An algorithm for 3D ultrasound tomography

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1 Introduction

We consider the inverse problem for the 3D Helmholtz equation

$$\begin{aligned} \Delta u^j + k^2(1+f)u^j &= 0 \\ u^j &= u_i^j + v^j \ , \end{aligned} \tag{1.1}$$

where u_i^j, $j = 1, \ldots, p$ are the incoming waves, v^j satisfies the Sommerfeld radiation condition and the function f vanishes outside the ball of radius ρ. We want to recover f numerically from knowing $u^j = g^j$ on the sphere of radius ρ for $j = 1, \ldots, p$ and a fixed frequency k.

This is a model for ultrasonic tomography [6]. However we point out that in a real ultrasonic scanner the irradiating waves are no longer plane waves but standing waves in a finite container.

We start with a short survey on the extensive literature on numerical methods. With the exception of methods which use the Born or Rytov approximation [4], it seems that the only method which actually has been tested numerically in 3D is the Newton method combined with a finite Fourier expansion of f [5]. The other methods have been used in 2D only, even though an extension to 3D is possible in principle. The dual space method [2] reduces the problem from the whole space $\mathbb{R}^3$ with the far field (which we do not use) as data to an overposed boundary value problem in a finite volume which in turn is solved by optimization.

Of course we can always try to compute the Born series [3]. For this purpose we write (1.1) as an integral equation

$$u^j(x) = u_i^j(x) - k^2 \int G(x,y)f(y)u^j(y)dy \ . \tag{1.2}$$

Here, G is the Green's function of $\Delta + k^2$ with the radiation condition at ∞. The Born series (f_ℓ) is now obtained by solving

$$g^j(x) = u_i^j(x) - k^2 \int G(x,y)f_\ell(y)u_\ell^j(y)dy \ , \quad |x| = \rho \ , \quad j = 1, \ldots, p$$

for f_ℓ, where $u_0^j = u_i^j$ and

$$\Delta u_{\ell+1}^j + k^2(1+f_\ell)u_{\ell+1}^j = 0$$

with $u_\ell^j = u_i^j + a$ function satisfying the radiation condition.

The generalized SOR-method of [7] also starts out from the integral equation (1.2). Writing for the integral operator in (1.2) simply G, this method minimizes the functional

$$\sum_{j=1}^{p} \left\{ \|g^j + u_i^j + k^2 G f u^j\|^2_{L_2(|x|=\rho)} + \gamma \|u^j - u_i^j + k^2 G f u^j\|^2_{L_2(|x|<\rho)} \right\} \tag{1.3}$$

with some weight factor γ. The minimization is done by

$$u_{\ell+1}^j = u_\ell^j + \alpha_\ell r_\ell^j \quad , \quad f_{\ell+1} = f_\ell + \beta_\ell d_\ell$$

where r_ℓ^j, d_ℓ are update directions and α_ℓ, β_ℓ are chosen so as to minimize (1.3).

A non iterative method has been suggested in [15]. With G_f the Green's function of $\Delta + k^2(1+f)$, we can rewrite (1.1) as

$$v^j(x) = -k^2 \int G_f(x,y) f(x) u_i^j(y) dy \ . \tag{1.4}$$

Now form a linear combination of the incoming waves such that the resulting field peaks at z, i.e.

$$\sum_{j=1}^{p} \alpha_j(z) u_i^j(x) \sim \delta(x-z) \ .$$

Then, from (1.4),

$$-k^2 G_f(x,z) f(z) \sim \sum_{j=1}^{p} \alpha_j(z) v^j(x) \tag{1.5}$$

is approximately known on $|x| = \rho$. From the identity

$$G_f(x,y) - G(x,y) = -k^2 \int\limits_{|z|<\rho} G_f(x,z) f(z) G(z,y) dy$$

we get an approximation to $G_f(x,y)$ for $|x| = \rho$ which together with (1.5) determines f. The method still has to be tested.

A very efficient code has been given in [6]. It is very similar to ours in that it uses initial value techniques and ignores in the Jacobian entries which correspond to different incoming waves. The initial value technique is based on factoring the Helmholtz equation into a product of two first order differential operators.

The purpose of this note is to extend the method of [14] to 3D and to conduct 3D numerical experiments. We illuminate the object by plane waves $u_i^j = e^{ikx\cdot\theta_j}$ where the unit vectors θ_j are lying in a plane. We use an initial value technique as in [6], but we do not rely on parabolic approximations [8] of the Helmholtz equation.

