A discrete Gelfand - Levitan theory

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1 Introduction

In the pioneering paper [5], Gelfand and Levitan introduced an elegant and explicit method for computing the potential of a Sturm-Liouville Operator from its eigenvalues and certain values of its eigenfunctions. It has been noted in Burridge [3] that the Gelfand-Levitan method is closely related to inversion methods for recovering the potential in a hyperbolic equation with focused initial state, thus giving a unified theory for the methods of Gelfand-Levitan, of Gopinath and Sondhi [8] and of Parijskij [11] and Blagoveshchenskij [1]; (see also Romanov [12, p. 39]), the latter ones going unnoticed in [3].

In the present note we analyse discrete analogues of these problems. The spacial differential operator of the Gelfand-Levitan theory is replaced by a symmetric tridiagonal matrix. It will turn out that in this setting the Gelfand-Levitan method reduces essentially to a Cholesky decomposition. The hyperbolic equation in the other theories is replaced by a recursion relation involving an arbitrary tridiagonal matrix, the essential step for the inversion being an LU-decomposition. In our approach, (approximate) transmutation operators as originally used in [5] (see Levitan [10] for a systematic study) play a paramount role.
We are aware of the fact that much work has been done on discrete inverse problems. We mention in particular Burridge [3, p. 514-537], Case and Kack [4], Bruckstein and Kailath [2], Landau [9], Gladwell and Willms [7]. Our approach differs from others in that it gives a unified treatment in terms of transmutation operators.
2 Transmutation matrices

Let $T, T_0$ be tridiagonal $(n,n)$-matrices, i.e.

$$T = \begin{pmatrix}
\alpha_1 & \beta_1 \\
\gamma_1 & \alpha_2 & \beta_2 \\
& \ddots & \ddots \\
& & \beta_{n-1} \\
& & \gamma_{n-1} & \alpha_n
\end{pmatrix}, \quad T_0 = \begin{pmatrix}
\alpha^0_1 & \beta^0_1 \\
\gamma^0_1 & \alpha^0_2 & \beta^0_2 \\
& \ddots & \ddots \\
& & \beta^0_{n-1} \\
& & \gamma^0_{n-1} & \alpha^0_n
\end{pmatrix}.$$

By a transmutation matrix for $T, T_0$ one usually means a $(n,n)$-matrix $L$ with $T_0 L = LT$. We shall use (approximate) transmutation matrices for which this holds only in rows 1 through $n - 1$. More precisely, with $E$ the projection

$$E = \begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1 \\
& & & 0
\end{pmatrix}$$

we define a transmutation $L$ by

$$E T_0 L = E L T. \tag{2.1}$$

**Theorem 2.1** Let $\beta^0_i \neq 0$, $i = 1, \ldots, n - 1$. Then there is a unique lower triangular transmutation $L = (\ell_{ij})$ whose $(1,1)$-entry is 1.

**Proof:** With $\ell_1, \ldots, \ell_n$ the rows of $L$, (2.1) reads

$$\alpha^0_1 \ell_1 + \beta^0_1 \ell_2 = \ell_1 T,$$
$$\gamma^0_{i-1} \ell_{i-1} + \alpha^0_i \ell_i + \beta^0_i \ell_{i+1} = \ell_i T, \quad i = 2, \ldots, n - 1.$$
Since $\beta^0_i \neq 0$ this recursion determines the $\ell_i$ uniquely once $\ell_1$ is given. For $\ell_1 = (1, 0, \ldots, 0)$, $\ell_i$ is nonzero only in its first $i$ components since $T$ is tridiagonal.

We remark that Theorem 2.1 holds also for lower Hessenberg matrices $T_0, T$. 

3 The inverse eigenvalue problem

Now assume $T$ to be symmetric with non-zero off-diagonal elements, i.e. $\gamma_i = \beta_i > 0, \ i = 1, \ldots, n-1$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues and $x(\lambda_1), \ldots, x(\lambda_n)$ the normalized eigenvectors of $T$, i.e.

$$T \ x(\lambda_k) = \lambda_k x(\lambda_k), \quad k = 1, \ldots, n.$$  

We consider the inverse problem: Determine $T$ from $\lambda_1, \ldots, \lambda_n$ and $x_1(\lambda_1), \ldots, x_1(\lambda_n)$. It is well known that this problem can easily be solved by the Lanczos method.

