An initial value approach to the inverse Helmholtz problem at fixed frequency

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Abstract: *We show that the Cauchy initial value problem for the inhomogeneous Helmholtz equation is stable for spatial frequencies below a natural limit. We use initial value methods in an iterative method for the inverse problem which is very similar to ART in computerized tomography. We give numerical results for a computer generated breast phantom.*

1 Introduction

The inverse Helmholtz problem calls for the determination of the function f in $\Omega \subseteq \mathbb{R}^n$ from the boundary values $g(\theta, x) = u(x)$, $x \in \partial\Omega$ of the solution u of the Helmholtz equation

$$\begin{aligned} \Delta u(x) + k^2(1 + f(x))u(x) &= 0 \\ u(x) &= e^{ikx\cdot\theta} + w(x) \end{aligned} \tag{1.1}$$

in $\mathbb{R}^n$, where w satisfies the Sommerfeld radiation condition. The number $k > 0$ is the frequency of the incident plane wave with direction $\theta \in S^{n-1}$. We assume $f = 0$ outside Ω. g is measured for a single fixed frequency k and all $\theta \in S^{n-1}$.

The mathematical theory of this inverse problem is fairly well developed [10]. There exists an extensive literature on numerical methods. If the Born or Rytov approximation is valid, the algorithms of diffraction tomography [3] can be used. For the general case, iterative ([1], [2], [4], [8]) and direct [13] methods have been suggested.

The purpose of the present paper is to give a rigourous mathematical justification of the method already described in [12] and to demonstrate its efficacy by computer simulations.

An essential feature in [12] is the use of initial value methods for the Helmholtz equation. Initial value methods, namely the parabolic equation approximation, are also used in [1]. In contrast to [1] we work with initial value problems of second order. The stability analysis is given in §2. In §3 we describe our method for the inverse problem in a continuous setting. The necessary discretizations have been described in [12]. In §4 we report the results of the reconstruction of a breast phantom from simulated data.

2 The initial value problem for the Helmholtz equation

We consider the $2D$ Helmholtz equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + k^2(1+f)u = r \tag{2.1}$$

in the upper half plane $x_2 > 0$, subject to the initial conditions

$$u = g \quad , \quad \frac{\partial u}{\partial x_2} = h \tag{2.2}$$

on $x_2 = 0$. This problem is notoriously unstable. However, this instability is a purely high frequency phenomenon. As soon as we restrict everything to sufficiently low frequencies, this instability disappears. More precisely, let u_κ be a low pass filtered version of u with cut off frequency κ, i.e.

$$\hat{u}_\kappa(\xi_1, x_2) = \begin{cases} \hat{u}(\xi_1, x_2) \ , & |\xi_1| < \kappa \ , \\ 0 & , \text{otherwise} \end{cases}$$

where $\hat{u}$ is the Fourier transform

$$\hat{u}(\xi_1, x_2) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{-ix_1 \cdot \xi_1} u(x_1, x_2) dx_1$$

of u with respect to x_1, then u_κ admits a perfectly reasonable estimate in terms of the initial values g, h provided κ is chosen properly. For $f = 0$, $r = 0$ and $\kappa < k$ this follows easily from the explicit solution

$$\hat{u}(\xi_1, x_2) = \hat{g}(\xi_1) \cos(x_2\sqrt{k^2 - \xi_1^2}) + \frac{\hat{h}(\xi_1)}{\sqrt{k^2 - \xi_1^2}} \sin(x_2\sqrt{k^2 - \xi_1^2}) \ .$$

In the following we extend this stability result to the general case.

We need the following estimate:

Proposition 1. *Let $v \in C_2([0, \infty), H)$ be a solution to*

$$v'' + A(t)v = r \ , \qquad t > 0 \tag{2.3}$$

where $A(t)$ is a linear bounded operator in the Hilbert space H with the following properties: We have $A = A_1 + A_2$ with $A_1^* = A_1$ and

$$\alpha_1^2(v, v) \leq (v, A_1 v) \leq \beta_1^2(v, v) \ , \tag{2.4}$$

$$(v, A_1' v) \leq \gamma_1(v, v) \ , \tag{2.5}$$

$$||A_2|| \leq \beta_2 \tag{2.6}$$

with $\alpha_1 > 0$. Let $\phi = (v', v') + (v, A_1 v)$. Then,

$$\phi(t) \leq \left(\phi(0) + \int_0^t ||r(\tau)||^2 d\tau \right) e^{(1+2\beta_2/\alpha_1+\gamma_1/\alpha_1^2)t} \ . \tag{2.7}$$