2 The initial value method

Let Ω_j be the cube circumscribed to the reconstruction region $|x| \le \rho$ with edges parallel to the coordinate axes and aligned with the direction $\theta_j = (\cos\varphi_j, \sin\varphi_j, 0)$ of the j-th incoming wave. Let $\Gamma_j^{\pm}$ be the face lying in the plane $x \cdot \theta_j = \pm\rho$, and let Γ_j be the other faces of Ω_j. Rather than working with the scattered fields v^j we use the scaled scattered fields $w^j = e^{-ikx\cdot\theta_j} v^j$ which satisfy

$$\Delta w^j + 2ik\theta_j \cdot \nabla w^j + k^2(1 + w^j) f = 0 \quad \text{in} \quad \Omega_j \ . \tag{2.1}$$

Note that we do not make the parabolic approximation [8], i.e. we do not assume that the second derivative of w^j in direction θ_j is small compared to $k\theta_j \cdot \nabla w^j$.

In [12] we haved analysed the stability of the initial value problem

$$w^j = g^j \quad \text{on} \quad \Gamma_j \cup \Gamma_j^- \ , \qquad \frac{\partial w^j}{\partial \nu} = \frac{\partial g^j}{\partial \nu} \quad \text{on} \quad \Gamma_j^- \tag{2.2}$$

for this elliptic differential equation. Let

$$\hat{w}^j(\xi, s) = \frac{1}{2\pi} \int\limits_{x\cdot\theta_j = s} e^{-ix\cdot\xi} w^j(x) dx \quad , \quad \xi \perp \theta_j$$

be the 2D Fourier transform of w in the plane $x \cdot \theta_j = s$. We found that $\hat{w}^j(\xi, s)$ depends in a perfectly stable way on the initial values for w^j on Γ_j^- for all frequencies $|\xi| < \kappa$ where κ is some number depending essentially on k and, to a minor extent, on f. In ultrasound tomography we can choose κ slightly smaller than k, typically $\kappa = 0.99k$.

Thus we may define a nonlinear map $R_j : L_2(|x| < \rho) \to L_2(\Gamma_j^+)$ by putting

$$R_j(f) = w_\kappa^j|_{\Gamma_j^+} \ . \tag{2.3}$$

The inverse scattering problem now calls for the solution of the nonlinear system

$$R_j(f) = g_j \quad , \quad g_j = g^j|_{\Gamma_j^+} \ , \qquad j = 1, \ldots, p \ . \tag{2.4}$$

As in [12] this is done by an ART-type procedure. Starting out from an initial approximation f^0, we put $f_0 = f^0$ and for $j = 1, \ldots, p$

$$f_j = f_{j-1} - \omega R_j'(f_{j-1})^* C_j^{-1}(R_j(f_{j-1}) - g_j) \ .$$

The first approximation f^1 is then defined to be f_p. For C_j we simply take the operator γI where γ is chosen such that, in the limit $k \to \infty$, $C_j \sim R_j'(0) R_j'(0)^*$, i.e. $\gamma = k^2/\rho$ [10]. The evaluation of $R_j'(f)^* g$ for some $g \in L_2(\Gamma_j^+)$ can be done as follows: Solve the initial value problem

$$\begin{aligned} \Delta z + 2ik\theta_j \cdot \nabla z - k^2 \overline{f} z &= 0 \quad \text{in} \quad \Omega_j \\ z = 0 \quad \text{on} \quad \Gamma_j \cup \Gamma_j^+ \ , \quad \frac{\partial z}{\partial \nu} &= g \quad \text{on} \quad \Gamma_j^+ \ . \end{aligned} \tag{2.5}$$

Then,

$$R_j^*(f)g = k^2(1+\overline{w}^j)z$$

where w^j is the solution of (2.1) - (2.2). Of course the stability properties of (2.5) are exactly as discussed above.