The Lanczos algorithm computes for a symmetric matrix $A$ a symmetric tridiagonal matrix $T$ and a unitary matrix $U$ such that

$$TU = UA$$

in the following way. With $u_1, \ldots, u_n$ the rows of $U$, (3.1) reads

$$\begin{align*}
\alpha_1 u_1 + \beta_1 u_2 &= u_1 A \\
\beta_1 u_1 + \alpha_2 u_2 + \beta_2 u_3 &= u_2 A \\
&\quad \vdots \\
\beta_{n-1} u_{n-1} + \alpha_n u_n &= u_n A
\end{align*}$$

Now let $u_1$ be arbitrary. Then, from equation 1,

$$\begin{align*}
\alpha_1 &= u_1 A u_1^* , \\
\tilde{u}_2 &= u_1 A - \alpha_1 u_1 , \quad \beta_1 = \|\tilde{u}_2\| \quad \text{(euclidean norm)} \\
u_2 &= \tilde{u}_2 / \beta_1.
\end{align*}$$

Thus $\alpha_1, \beta_1, u_1, u_2$ are determined. Likewise, from equation 2,

$$\begin{align*}
\alpha_2 &= u_2 A u_2^* \\
\tilde{u}_3 &= u_2 A - \beta_1 u_1 - \alpha_2 u_2 , \quad \beta_2 = \|\tilde{u}_3\| \\
u_3 &= \tilde{u}_3 / \beta_2.
\end{align*}$$

yielding $\beta_2, u_3$. Proceeding in this fashion in equations up to and including $n-1$ we obtain $\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}$ and $u_1 \ldots u_n$. $\alpha_n$ is obtained from equation $n$. 

In order to solve our inverse problem we simply apply the Lanczos method to the matrix $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$. The same problem can be solved by the Gelfand-Levitan method in the following way.

Let $T_0$ be an arbitrary known symmetric tridiagonal matrix, subject only to the condition $\beta_0^i \neq 0$, $i = 1, \ldots, n - 1$. We introduce solutions $\varphi(\lambda)$, $\varphi^0(\lambda) \in \mathbb{R}^n$ of

\[
\begin{align*}
\varphi_1(\lambda) &= 1 , \\
\alpha_1 \varphi_1(\lambda) + \beta_1 \varphi_2(\lambda) &= \lambda \varphi_1(\lambda) , \\
\beta_{i-1} \varphi_{i-1}(\lambda) + \alpha_i \varphi_i(\lambda) + \beta_i \varphi_{i+1}(\lambda) &= \lambda \varphi_i(\lambda) , & i = 2, \ldots, n - 1
\end{align*}
\]

and correspondingly for $\varphi^0$ using $T_0$ instead of $T$. In other words, $\varphi(\lambda)$, $\varphi^0(\lambda)$ satisfy

\[
\begin{align*}
ET \varphi(\lambda) &= \lambda E \varphi(\lambda) , & \varphi_1(\lambda) &= 1 \quad \text{(3.2)} \\
ET_0 \varphi^0(\lambda) &= \lambda E \varphi^0(\lambda) , & \varphi^0_1(\lambda) &= 1 . \quad \text{(3.3)}
\end{align*}
\]

Note that for $\lambda = \lambda_k$, $\varphi(\lambda_k)$ is an eigenvector of $T$, hence

\[
T \varphi(\lambda_k) = \lambda_k \varphi(\lambda_k) , \quad k = 1, \ldots, n ,
\]

i.e. the projection operator $E$ can be dropped in that case.

**Theorem 3.1** Let $\beta_0^i \neq 0$, $i = 1, \ldots, n - 1$ and let $L$ be the transmutation of Theorem 2.1 Then,

\[
\varphi^0(\lambda_k) = L \varphi(\lambda_k) , \quad k = 1, \ldots, n .
\]

**Proof:** Let $\psi^0 = L \varphi(\lambda_k)$. Then, because of (2.1), (3.4),

\[
ET_0 \psi^0 = ET_0 L \varphi(\lambda_k) = ELT \varphi(\lambda_k) = \lambda_k EL \varphi(\lambda_k) = \lambda_k E \psi^0 .
\]

Since $\ell_{11} = 1$, $\psi_1^0 = 1$. Thus $\psi^0$ and $\varphi^0(\lambda_k)$ both satisfy (3.3) with $\lambda = \lambda_k$. Because of $\beta_0^i \neq 0$, (3.3) is uniquely solvable. Hence $\psi^0 = \varphi^0(\lambda_k)$.
Now we come to the core of the Gelfand-Levitan method. We introduce the matrices

$$
\Phi = (\varphi(\lambda_1), \ldots, \varphi(\lambda_n)), \quad \Phi_0 = (\varphi^0(\lambda_1), \ldots, \varphi^0(\lambda_n)),
$$

$$
P = \begin{pmatrix}
p_1 \\
p_2 \\
\vdots \\
p_m
\end{pmatrix}, \quad p_k = \|\varphi(\lambda_k)\| \quad \text{(euclidean norm)}.
$$