Proof: We use the energy method see e.g. [5]. Multiplying (2.3) with v' we obtain

$$(v', v'') + (v', Av) = (v', r) \ . \tag{2.8}$$

Using

$$\frac{1}{2} \frac{d}{dt}(v', v') = Re(v', v'') \ ,$$
$$\frac{1}{2} \frac{d}{dt}(v, A_1 v) = Re(v', A_1 v) + \frac{1}{2}(v, A_1' v)$$

we can rewrite the real part of (2.8) as

$$\frac{1}{2} \frac{d}{dt}(v', v') + \frac{1}{2} \frac{d}{dt}(v, A_1 v)$$
$$= Re(v', r) + \frac{1}{2}(v, A_1' v) - Re(v', A_2 v) \ .$$

Integrating we obtain

$$\phi(t) = \phi(0) + 2 \int_0^t \left\{ Re(v', r) + \frac{1}{2}(v, A_1' v) - Re(v', A_2 v) \right\} dt \ .$$

Cauchy-Schwarz and (2.5), (2.6) yield

$$\phi(t) \le \phi(0) + \int_0^t \{ \ 2\|v'\|\|r\| + \gamma_1 \|v\|^2 + 2\beta_2 \|v'\|\|v\| \} \ dt \ .$$

By an appropriate use of the inequality

$$ab \le \frac{1}{2}(\delta^2 a^2 + \delta^{-2} b^2) \ , \quad \delta > 0$$

we obtain

$$\phi(t) \le \phi(0) + \int_0^t \left\{ \|v'\|^2 + \|r\|^2 + \gamma_1 \|v\|^2 + \frac{\beta_2}{\alpha_1} \|v'\|^2 + \beta_2 \alpha_1 \|v\|^2 \right\} dt$$
$$= \phi(0) + \int_0^t \|r\|^2 dt + \int_0^t \left\{ \left(1 + \frac{\beta_2}{\alpha_1} \right) \|v'\|^2 + (\gamma_1 + \beta_2 \alpha_1) \|v\|^2 \right\} dt \ .$$

Using

$$\phi \ge \|v'\|^2 + \alpha_1^2 \|v\|^2$$

we obtain

$$\phi(t) \le c_1(t) + c_2 \int_0^t \phi(t) dt \ ,$$

$$c_1(t) = \phi(0) + \int_0^t \|r\|^2 dt \ , \quad c_2 = 1 + 2\frac{\beta_2}{\alpha_1} + \frac{\gamma_1}{\alpha_1^2} \ .$$

Now (2.7) follows from Gronwall's inequality (see e.g. [6], p. 24).

□

Now we can prove our stability estimate for u_κ.

Theorem 2. *Let $f \in C^1(\mathbb{R}^2)$ and $f = f_1 + \frac{i}{k} f_2$ with f_1, f_2 real, and let m_ν, M_ν be constants such that*

$$-1 < m_1 \le f_1 \le M_1\ , \quad \left|\frac{\partial f_1}{\partial x_2}\right| \le M_1\ , \quad |f_2| \le M_2\ .$$

Then, for $\kappa < k\sqrt{1+m_1}$, we have

$$\|u_\kappa'(\cdot,x_2)\|^2 \le e^{\alpha x_2}\left(\|h\|^2 + k^2(1+M_1)\|g\|^2 + 2\|r\|^2 + 2k^4\|f(u-u_\kappa)(\cdot,x_2)\|^2\right) \tag{2.9}$$

where

$$\alpha = 1 + \frac{2M_2}{\vartheta} + \frac{M_1}{\vartheta^2}\ , \quad \vartheta = \sqrt{1 + m_1 - \left(\frac{\kappa}{k}\right)^2}\ .$$

Proof: Taking in (2.1) the Fourier transform with respect to x_1 we obtain

$$(k^2 - \xi_1^2)\hat{u}(\xi_1,x_2) + \frac{\partial^2}{\partial x_2^2}\hat{u}(\xi_1,x_2) + (2\pi)^{-1/2}k^2 \int_{-\infty}^{+\infty} \hat{f}(\xi_1-\eta_1,x_2)\hat{u}(\eta_1,x_2)d\eta_1$$
$$= \hat{r}(\xi_1,x_2)\ .$$

We consider this equation only for $|\xi_1| \le \kappa$ and decompose the integral accordingly, i.e.

