We have approximately

$$T_{r,s} = \int_s^r \frac{dx}{c} dx = \frac{1}{c_0} \int_s^r \sqrt{1+f} dx = \frac{1}{c_0} \int_s^r (1+f/2) dx$$

hence

$$\int_s^r f dx = 2c_0(T_{r,s} - |r-s|). \tag{9}$$

Thus we get an approximation for the fan beam transform of f. With the help of Radon's inversion formula we obtain an approximation for f which we in turn use as an initial approximation for the Kaczmarz method.

5 Finite difference methods

The initial and final value problems are solved numerically by finite difference methods which can be found in every text on numerical analysis. The only point that needs attention is the discretisation of the distribution g_Γ in (7). Viewing g_Γ as a function and discretizing the left hand side of (6) on a grid with step size h in space and δt in time, the right hand side by means of a quadrature rule with the receivers r_l as nodes and weights W_l we obtain

$$h^2 \delta t \sum_{j,k,n} g_{\Gamma,j,k,n} \Phi_{j,k,n} = \delta t \sum_{l,n} W_l g_{l,n} \Phi_{l,n}.$$

Hence $g_{\Gamma,j,k,n} = W_l g_{l,n}/h^2$ if j,k point to a receiver r_l and zero otherwise. For Γ a circle of radius ρ_0 and $2P$ receivers uniformly distributed on Γ we have $W_l = \rho_0 \pi / P$.

6 Numerical experiments

We created a breast phantom patterned after the one in [1]. We omitted the attenuation (see the next section) but made it more realistic by making it more inhomogeneous; see Fig. 1. The breast is imbedded in a square of side length 20 cm. The four tumors have diameters 2, 4, 6 and 10 mm. The function f assumes the values around 0.08 in the tumors and oscillates

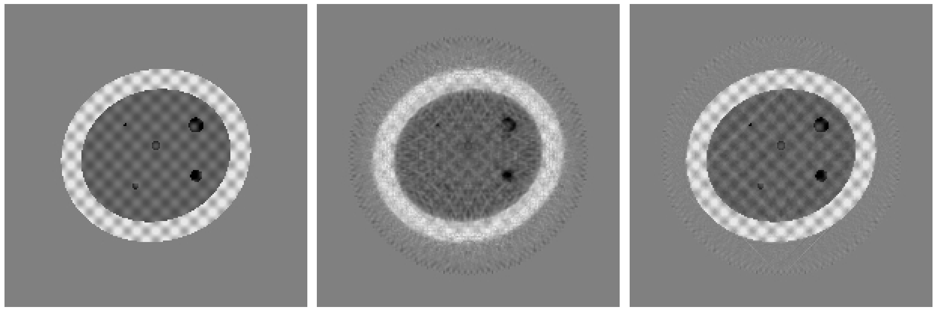

Fig.1: Left: Breast phantom. Reconstructions with 250 kHz and 500 kHz middle and right, respectively.

around the values -0.03 and 0.06 in the glandular tissue and in the fat, respectively. The transducers are sitting on a circle with diameter 16 cm. The wavelet q is

$$q(t) = e^{-t^2/(2\tau^2)} \cos \omega_c t, \ \tau = \pi/\omega_c.$$

According to the $1/e$ convention (bandwidth defined as decay of $|\hat{q}|$ to $1/e$ of the central value) the bandwidth of q is $2\sqrt{2}\omega_c/\pi$, which corresponds to a bandwidth of 90% of the central frequency. We worked with a central frequency of f=500 kHz, i. e. $\omega_c = 2\pi$f. Assuming the ambient speed of sound c_0 to be 1500 m/s this corresponds to a wavelength of 3 mm. Hence we expect a resolution of 1.5 mm. We chose a pixel size of 1 mm, i. e. we discretized f on a 200×200 grid.

For $f_0 = 0$ the condition (8) reads $|Rf| \leq \lambda$ with R the Radon transform. We have $\max |Rf| = 5.4$ mm and, for 500 kHz, λ =3 mm. Hence (8) is not satisfied, and we don't expect convergence. In fact the iteration gets stuck in an object far away from the true f. However for 250 kHz, the wavelength is 6 mm, and (8) is satisfied. The reconstruction with $f_0 = 0$ and 250 kHz converges and is displayed in Fig. 1. It is completely useless for diagnostic purposes. However it gives a rough estimate for the geometry and the velocity of the breast. We use it as initial approximation f_0 for the reconstruction with the correct frequency of 500 kHz. The result is also shown in Fig. 1. We see that the smallest tumor is clearly resolved. Since the theoretical resolution, as shown above, is 1.5 mm, this is what we expect.

In both computations it turned out to be advantages to let the parameter α in (3) vary with depth. With $r_0 = 0.08$ the radius of the circle of sources we put $\alpha = \alpha(x) = \alpha_0(\rho_0^2 - |x|^2)$ with $\alpha_0 = 2 \times 10^3$. We did 3 sweeps.

At a first glance it looks odd to do do the preliminary reconstruction with the false frequency. However, this is very similar to regularization: In regularization we replace an operator which we can't invert (because of lack of stability) by one that is stably invertible and apply this false operator to the measured data. Here we replace an operator which we can't invert (because of lack of an initial approximation) by one for which we can find an initial approximation and apply this false operator to the measured data.

Alternatively one can find the initial approximation by extracting travel times from the data as described in (9). We did not get useful results in this way.

The central wavelength in this example is 3 mm, while the step size in the spatial discretisation is 1 mm. It is clear that for larger frequencies, e. g. 1.5 MHz with a wavelength of 1 mm, a much smaller step size is needed.

7 Attenuation

Including attenuation is easy. The wave equation including attenuation reads

$$c^{-2}\frac{\partial^2 u}{\partial t} + a\frac{\partial u}{\partial t} = \Delta u + q(t)p(x-s). \tag{10}$$

The solution operator R now maps the pair (f,a) to the boundary values of u on Γ: $R(f,a) = u|_\Gamma$. Its derivative is given by $R'(f,a)(h,k) = w|_\Gamma$ where w is the solution of

$$c^{-2}\frac{\partial^2 w}{\partial t^2} + a\frac{\partial w}{\partial t} = \Delta w - \frac{h}{c_0^2}\frac{\partial^2 u}{\partial t^2} - k\frac{\partial u}{\partial t}.$$

The adjoint is now an operator from $L_2(\Gamma \times (0,T))$ into $L_2(\Omega) \times L_2(\Omega)$. For $g \in L_2(\Gamma \times (0,T))$ it is given by

$$R'(f,a)^* g = -\int_0^T \left(\frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}\right) z dt$$

where z is the solution of the final value problem

$$c^{-2}\frac{\partial^2 z}{\partial t^2} - a\frac{\partial z}{\partial t} = \Delta z + g_\Gamma.$$

The Kaczmarz method can now be done exactly as in (3) with simultaneous updates for f, a.

Doing a Fourier transform with respect to time we get from (10)

$$\Delta\hat{u} + \frac{\omega^2}{c_0^2}(1 + f - iac_0^2/\omega)\hat{u} = -\hat{q}p$$

with ω the frequency variable. This is the starting point for frequency domain methods such as the ones in [1], [5].

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