Assume that $x_1(\lambda_k) > 0$, $k = 1, \ldots, n$. Then, $\varphi(\lambda_k) = \frac{1}{x_1(\lambda_k)} x(\lambda_k)$, hence $p_k = \frac{1}{x_1(\lambda_k)}$ and

$$
\Phi = XP, \quad X = (x(\lambda_1), \ldots, x(\lambda_n)).
$$

Since $XX^t = I$,

$$
\Phi P^{-2} \Phi^t = I. \quad (3.5)
$$

Theorem 3.1 means that

$$
\Phi_0 = L \Phi. \quad (3.6)
$$

Inserting this into (3.5) yields

$$
L^{-1} \Phi_0 P^{-2} \Phi_0^t L^{-t} = I,
$$

$$
LL^t = \Phi_0 P^{-2} \Phi_0^t.
$$

Since $\Phi_0, P$ are known from the data, $L$ can be computed by a Cholesky decomposition of $\Phi_0 P^{-2} \Phi_0^t$. Once $L$ is determined, $T$ can be computed from

$$
T = \Phi \Lambda \Phi^{-1} = L^{-1} \Phi_0 \Lambda \Phi_0^{-1} L, \quad \Lambda = \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_n
\end{pmatrix}.
$$

This solves the inverse eigenvalue problem.
4 The inverse evolution problem

Now let $T = T_0 + Q$ with $T_0$ known and $Q$ a diagonal matrix whose entries we denote by $q_1, \ldots, q_n$. Let $u^\ell$ be a solution to

$$u^{\ell+1} = Tu^\ell, \quad \ell = 0, 1, \ldots, u^0 = e_1$$

(4.1)

with $e_1$ the first unit vector. We consider the inverse problem of recovering $Q$ from

$$u_1^\ell = g^\ell, \quad \ell = 0, \ldots, 2n - 1.$$ 

Again we make use of the transmutation $L$ from Theorem 2.1. Multiplying (4.1) with $EL$ we obtain from (2.1)

$$ELu^{\ell+1} = ELu^\ell = ET_0Lu^\ell.$$ 

Thus the vectors $z^\ell = Lu^\ell$ satisfy

$$Ez^{\ell+1} = ET_0z^\ell, \quad \ell = 0, 1, \ldots,$$

(4.2)

i.e. $z^{\ell+1} = T_0z^\ell$ in components $1, \ldots, n - 1$. Since $T_0$ is tridiagonal we can show that for $k = 0, \ldots, n - 1$

$$z_{i}^{\ell+k} = (T_0^k z^\ell)_i, \quad i = 1, \ldots, n - k.$$ 

(4.3)

We use induction with respect to $k$. The case $k = 0$ is obvious. Assume that (4.3) is correct for some $k$ with $0 \leq k < n - 1$. From (4.2) we get

$$z_{i}^{\ell+k+1} = (T_0 z^{\ell+k})_i, \quad i = 1, \ldots, n - 1.$$ 

For the evaluation of $(T_0 z^{\ell+k})_i, \ i = 1, \ldots, n - k - 1$, we need only the first $n - k$ components of $z^{\ell+k}$ since $T_0$ is tridiagonal. Thus, by the induction hypothesis,

$$z_{i}^{\ell+k+1} = (T_0 T_0^k z^\ell)_i = (T_0^{k+1} z^\ell)_i, \quad i = 1, \ldots, n - k - 1.$$ 

This is (4.3) for $k + 1$. Hence (4.3) is established.

Since the first row of $L$ is $(1, 0, \ldots, 0)$ we also have

$$z_1^\ell = g^\ell, \quad \ell = 0, 1, \ldots.$$
This combines with (4.3) to yield, for \( \ell = 0, 1, \ldots, n - 1, \)
\[
g^{\ell+k} = (T_0^k z^\ell)_1, \quad k = 0, 1, \ldots, n - 1 .
\]

Introducing the row vector
\[
(u_0^k)^t = e_1^t T_0^k
\]
we have
\[
g^{\ell+k} = (u_0^k)^t z^\ell, \quad k = 0, 1, \ldots, n - 1 \tag{4.4}
\]
and
\[
u_0^{k+1} = T_0^t u_0^k, \quad k = 0, 1, \ldots, \quad u_0^0 = e_1 . \tag{4.5}
\]