$$(k^2 - \xi_1^2)\hat{u}_\kappa(\xi_1,x_2) + \frac{\partial^2}{\partial x_2^2}\hat{u}_\kappa(\xi_1,x_2) + (2\pi)^{-1/2}k^2 \int_{-\infty}^{+\infty} \hat{f}(\xi_1-\eta_1,x_2)\hat{u}_\kappa(\eta_1,x_2)d\eta_1$$
$$= \hat{r}(\xi_1,x_2) + \hat{\varepsilon}(\xi_1,x_2)\ , \qquad \hat{\varepsilon}(\xi_1,x_2) = -(2\pi)^{-1/2}k^2 \int_{|\eta_1|\ge\kappa} \hat{f}(\xi_1-\eta_1,x_2)\hat{u}(\eta_1,x_2)d\eta_1\ .$$

Obviously,

$$\varepsilon = -k^2 f(u-u_\kappa)\ .$$

For fixed ξ_1 we rewrite the last equation for $\hat{u}_\kappa$ as

$$\hat{u}_\kappa''(x_2) + A(x_2)\hat{u}_\kappa(x_2) = \hat{r}(x_2) + \hat{\varepsilon}(x_2) \tag{2.10}$$

In order to apply Proposition 1 to (2.9) we only have to find the constants in (2.4 - 2.6) for the operator A from (2.10). We have

$$\begin{aligned}(A\hat{u},\hat{v}) &= ((k^2-\xi_1^2)\hat{u},\hat{v}) + (2\pi)^{-1/2}k^2 \int_{-\infty}^{+\infty} (\hat{f} \star \hat{u})\overline{\hat{v}}d\xi_1 \\ &= ((k^2-\xi_1^2)\hat{u},\hat{v}) + k^2 \int_{-\infty}^{+\infty} \widehat{fu}\overline{\tilde{\hat{v}}}d\xi_1 \\ &= ((k^2-\xi_1^2)\hat{u},\hat{v}) + k^2 \int_{-\infty}^{+\infty} fu\overline{v}dx_1\end{aligned}$$

where we have used Parseval's relation. Writing f as $f_1 + \frac{i}{k} f_2$ it readily follows that

$$(A_1\hat{u}, \hat{u}) = ((k^2 - \xi_1^2)\hat{u}, \hat{u}) + k^2 \int\limits_{-\infty}^{+\infty} f_1 |u|^2 dx_1 \ ,$$

$$(A_2\hat{u})^{\sim} = ikf_2 u \ .$$

From these relations we easily read

$$\alpha_1^2 = k^2 - \kappa^2 + k^2 m_1 \ , \quad \beta_1^2 = k^2(1 + M_1) \ , \quad \gamma_1 = k^2 M_1 \ , \beta_2 = kM_2 \ .$$

Inserting this into (2.7) and using

$$\|\hat{u}_\kappa'(\cdot, x_2)\|^2 \le \phi(x_2) \le \|\hat{u}_\kappa'(\cdot, x_2)\|^2 + k^2(1 + M_1)\|\hat{u}_\kappa(\cdot, x_2)\|^2$$

yields (2.9).

□

The estimate of Theorem 2 makes sense only if $u - u_\kappa$ is small in some sense. In the next theorem we show that this is in fact the case for the scattered wave w from (1.1) with $\theta = \binom{0}{1}$. For w we have the Lippmann-Schwinger equation

$$w(x) = \frac{-i}{4} k^2 \int\limits_{\mathbf{R}^2} H_0(k|x - x'|) f(x')(e^{ikx_2'} + w(x'))dx'$$

with the Hankel function of the first kind H_0. Taking the Fourier transform with respect to x_1 we obtain

$$\begin{aligned} &\hat{w}(\xi_1, x_2) \\ = -&\frac{i}{4\sqrt{2\pi}} k^2 \int\!\!\int\!\!\int H_0(k\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2}) e^{-i\xi_1 x_1} dx_1 \\ &(e^{ikx_2'} + w(x_1', x_2')) f(x_1', x_2') dx_1' dx_2' \ . \end{aligned}$$

The x_1 integral can be evaluated as follows. With $a(\xi) = \sqrt{k^2 - \xi^2}$ we have

$$H_0(k\sqrt{u^2 + v^2}) = \frac{-i}{4\pi} \int e^{i(|u|a(\xi) + v\xi)} \frac{d\xi}{a(\xi)} \ ,$$

see e.g. [9], p. 123. Hence

$$\begin{aligned} \int H_0\left(k\sqrt{(x_1 - x_1')^2 - (x_2 + x_2')^2}\right) e^{-i\xi_1 x_1} dx_1 &= \frac{-i}{4\pi} \int\int e^{i(|x_2 - x_2'|a(\xi) + (x_1 - x_1')\xi) - i\xi_1 x_1} \frac{dx_1 d\xi}{a(\xi)} \\ &= \frac{-i}{2} \int e^{i(|x_2 - x_2'|a(\xi) - x_1'\xi)} \delta(\xi - \xi_1) \frac{d\xi}{a(\xi)} \\ &= -\frac{i}{2} e^{i(|x_2 - x_2'|a(\xi_1) - x_1'\xi_1)} \frac{1}{a(\xi_1)} \ . \end{aligned}$$

