Numerical Solution of Bilinear Inverse Problems

Frank Natterer

November 1995

1 Introduction

This note is concerned with a quite down-to-earth approach to the numerical solution of inverse problems. These problems often have a bilinear structure. We assume that our inverse problem is governed by a family of bilinear equations

$$A_j u_j + B_j(f, u_j) = a_j + b_j f \quad , \quad j = 1, \ldots, p \tag{1.1}$$

where $A_j : U_j \to Y_j$ is linear, $B_j : F \times U_j \to Y_j$ bilinear, $a_j \in Y_j$, $b_j : F \to Y_j$ is linear. Here, U_j is the Hilbert space of states, F the Hilbert space of parameters, and Y_j another Hilbert space. The subscript j refers to a projection direction or source position in transmission tomography or to a electrode pair in impedance tomography. Considering only one of the equations (1.1) at a time is an essential feature of our method. We assume that (1.1) is uniquely solvable for u_j if f is given. We consider the problem of determining f from the additional equations

$$C_j u_j + D_j(f, u_j) = c_j + d_j f \quad , \quad j = 1, \ldots, p \, . \tag{1.2}$$

C_j, D_j, c_j and d_j are exactly as A_j, B_j, a_j, b_j in (1.1) with Y_j replaced by a third Hilbert space Z_j, the space of observations. Let $u_j(f)$ be the solution of (1.1) for a certain $f \in F$. Then our problem amounts to solving the equations

$$R_j(f) = 0 \, , \qquad j = 1, \ldots, p \, , \tag{1.3}$$

$$R_j(f) = C_j u_j(f) + D_j(f, u_j(f)) - c_j - d_j f$$

for f.

A standard method for solving the nonlinear problem (1.3) is the Newton or Gauss-Newton method. These methods necessitate the computation of the operators $(R_j'(f)^* R_k'(f))^{-1}_{j,k=1,\dots,p}$, where $R_j'(f)$ is the Fréchet derivative of $R_j(f)$. For problems of practical relevance this is totally out of question. An obvious way to circumvent this difficulty would be to compute a minimal norm solution to just one of the linearized equations

$$R_j(f) + R_j'(f)h_j = 0 \ ,$$

i.e.

$$h_j = -R_j'(f)^* \mathcal{C}_j^{-1} R_j(f) \quad , \qquad \mathcal{C}_j = R_j'(f) R_j'(f)^*$$

and to update f according to $f \leftarrow f + \omega h_j$ with a relaxation factor ω. However, the computational burden for $\mathcal{C}_j$ is still by far too high. Thus we go one step further, replacing $\mathcal{C}_j$ by an easy to evaluate operator, e.g. by the identity operator. Then, one step of our algorithm, transforming the approximation f^0 into a new approximation f^1, reads:

$$\begin{array}{ll} \text{(i)} & f_0 = f^0 \\ \text{(ii)} & \text{For } \ j = 1, 2, \dots, p \\ & \qquad f_j = f_{j-1} - \omega h_j \ , \quad h_j = R_j'(f_{j-1})^* \mathcal{C}_j^{-1} R_j(f_{j-1}) \end{array} \tag{1.4}$$

$$\text{(iii)} \quad f^1 = f_p \ . \tag{1.5}$$

If R_j is a linear operator and $\mathcal{C}_j$ is simply the diagonal of $R_j R_j^*$, this iteration is the well-known algebraic reconstruction technique of computerized tomography, or the (block) Kaczmarz method, for solving underdetermined linear systems. We shall prove that (1.4) converges subject to natural assumptions for $0 < \omega < 2$ provided that $\mathcal{C}_j$ is positive definite and $\mathcal{C}_j \geq R_j'(f) R_j'(f)^*$. In the general nonlinear case, we do not have any convergence results. But it is easy to see what the method actually does. Suppose the problem (1.3) has a solution f - not necessarily unique. Then, in F,

$$\|f_j - f\|^2 = \|f_{j-1} - f - \omega h_j\|^2 = \|f_{j-1} - f\|^2 - 2\omega(f_{j-1} - f, h_j) + \omega^2 \|h_j\|^2 \ .$$

