Initial value techniques for the Helmholtz and Maxwell equations

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Abstract: We study the initial value problem of the Helmholtz equation with spatially variable wave number. We show that it can be stabilized by suppressing the evanescent waves. The stabilized Helmholtz equation can be solved numerically by a marching scheme combined with FFT. The resulting algorithm has complexity $n^2 \log n$ on a $n \times n$ grid. We demonstrate the efficacy of the method by numerical examples with caustics. For the Maxwell equation the same treatment is possible after reducing it to a second order system. We show how the method can be used for inverse problems arising in acoustic tomography and microwave imaging.

1 Introduction

The initial value problem for elliptic equations, such as the Helmholtz and the Maxwell equations, are notoriously unstable. There exists a huge literature on stabilizing these initial value problems. Common features of these works are the use of a-priori information about the exact solution and conditional stability estimates; see [1]. For a recent paper that provides an overview and the spirit of these works see [2].

In this paper we follow a completely different route. We consider differential equations of the form

$$\Delta u + k^2(1 + f(x))u = 0$$

(1.1)
with a large parameter \( k \) and show that the Cauchy initial value problem for this equation is perfectly stable, provided we restrict ourselves to low frequencies, i. e. the part of the solution \( u \) that is obtained by low-pass filtering \( u \) with a cut-off frequency near \( k \). In other words, the instability is a pure high frequency phenomenon and disappears as soon as the high frequencies are removed. We do not need a-priori assumptions, and our estimates are linear. Physically our stabilization means the removal of the evanescent waves.

Estimates of this type were derived in [7] for the Helmholtz equation and in [6] for the Maxwell equations by energy estimates. These estimates contain factors of \( k \) which make the application to high frequency imaging questionable. In section 2 we derive new estimates with a much better behavior in terms of \( k \). These estimates are based on the thesis [10] and can be viewed as the analogue of the famous \( 1/k \) estimates for the inverse Helmholtz operator of [4] and [3]; they are also reminiscent of the recent work [9]. In section 3 we give numerical examples for initial value problems with a focal point. In section 4 we demonstrate the usefulness of the initial value approach to inverse problems.

Methods competing with our initial value technique are the various forms of one-way wave equations and the paraxial or parabolic approximation to the wave equation [5]. The advantage of our approach is its simplicity and its ability to handle backscatter. The disadvantage is that it is restricted to the solution of inverse problems, since for the direct boundary value problem Cauchy data are not available.

## 2 Stability estimates

For simplicity we restrict ourselves to the Helmholtz case with zero initial values, i. e. we consider the initial value problem

\[
\Delta u + k^2 (1 + f) u = r, x_2 > 0, u(x_1, 0) = 0, \frac{\partial u}{\partial x_2} (x_1, 0) = 0.
\]

**Theorem 1.** Let \( f \in C^1(\mathbb{R}^2) \) be real valued and supported in \([-\rho, \rho] \times [0, \infty]\), and let \( m \) be a constant such that \(-1 < m \leq f\). Then, for \( \kappa = \theta k \sqrt{1 + m}, 0 < \theta < 1 \), there exists a constant \( c \) such that

\[
\|u_{k\theta}(\cdot, x_2)\|_{L^2(-\rho, \rho)} \leq \frac{\sqrt{\text{det} C}}{k\theta} \|r\|_{L^2(-\rho, \rho) \times (0, \rho)}
\]

(2.2)
where
\[ \vartheta = \sqrt{1 + m\sqrt{1 - \theta^2}}. \] (2.3)

**Proof:** In a first step we assume \( f \) to be piecewise constant as a function of \( x_2 \), i.e.,

\[ f(x_1, x_2) = f_i(x_1), \quad ih \leq x_2 \leq (i + 1)h \]

with some \( h > 0 \). Fourier transforming (2.1) with respect to \( x_1 \) yields for \( ih \leq x_2 \leq (i + 1)h \)

\[ \frac{d^2}{dx_2^2} \hat{u}(\cdot, x_2) + A_i \hat{u}(\cdot, x_2) = \hat{r}(\cdot, x_2), \] (2.4)

the operator \( A_i \) in \( L_2(\mathbb{R}^1) \) being defined by

\[ (A_i v)(\xi) = (k^2 - \xi^2) v(\xi) + (2\pi)^{-1/2} k^2 (\hat{f}_i * v)(\xi) \]

with \( * \) the convolution in \( \mathbb{R}^1 \). Since \( f \) is real, \( A_i \) is selfadjoint. We have by Parseval’s relation

\[ (\hat{f}_i * v, v)_{L_2(\mathbb{R}^1)} = \int_{-\infty}^{+\infty} (\hat{f}_i * v) \bar{v} d\xi_1 = \int_{-\infty}^{+\infty} (\hat{f}_i * v) \bar{v} dx_1 \]

\[ = (2\pi)^{1/2} \int_{-\infty}^{+\infty} f_i |\bar{v}|^2 dx_1 \geq (2\pi)^{1/2} m(v, v)_{L_2(\mathbb{R}^1)}. \]