It follows that

$$\hat{w}(\xi_1, x_2)$$
$$= \frac{k^2}{8a(\xi_1)} \int\int e^{i(|x_2 - x_2'|a(\xi_1) - x_1'\xi_1)} \left(e^{ikx_2'} + w(x_1', x_2') \right) f(x_1', x_2') dx_1' dx_2' \ .$$

Now assume that the straight line $x_2' = x_2$ misses $\mathrm{supp}(f)$ by the quantity $\varepsilon > 0$ and $\xi_1 > k$. Then,

$$|\hat{w}(\xi_1, x_2)| \leq \frac{k^2}{8\sqrt{\xi_1^2 - k^2}} \, e^{-\varepsilon\sqrt{\xi_1^2 - k^2}} (1 + \|w\|_\infty) \|f\|_1 \ .$$

Hence we have for arbitrary θ:

Theorem 3. *Let w be the scattered wave for incident direction θ, and let L be a straight line perpendicular to θ which misses the support of f. Then, the Fourier transform of w along L decays exponentially beyond k.*

We shall apply Theorem 2 with m_1 negative but close to zero (typically $m_1 = -0.01$). Then, Theorem 2 requires κ to be slightly smaller than k, while, according to Theorem 3, $u - u_\kappa$ in Theorem 2 is small if κ is slightly bigger than k. In practice this conflict did not cause any problems. With $\kappa = 0.99k$ we observed both stability and accuracy.

3 The inversion method

We consider the case $n = 2$. Assume that Ω is the ball of radius ρ centered at the origin. Let the scattered field w_j be measured on $\partial\Omega$ for p directions θ_j. We first extend w_j to the exterior of Ω. Since w_j satisfies there the homogeneous Helmholtz equation $\Delta w_j + k^2 w_j = 0$, this is easily done by expanding $w_j(x)$ for $|x| > \rho$ as

$$w_j(r\cos\varphi, r\sin\varphi) = \sum_\ell c_\ell e^{i\ell\varphi} H_\ell(kr)$$

with H_ℓ the first kind Hankel function of order ℓ. The coefficients c_ℓ are readily obtained from the data by

$$c_\ell = \frac{1}{2\pi H_\ell(k\rho)} \int_0^{2\pi} w_j(\rho\cos\varphi, \rho\sin\varphi) d\varphi \ .$$

Now let Q_j be the square circumscribed to Ω, two of its edges, named Γ_j, parallel to θ_j. The other edges are denoted by Γ_j^-, Γ_j^+ with θ_j pointing from Γ_j^- to Γ_j^+. We define an operator $R_j : L_2(\Omega) \to L_2(\Gamma_j^+)$ by solving the initial value problem

$$\Delta w + k^2(1 + f)w = -k^2 f e^{ikx\cdot\theta} \quad \text{in } Q_j \ ,$$
$$w = w_j \ , \ \ \frac{\partial w}{\partial \nu} = \frac{\partial w_j}{\partial \nu} \quad \text{on } \Gamma_j^- \ , \ \ w = w_j \ \text{ on } \Gamma_j$$

and putting $R_j(f) = w$ on Γ_j^+. We also put $g_j = w_j|_{\Gamma_j^+}$.

Now we have to solve the nonlinear system

$$R_j(f) = g_j \ , \quad j = 1, \ldots, p$$

for f. We do this by an ART type algorithm.

With f^0 an initial approximation to f we compute a new approximation f^1 by

$$\begin{aligned} f_0 &= f^0 \quad , \\ f_j &= f_{j-1} + \gamma R_j'^*(f_{j-1})(g_j - R_j(f_{j-1})) \ , \ \ j = 1, \ldots, p \\ f^1 &= f_p \ . \end{aligned}$$

γ is a relaxation parameter.

$R_j'^*$ can be evaluated by solving an initial value problem with initial values on Γ_j^+. This can done by the usual five point finite difference star, combined with filtering along the grid lines perpendicular to θ_j. For details see [12].