With the \((n,n)\)-matrices
\[
Z = (z^0, \ldots, z^{n-1}), \quad G = \begin{pmatrix} g^0 & g^1 & \cdots & g^{n-1} \\ \vdots & \vdots \\ g^{n-1} & g^n & \cdots & g^{2n-2} \end{pmatrix}, \quad U_0 = (u_0^0, \ldots, u_0^{n-1})
\]
(4.4) simply reads
\[
G = U_0^t Z .
\]
Note that \( U_0 \) is upper triangular with diagonal elements
\[
\prod_{i=1}^k \beta_i^0, \quad k = 0, \ldots, n - 1 .
\]
Thus \( U_0 \) is invertible provided that \( \beta_i^0 \neq 0, \quad i = 1, \ldots, n - 1 \). Hence
\[
Z = U_0^{-t} G
\]
which determines \( Z \) by the data.

The relations \( z^\ell = Lu^\ell \) can be written as
\[
Z = LU , \quad U = (u^0, \ldots, u^{n-1}) .
\]

We finally obtain
\[
LU = U_0^{-t} G .
\]
Since \( T \) is tridiagonal and \( u^0 = e_1 \), \( U \) is upper triangular, its diagonal entries being
\[
\prod_{i=1}^k \gamma_i = \prod_{i=1}^k \gamma_i^0, \quad k = 0, \ldots, n - 1 .
\]
Thus $L, U$ can be determined simply by doing an $LU$-decomposition on the matrix $U_0^{-1}G$, with the diagonal of $U$ being known.

Once $L, U$ are known there is a variety of ways to find $Q$. For instance we can compute $u^n$ from $Lu^n = z^n$ and

$$
\begin{pmatrix}
g^n \\
\vdots \\
g^{2n-1}
\end{pmatrix} = U_0^t z^n,
$$

which is (4.4) for $\ell = n$. Then,

$$
u^\ell+1 = T_0 u^\ell + \begin{pmatrix}
q_1 u_1^\ell \\
\vdots \\
q_{\ell+1} u_{\ell+1}^\ell \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad \ell = 0, \ldots, n - 1
$$

from which $q_1, \ldots, q_n$ can be computed recursively since $u_{\ell+1}^\ell \neq 0$, $\ell = 0, \ldots, n - 1$. This solves the inverse evolution problem.
5 A second order inverse evolution problem

Let $T$ be as in the preceding section, and let $u^\ell$ be a solution to

$$u^{\ell+1} - 2u^\ell + u^{\ell-1} = Tu^\ell, \quad \ell = 0, 1, 2, \ldots$$

$$u^{-1} = 0, \quad u^0 = e_1.$$  \hfill (5.1)

We consider the inverse problem of recovering the diagonal matrix $Q$ in $T = T_0 + Q$ from

$$u^\ell_1 = g^\ell, \quad \ell = 0, 1, \ldots, 2n.$$  

Again we put $z^\ell = Lu^\ell$ with $L$ from Theorem 2.1. We obtain

$$Ez^{\ell+1} - 2Ez^\ell + Ez^{\ell-1} = ET_0z^\ell, \quad \ell = 0, 1, 2, \ldots$$

$$z^{-1} = 0, \quad z^0 = Le_1.$$  

With $S = 2I + T_0$ the recursion can be written as

$$z^{\ell+1}_i = (Sz^\ell)_i - z^{\ell-1}_i, \quad i = 1, \ldots, n - 1.$$  \hfill (5.2)

We claim that for $k = 0, \ldots, n - 1$ and $\ell = 0, 1, 2, \ldots$

$$z^{\ell+k}_i = (S_kz^\ell)_i - (S_{k-1}z^{\ell-1})_i, \quad i = 1, \ldots, n - k$$  \hfill (5.3)

where

$$S_{-1} = 0, \quad S_0 = I, \quad S_{k+1} = SS_k - S_{k-1}, \quad k = 0, \ldots, n - 1.$$  

We prove (5.3) by induction with respect to $k$. The case $k = 0$ is obvious. Assume (5.3) to be correct up to some $k$ with $0 \leq k < n - 1$. From (5.2) we get

$$z^{\ell+k+1}_i = (Sz^{\ell+k})_i - z^{\ell+k-1}_i, \quad i = 1, \ldots, n - k - 1.$$  