For the restriction of \( A_i \) to \( L_2(-\kappa, \kappa) \) - again denoted by \( A_i \) - we thus have

\[ (A_i v, v)_{L_2(-\kappa, \kappa)} \geq (k^2 - \kappa^2 + k^2 m)(v, v)_{L_2(-\kappa, \kappa)}. \]

Integrating (3.2) over \([ih, x_2]\) we obtain

\[ \hat{u}(\cdot, x_2) = \cos(K_i(x_2 - \cdot)) \hat{u}(\cdot, x_2) + K_i^{-1} \sin(K_i(x_2 - \cdot)) \frac{\partial \hat{u}}{\partial \xi_1}(\cdot, x_2) \]

\[ + \int_{ih}^{x_2} K_i^{-1} \sin(K_i(x_2 - x_2')) \hat{r}(\cdot, x_2') dx_2' \] (2.5)

where \( K_i = \sqrt{A_i} \). For this to make sense we assume that \( A_i \) is positive definite, this being the case for \( k^2 - \kappa^2 + \hbar^2 m > 0 \), i.e., for \( \kappa < k\sqrt{1 + m} \). For
\[ \kappa = \theta \sqrt{k^2 + m^2}, 0 < \theta < 1 \] the eigenvalues of \( K_i \) are \( \geq k\sqrt{1 + m - \theta^2} = k\theta \). For \( |\xi_1| < \kappa \) we put
\[ U_i(\xi_1) = \left( \begin{array}{c} \hat{u}(\xi_1, ih) \\ \frac{\partial \hat{u}}{\partial x_2}(\xi_1, ih) \end{array} \right) \]

From (2.5) we obtain
\[ U_{i+1} = L_i U_i + \int_{ih}^{(i+1)h} J_i(x'_2) \hat{r}(\cdot, x'_2) dx'_2 \] (2.6)

\[ L_i = \left( \begin{array}{cc} \cos(K_i h), & K_i^{-1} \sin(K_i h) \\ -K_i \sin(K_i h), & \cos(K_i h) \end{array} \right), \]
\[ J_i(x'_2) = \left( \begin{array}{c} K_i^{-1} \sin(K_i((i + 1)h - x'_2)) \\ \cos(K_i((i + 1)h - x'_2)) \end{array} \right). \]

Solving the recursion (2.6) yields
\[ U_i = L_{i-1} \cdots L_0 U_0 + \sum_{j=0}^{i-1} L_i \cdots L_{j+1} \int_{jh}^{(j+1)h} J_j(x'_2) \hat{r}(\xi_1, x_2) dx'_2. \]

Obviously \( L_i = D_i Q_i D_i^{-1} \) with
\[ Q_i = \left( \begin{array}{cc} \cos(K_i h) & \sin(K_i h) \\ -\sin(K_i h) & \cos(K_i h) \end{array} \right), \]
\[ D_i = \left( \begin{array}{cc} I & 0 \\ 0 & K_i \end{array} \right). \]

With \( \| \cdot \| \) the euclidean norm we have
\[ \|Q_i\| = 1, \|D_i^{-1} D_{i-1}\| \leq 1 + ch \]
where \( c > 0 \) is a constant. Hence
\[ L_i \cdots L_{j+1} J_j(x'_2) = \]
\[ D_i Q_i D_i^{-1} D_{i-1} Q_{i-1} D_{i-1}^{-1} D_{i-2} \cdots D_{j+2}^{-1} D_{j+1} Q_{j+1} \left( \begin{array}{c} K_j^{-1} \sin(K_j((j + 1)h - x'_2)) \\ K_j^{-1} \cos(K_j((j + 1)h - x'_2)) \end{array} \right) \]
\[ = D_i M_{ij} \]
where
\[ \|M_{ij}\| \leq \frac{(1 + ch)^{i-j+1}}{k\theta}. \]
Since $U_0 = 0$ we finally get

$$U_i = D_i \sum_{j=0}^{i-1} \int_{jh}^{(j+1)h} M_{ij}(x'_2) \hat{r}(jx'_2)dx'_2.$$ 