4 Numerical experiment

We did reconstructions from data obtained from a computer generated breast phantom which has been patterned after a phantom created by Borup et al. [1]. With c the sound speed in the breast tissue, $c_0 = 1500\frac{m}{sec}$ the sound speed in the surrounding water, and α the attenuation coefficient in the breast tissue we have

$$f = \frac{c_0^2}{c^2} - 1 - i\frac{2\alpha c_0}{kc} \quad , \quad \omega = c_0 k \ . \tag{4.1}$$

The phantom is made up of four different kinds of tissue: fat, glandular tissue, tumor, and cyst. The values of c and α (at 1MHZ) are

tissue	$c[\frac{m}{sec}]$	$\alpha[\frac{db}{m}]$	Ref	Imf
fat	1458	41	0.058	$-9.4/k$
glandular tissue	1519	80	-0.025	$-18.4/k$
tumor	1564	118	-0.080	$-27.2/k$
cyst	1568	10	-0.084	$-2.3/k$

The value of α in column 2 have to be converted to those in formula (4.1) by dividing with the factor $20\log_{10}(e)$, yielding α in units $[\frac{1}{m}]$. Therefore, k in column 4 is in units of $[\frac{1}{m}]$, too.

We generated data by solving the forward problem with the initial value method, assuming the backscatter to be zero. We worked on a 128×128 grid, using $p = 128$ equally spaced directions in $[0, 2\pi)$. The frequency of the irradiating plane waves was 1MHZ, i.e. $\omega = 2\pi \cdot 10^6 \ \mathrm{sec}^{-1}$, hence $k = 4189 m^{-1}$.

The condition for the Born approximation to hold is

$$R|f| \ll \frac{2\pi}{k}$$

where R is the Radon transform. This is a slight generalisation of the condition given by Kak and Slaney [7]. It is obviously far from being satisfied. As has been discussed in [12], we can't use 0 as initial approximation in this case. We used the reconstruction for the smaller value $k = 1000\ m^{-1}$ instead which can be computed starting out from 0. A smoothed version of the initial approximation thus obtained is shown in Fig. 2.

The reconstruction after 6 sweeps is shown in Fig. 3. The total computation time on a SPARC 20 was 5 minutes. The value of γ has been chosen in the following way. First we determined in [11] the asymptotic value $1/(\rho k^2)$ of the operator $(R_j'(0)R_j'^*(0))^{-1}$ for large k. Then we put $\gamma = \gamma'/(\rho k^2)$ where, in agreement with the convergence properties of the Kaczmarz procedure, $0 < \gamma' < 2$. We used $\gamma' = 0.5$.

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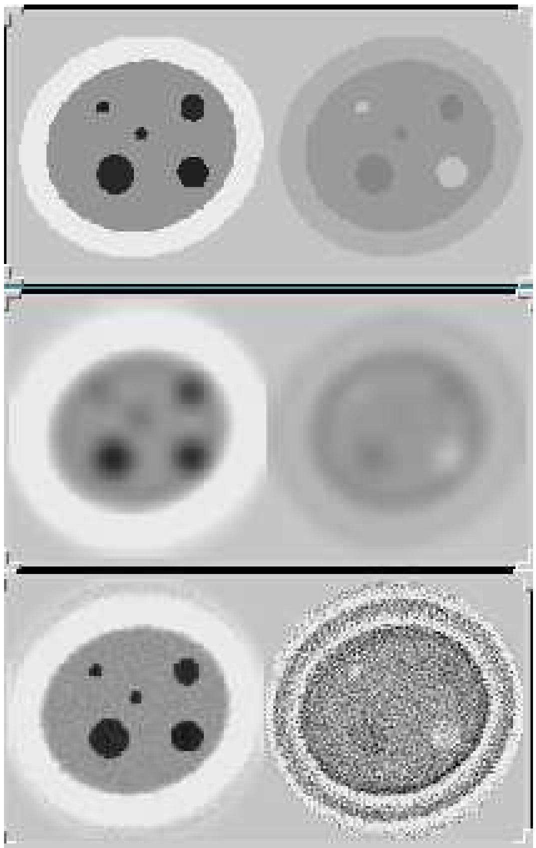

Figure captions

Fig. 1 (top) : Breast phantom. Real part left, imaginary part right. Since the imaginary part is much smaller than the real part it has been scaled. The location of the 3 tumors and 2 cysts is clear from the real part, while they can be distinguished by looking at the imaginary part.

Fig. 2 (middle) : Initial approximation which has been computed from data corresponding to $k = 1000\ m^{-1}$ using 0 as initial approximation.

Fig. 3 (bottom) : Reconstruction after 6 sweeps with $\gamma' = 0.5$.