The factor of ω is

$$\begin{aligned} (f_j - f, h_j) &= (\mathcal{C}_j^{-1} R_j'(f_{j-1})(f_{j-1} - f) \ , \ R_j(f_{j-1})) \\ &= (\mathcal{C}_j^{-1} R_j(f_{j-1}) \ , \ R_j(f_{j-1})) + \cdots \end{aligned}$$

where the dots stand for higher order terms in $f_{j-1} - f$. Thus, if $\mathcal{C}_j$ is positiv definit and ω sufficiently small, we can expect f_j to be closer to f than f_{j-1}, provided $R_j(f_{j-1}) \neq 0$. The decrease of the error is best possible if

$$\omega = \frac{(\mathcal{C}_j^{-1} R_j(f_{j-1}), R_j(f_{j-1}))}{\| R_j'(f_{j-1})^* \mathcal{C}_j^{-1} R_j(f_{j-1}) \|^2}$$

However, the emphasis of this paper is not so much on theoretical questions such as convergence. Rather we describe the method to some detail for a variety of inverse problems for partial differential equations. We also present numerical examples which demonstrate that the method is quite easy to implement and, compared to other methods, very efficient. Its success depends of course on the degree of ill-posedness of the inverse problem which varies very much within the class of problems treated here. Also, the method is quite plausible, making use of vivid phenomena such as backpropagation. The method is closely related to the method of adjoint fields which is used in the engineering literature [1].

We do not claim any originality of our approach. Similar methods have been used in thousends of papers. A typical example is [2]. An implementation for ultrasound tomography is given in [4].

2 Linerarization

We linearize (1.1)-(1.2) about a solution $u_j = u_j(f)$ of (1.1). Replacing u_j by $u_j + w_j$, f by $f + h$, (1.1) reads

$$A_j(u_j + w) + B_j(f + h, u_j + w) = a_j + b_j(f + h) \ .$$

Neglecting higher order terms and using that u_j, f solve (1.1) we obtain

$$A_j w_j + B_j(f, w_j) = b_j h - B_j(h, u_j) \ . \tag{2.1}$$

Similarly, neglecting higher order terms in (1.2) yields

$$R_j(f + h) = R_j(f) + C_j w_j + D_j(f, w_j) + D_j(h, u_j) - d_j h \ .$$

Thus the operator $R_j'(f) : F \to Z_j$ is given by

$$R_j'(f)h = C_j w_j + D_j(f, w_j) + D_j(h, u_j) - d_j h \tag{2.2}$$

where w_j is the solution of (2.1).

In the application to concrete problems, the bulk of the analytical work consists in determining the adjoint operator $(R_j'(f))^* : Z_j \to F$. While an explicit formula for this operator in the functional analytic framework is easy to derive, this formula is not very useful in practice. Thus we derive $(R_j'(f))^*$ for specific cases in the next section.

3 Special inverse problems

Now we derive the operator $(R_j'(f))^*$ for special inverse problems of partial differential equations. The derivation is in no way rigouros. We neglect subtleties such as the exact domains of definition. The essential tools are Gauss' integral theorem and Green's formula.

1. Impedance tomography.

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be sufficiently regular, and let σ be a function in Ω, the conductivity of the material in Ω. We want to determine σ from measurements of currents f_j and voltages g_j, $j = 1, \ldots, p$, on $\partial\Omega$. The currents are usually applied at pairs of electrodes on $\partial\Omega$.

The mathematical model is as follows. The potential u_j satisfies

$$\operatorname{div}(\sigma \nabla u_j) = 0 \quad \text{in} \quad \Omega \, , \quad \sigma \frac{\partial u_j}{\partial \nu} = f_j \quad \text{on} \quad \partial\Omega \, . \tag{3.1}$$

We normalize u_j by requiring that the mean value on $\partial\Omega$ is zero. The measurements provide us with the values of

$$u_j = g_j \quad \text{on} \quad \partial\Omega \, . \tag{3.2}$$

We have to determine σ from (3.1)-(3.2) for $j = 1, \ldots, p$. We assume σ to be known on $\partial\Omega$.