For the first component of $U_i$ this implies

$$|\hat{u}(\xi_1, ih)| \leq \frac{1}{k^2 \vartheta} \sum_{j=0}^{i-1} (1 + ch)^{i-j+1} \int_{jh}^{(j+1)h} |\hat{r}(\xi_1, x'_2)|dx'_2$$

$$\leq \frac{1}{k^2 \vartheta} (1 + ch)^i \int_0^{ih} |\hat{r}(\xi_1, x'_2)|dx'_2$$

$$\leq \frac{1}{k^2 \vartheta} e^{cih} \sqrt{ih} \left( \int_0^{ih} |\hat{r}(\xi_1, x'_2)|^2dx'_2 \right)^{1/2}.$$ 

For $ih = x_2 \leq \varrho$ it follows from $(1 + \frac{\varrho}{i})^i \leq e^\varrho$ that

$$|\hat{u}(\xi_1, x_2)|^2 \leq \frac{\varrho e^{2\varrho c}}{k^2 \vartheta^2} \int_0^{\varrho} |\hat{r}(\xi_1, x'_2)|dx'_2$$

$$||u_{k\vartheta}(\cdot, x_2)||_{L_2([-\varrho, \varrho])} \leq \frac{\varrho e^{2\varrho c}}{k^2 \vartheta^2} ||r||_{L_2([e, e] \times [0, e])}.$$ 

So far the piecewise constant case. In order to get the general case we only have to let $h$ tend to 0.

Theorem 1 is a simplified version of a result of [10]. It is especially useful for large values of $k$. In fact it has been used in [10] for high frequency estimates for the geometric optics approximation to the inverse Helmholtz problem.

### 3 Numerical example

It is easy to derive a numerical method from Theorem 1: We simply discretize (2.1) on a cartesian grid, march it in the $x_2$ direction, and apply a low pass filter in the $x_1$ direction with cut-off $k^2 \vartheta$ after each marching step. From Theorem 1 we expect this procedure to be stable, and numerical evidence
shows that this is in fact the case. The resulting algorithm needs \( n^2 \log n \) on a \( n \times n \) grid. This is only slightly more than for the algorithms that are based on the one way wave equation [5].

As an example we treat the scattering of a plane wave by the Luneberg lense

\[
f(x) = 1 - 4|x|^2
\]

in the unit ball and and \( f = 0 \) outside. It is well known that under plane wave insonification the Luneberg lens generates a focal point at its rim [11]. Due to this focal point we consider the Luneberg lens as a challenging test problem for a Helmholtz solver at high frequency. We are seeking a solution \( u \) of (1.1) that is of the form

\[
u = e^{ikx_2} + u_s
\]

where the scattered field \( u_s \) satisfies the Sommerfeld radiation condition.

For our numerical experiment we used the wave number \( k = 100 \) and the cut-off frequency \( \kappa = 90 \). The computations are done in a square of side length 1. Real part and imaginary part of the scattered field are displayed in Figure 1. The focal point is clearly visible. The field was computed by a time domain finite difference method, followed by a Fourier transform.

![Figure 1: Scattered field of Luneberg lens with plane wave illumination (plane wave coming in from top). Real part left, imaginary part right.](image)

In Figure 2 we display central vertical cross sections through the real part of the exact field and of the approximate scattered field computed by the initial value technique, the Cauchy initial values being stipulated at the top boundary of the reconstruction region. We see that the agreement of exact and approximate fields is almost perfect.
4 Application to inverse problems

Since we need Cauchy initial values, i. e. values for $u$ as well as for $\frac{\partial u}{\partial x_2}$ on the initial manifold, initial value techniques can’t be used for the direct boundary value problem. However, in inverse problems, $u$ is measured on the boundary of the reconstruction region, while $\frac{\partial u}{\partial x_2}$ is obtained by the Dirichlet-to-Neumann-map. Thus, Cauchy initial values are available, and initial value techniques can be used for the forward and for the adjoint problems. This applies also to the Maxwell equations

$$curlE - ikH = 0, curlH + ik(1 + f)E = 0$$

(4.1)

where we have written $n = 1 + f$ for the refractive index. We employ the usual procedure for eliminating $H$: With $\text{div}(nE) = 0$ we obtain for $E$ the second order equation

$$\Delta E - \nabla \text{div}E + k^2(1 + f)E = 0.$$  (4.2)

This equation can be treated exactly as the Helmholtz equation; see [6]. Numerical results for Helmholtz and Maxwell are presented in [7], [6].

References