Since $S$ is tridiagonal, only components 1 through $n - k$ of $z^{\ell+k}$ enter $(Sz^{\ell+k})_i$ for $i < n - k$. Hence, by the induction hypothesis,

$$z^{\ell+k+1}_i = (S(S_kz^\ell - S_{k-1}z^{\ell-1}))_i - (S_{k-1}z^\ell - S_{k-2}z^{\ell-1})_i$$  

$$= (S_{k+1}z^{\ell})_i - (S_kz^{\ell-1})_i, \quad i = 1, \ldots, n - k - 1.$$  

This is (5.3) with $k$ replaced by $k + 1$. Thus (5.3) is established.
We use (5.3) for \( i = 1 \) only, yielding for \( \ell = 1, 2, \ldots \)

\[
g^{\ell+k} = (S_k z^\ell)_1 - (S_{k-1} z^{\ell-1})_1, \quad k = 0, \ldots, n - 1.
\]

Introducing the row vectors

\[
(u_0^k)^t = e_1^t S_k, \quad k = -1, \ldots, n - 1
\]

we have

\[
g^{\ell+k} = (u_0^k)^t z^\ell - (u_0^{k-1})^t z^{\ell-1}, \quad k = 0, \ldots, n - 1
\]

and, since \( S, S_k \) commute,

\[
(u_0^{k+1})^t = e_1^t S_{k+1} = e_1^t (S S_k - S_{k-1}) = e_1^t (S_k S - S_{k-1}) = (u_0^k)^t S - (u_0^{k-1})^t.
\]

Thus \( u_0^k \) is the analogue to \( u^k \) with \( T \) replaced by \( T_0^t \):

\[
\begin{align*}
 u_0^{k+1} - 2u_0^k + u_0^{k-1} &= T_0^t u_k, \quad k = 0, 1, 2, \ldots, \\
 u_0^{-1} &= 0, \quad u_0^0 = e_1.
\end{align*}
\]

This shows that the matrix \( U_0 = (u_0^0, \ldots, u_0^{n-1}) \) is upper triangular with diagonal elements

\[
\prod_{i=0}^{k} \beta_i^0, \quad k = 0, \ldots, n - 1.
\]

It follows that the systems

\[
\begin{pmatrix}
g^\ell \\
\vdots \\
g^{\ell+n-1}
\end{pmatrix} = U_0 z^\ell - U_{-1} z^{\ell-1}, \quad \ell = 1, \ldots, n
\]

where \( U_{-1} = (u_0^{-1}, \ldots, u_0^{n-2}) \) are uniquely solvable for \( z^\ell \) provided that \( \beta_i^0 \neq 0, \quad i = 1, \ldots, n - 1. \)

Given that \( z^{-1} = 0 \), we can determine the matrix \( Z = (z^0, \ldots, z^{n-1}) \) from our data. The matrix \( U = (u^0, \ldots, u^{n-1}) \) is upper triangular with diagonal elements

\[
\prod_{i=1}^{k-1} \gamma_i = \prod_{i=1}^{k-1} \gamma_i^0, \quad k = 1, \ldots, n.
\]
Thus computing $L, U$ amounts to doing an $LU$-Decomposition on the matrix $Z$,

$$Z = LU,$$

with the diagonal of $U$ being known. Once $L, U$ are known, $Q$ is computed very much in the same fashion as in the preceding section.
6 Links to other work

It has been noticed in Landau [9] that the inverse eigenvalue problem is intimately related to orthogonal polynomials. Our treatment in section 2 can also be interpreted that way.

To begin with it is clear that $\varphi_k(\lambda), \varphi_0^k(\lambda)$ are polynomials of degree $k - 1$ in $\lambda$. Thus (3.5) simply states that the polynomials $\varphi_k, \ k = 1, \ldots, n$, are the orthonormal polynomials with respect to the scalar product

$$ (p, q) = \sum_{k=1}^{n} p_k^{-2} p(\lambda_k) q(\lambda_k) $$

(6.1)

in the space of polynomials of degree $n - 1$. Writing $\varphi_0^k$ in terms of the orthogonal polynomials we get

$$ \varphi_0^\ell = \sum_{k=1}^{\ell} \ell_{\ell k} \varphi_k $$

or

$$ \Phi_0 = L\Phi $$

with a lower triangular matrix $L$. This is (3.6). Thus solving the inverse eigenvalue problem, i.e. finding $L$, boils down to computing the orthogonal polynomials with respect to the scalar product (6.1).

The approach of Gladwell and Willms [7] is essentially equivalent to our treatment of the inverse eigenvalue problems, except that we made the role of transmutation more explicit. For instance, equation (10) of [7] is precisely our relation (3.6) in the form $\Phi = L^{-1}\Phi_0$. 
References