It is clear that (3.1) can be written in the form of (1.1) with $U_j = L_2(\Omega)$, $Y_j = L_2(\Omega) \times L_2(\Omega)$, $F = L_2(\Omega)$, and σ playing the role of f. Also, (3.2) is a special case of (1.2) with $Z_j = L_{2,\sigma}(\partial\Omega)$ which is simply $L_2(\partial\Omega)$ with the (known) scalar product $\int_{\partial\Omega} \sigma u v ds$, and we have

$$R_j(\sigma) = u_j|_{\partial\Omega} - g_j$$

where u_j is the solution of (3.1). Hence, (2.2) reads

$$R_j'(\sigma)\eta = w_j|_{\partial\Omega} \, ,$$

$$\operatorname{div}(\sigma \Delta w_j) = -\operatorname{div}(\eta \nabla u_j) \ \text{ in } \ \Omega \, , \quad \sigma \frac{\partial w_j}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega \, . \tag{3.3}$$

Theorem 3.1 *The operator* $(R_j'(\sigma))^* : L_{2,\sigma}(\partial\Omega) \to L_2(\Omega)$ *is given by*

$$(R_j'(\sigma))^* g = \nabla u_j \cdot \nabla z \tag{3.4}$$

where z *is the solution of*

$$div(\sigma \nabla z) = 0 \;\; in \;\; \Omega \; , \quad \sigma \frac{\partial z}{\partial \nu} = -g \;\; on \;\; \partial\Omega \; . \tag{3.5}$$

Proof: For any z, w we have

$$\int_\Omega (\mathrm{div}(\sigma\nabla w) z - w \; \mathrm{div}(\sigma \nabla z)) dx = \int_{\partial\Omega} \sigma \left(\frac{\partial w}{\partial \nu} z - w \frac{\partial z}{\partial \nu} \right) ds \; .$$

Choosing $w = w_j$ from (3.3) and z from (3.5) we obtain

$$- \int_\Omega \mathrm{div}(\eta \nabla u_j) z dx = - \int_{\partial\Omega} \sigma (R_j'(\sigma)\eta) g ds \; .$$

Since $\eta = 0$ on $\partial\Omega$ an integration by parts yields

$$\int_\Omega \eta \nabla u_j \cdot \nabla z dx = \int_{\partial\Omega} \sigma (R_j'(\sigma)\eta) g ds$$

or

$$(\eta, \nabla u_j \cdot \nabla z)_{L_2(\Omega)} = (R_j'(\sigma)\eta, g)_{L_{2,\sigma}(\partial\Omega)} \; .$$

This holds for each η vanishing on $\partial\Omega$, verifying (3.4).

$\square$

In the light of the theorem, the iteration (1.4) proceeds as follows.

For $j = 1, 2, \ldots, p$

Solve div $(\sigma_{j-1} \nabla u_j) = 0$ in Ω , $\quad \sigma_{j-1} \dfrac{\partial u_j}{\partial \nu} = f_j$ on $\partial\Omega$.

Solve div $(\sigma_{j-1} \nabla z_j) = 0$ in Ω , $\quad \sigma_{j-1} \dfrac{\partial z_j}{\partial \nu} = u_j - g_j$ on $\partial\Omega$.

Put $\sigma_j = \sigma_{j-1} + \omega \nabla u_j \cdot \nabla z_j$.

We see that each sweep of the iteration requires the solution of $2p$ boundary value problems of nearly identical shape. This is not only easy to program but also very cheap computationally, compared to other methods.

2. A parabolic inverse problem.

Consider the initial-boundary value problem

$$\begin{aligned} &\frac{\partial u_j}{\partial t} - \operatorname{div}(\sigma \nabla u_j) = q_j \quad \text{in} \quad \Omega \times [0,T] \\ &u_j(x,0) = 0 \quad \text{for} \ \ x \in \Omega \ , \quad \frac{\partial u_j}{\partial \nu}(x,t) = 0 \ \text{ for } \ x \in \partial\Omega \ . \end{aligned} \tag{3.6}$$

The source terms q_j are assumed to be known, while the diffusion coefficient $\sigma(x)$ has to be determined from the knowledge of u_j on $\partial\Omega$:

$$u_j(x,t) = g_j(x,t) \ , \quad x \in \partial\Omega \ , \ \ t \in [0,T] \ . \tag{3.7}$$

Again we assume σ to be known on $\partial\Omega$.
This problem is of the form (1.1)-(1.2). We have

$$\begin{array}{rcllcl} U_j &=& \{u \in L_2(\Omega \times [0,T]) & : \ u &=& 0 \ \text{ for } \ t = 0 \ , \ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega\} \ , \\ F &=& L_2(\Omega) \ , & Z_j &=& L_{2,\sigma}(\partial\Omega \times [0,T]) \ . \end{array}$$