3 The finite difference method

The numerical solution of the initial value problems (2.1) - (2.2) and (2.5) can efficiently and conveniently be done by a finite difference method. In view of our stability result (see previous section) this suggestions itsself, but it has been used already in [9]. We simply use the usual five point discretization on the grid $\Omega_j^h = \{h\ell\theta_j + hm\theta_j^\perp + hne_3 : \ell, m, n = -q, \ldots, q\}$ where $\theta_j^\perp = (-\sin\varphi_j, \cos\varphi_j, 0)$ and $e_3 = (0,0,1)$ and $h = \rho/q$. Then, the finite difference approximation to (2.1) - (2.2) reads (the superscript j is omitted)

$$\begin{aligned}
&w_{\ell+1,m.n} + w_{\ell-1,m,n} + w_{\ell,m+1,n} + w_{\ell,m-1,n} + w_{\ell,m,n+1} + w_{\ell,m,n-1} \\
&-6w_{\ell,m,n} + i\varepsilon(w_{\ell+1,m,n} - w_{\ell-1,m,n}) + \varepsilon^2(1+w_{\ell,m,n})f_{\ell,m,n} = 0\ , \\
&\qquad \ell, m, n = -q+1, \ldots, q-1\ , \qquad (3.1) \\
&w_{\ell,m,n} = g_{\ell,m,n} \quad \text{for} \quad |m| = q \ \text{ or } \ |n| = q \ \text{ and for } \ \ell = -q, -q+1\ .
\end{aligned}$$

The boundary conditions in (3.1) assume that the field can be measured everywhere on each of the faces of Ω_j. The values of $g_{\ell,m,n} = f^j(h\ell\theta_j^1 + hn\theta_j^2)$ for $\ell = q-1$ have to be computed from the field in $|x| \geq \rho$ by numerical differentiation. (3.1) is solved in a recursive way, i.e. if w in known on the levels ℓ, $\ell-1$ we compute it for $\ell+1$ by (3.1). In order to preserve stability we have to filter out the frequencies greater than κ at each stage of this process. This can be done by doing a 2D FFT on each matrix $(w_{\ell,m,n})_{m,n=-q,\ldots,q}$ as soon as it is computed, zeroing all the entries with $\sqrt{m^2+n^2} > 2\rho\kappa/\pi$, followed by a 2D inverse FFT. The total operator count for solving (2.1) - (2.2) once is $\mathrm{O}(q^3 \log q)$. For details see [16]. (2.5) is solved exactly in the same way.

4 Numerical experiments

In a first test we checked the initial value method for accuracy. We solved the forward problem (1.1) for the function

$$f(x) = \begin{cases} 0.2 & , \quad |x| \leq 0.5 \\ 0 & , \quad \text{otherwise} \end{cases} \qquad (4.1)$$

and $k = 50$ analytically and compared the exact solution with the approximate solution of the initial value method for $h = 32$. We found satisfactory agreement.

In a second test we used the exact data for (4.1) and $k = 50$ as input to our reconstruction method with $p = 25$, $q = 32$. As initial approximation we chose (4.1) with 0.2 replaced by 0.1. After 3 sweeps of our algorithm we obtained a reconstruction very similar to a low pass filtered version of f with cut off 50.

Finally we created a 3D breastphantom, see Fig. 1. It consists of fat, glandular tissue, a tumor and a cyst. The breast is suspended in a cube of sidelength 12 cm which is filled with water. f is given by

$$f = \frac{c_0^2}{c^2} - 1 - i\frac{2\alpha c_0}{kc} \tag{4.2}$$

with $c_0 = 1500$ m sec^{-1} the speed of sound in water. The values of c and α (at 1MHz) are

tissue	$c[\frac{m}{sec}]$	$\alpha[\frac{db}{m}]$	$Re f$	$Im f$
fat	1458	41	0.058	$-9.4/k$
glandular tissue	1519	80	-0.025	$-18.4/k$
tumor	1564	118	-0.080	$-27.2/k$
cyst	1568	10	-0.084	$-2.3/k$

Since the top face of the cube is not accessible, we have to modify the finite difference method (3.1). We stipulate the boundary condition $\partial w/\partial\nu = 0$ on the top face of Ω_j, i.e. we let n run from $-q$ to q and put $w_{\ell,m,q+1} = w_{\ell,m,q-1}$. The boundary value problem (2.5) for z has to be changed accordingly. Of course this procedure is questionable, but at the present state of our work we just don't know anything better.

We generated data for $p = 32$ equally spaced directions in $[0, 2\pi]$ using our initial value method with $q = 32$, i.e. $h = 6\text{cm}/32 = 1.875\text{mm}$. The frequency of the iradiating waves was chosen to be 250 KHz, i.e. $k = 10.47\text{cm}^{-1}$. This corresponds to a wavelength of 6mm. After three sweeps of our algorithm we obtained a reconstruction in which the cyst and the tumor where clearly visible and distinguishable. The computing time per sweep on a SPARC 20 was 10 minutes.

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Fig.: Breast phantom at 200 KHz. Real part right, imaginary part left. Note that cyst and tumor can be distinguished clearly in the imaginary part.