The operator $R_j : L_2(\Omega) \to L_{2,\sigma}(\partial\Omega \times [0,T])$ is given by

$$R_j(\sigma) = u_j - g_j \quad \text{on} \ \ \partial\Omega \times [0,T]$$

where u_j is the solution of (3.6). According to (2.2), the Fréchet derivative is

$$R_j'(\sigma)\eta = w_j\Big|_{\partial\Omega \times [0,T]} \ ,$$

$$\begin{aligned} &\frac{\partial w_j}{\partial t} - \operatorname{div}(\sigma \nabla w_j) = \operatorname{div}(\eta \nabla u_j) \quad \text{in} \ \ \Omega \times [0,T] \ , \\ &w_j = 0 \quad \text{for} \ \ t = 0 \ , \quad \frac{\partial w_j}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega \times [0,T] \ . \end{aligned} \tag{3.8}$$

Theorem 3.2 *The operator* $(R_j'(\sigma))^* : L_{2,\sigma}(\partial\Omega \times [0,T]) \to L_2(\Omega)$ *is given by*

$$(R_j'(\sigma))^* g(x) = \int_0^T \nabla u_j \cdot \nabla z(x,t) dt$$

where z *is the solution of*

$$\begin{aligned} &\frac{\partial z}{\partial t} + div(\sigma \nabla z) = 0 \quad in \;\; \Omega \times [0,T] \, , \\ &z = 0 \quad for \;\; t = T \, , \quad \frac{\partial z}{\partial \nu} = -g \;\; on \;\; \partial\Omega \times [0,T] \, . \end{aligned} \tag{3.9}$$

Proof: For any z, w we have

$$\begin{aligned} &\int_0^T \int_\Omega \left\{ (\frac{\partial w}{\partial t} - \mathrm{div}(\sigma \nabla w)) z + (\frac{\partial z}{\partial t} + \mathrm{div}(\sigma \nabla z)) w \right\} dx dt \\ = \;\; &\int_\Omega ((wz)(x,T) - (wz)(x,0)) dx + \int_0^T \int_{\partial\Omega} \sigma (\frac{\partial z}{\partial \nu} w - z \frac{\partial w}{\partial \nu}) ds dt \, . \end{aligned}$$

For w_j, z from (3.8)-(3.9) this reads

$$\int_0^T \int_\Omega \mathrm{div}(\eta \nabla u_j) z dx dt = - \int_0^T \int_{\partial\Omega} \sigma g R_j'(\sigma) \eta ds dt \, .$$

An integration by parts on the left hand side turns this into

$$\int_\Omega \eta(x) \int_0^T \nabla u_j \cdot \nabla z dt dx = \int_0^T \int_{\partial\Omega} \sigma g R_j'(\sigma) \eta ds dt \, ,$$

hence the theorem.

□

The algorithm (1.4) is as follows: For $j = 1, 2, \ldots, p$

Solve

$$\frac{\partial u_j}{\partial t} - \operatorname{div}(\sigma_{j-1}\nabla u_j) = q_j \ \text{ in } \ \Omega \times T \ , \quad u_j = 0 \ \text{ for } \ t = 0 \ , \ \ \frac{\partial u_j}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega$$

and

$$\frac{\partial z_j}{\partial t} + \operatorname{div}(\sigma_{j-1}\nabla z_j) = 0 \ \text{ in } \ \Omega\times T \ , \ \ z_j = 0 \ \text{ for } \ t = T \ , \ \ \frac{\partial z_j}{\partial \nu} = g_j - u_j \ \text{ on } \ \partial\Omega \ .$$

Put

$$\sigma_j(x) = \sigma_{j-1}(x) + \omega \int_0^T \nabla u_j(x,t) \cdot \nabla z_j(x,t) dt \ .$$

Alternatively we may consider the boundary condition $u_j = 0$ on $\partial\Omega \times [0,T]$ instead of $\partial u_j/\partial\nu = 0$ in (3.6), and $\partial u_j/\partial\nu = g_j$ on $\partial\Omega \times [0,T]$ as data in (3.7). Then,

$$R_j(\sigma) = \frac{\partial u_j}{\partial \nu} - g_j \ ,$$

and

$$R'(\sigma)\eta = \left.\frac{\partial w_j}{\partial \nu}\right|_{\partial\Omega\times[0,T]}$$

where now w_j satisfies (3.8) with the boundary condition $w_j = 0$ on $\partial\Omega \times [0,T]$. The adjoint is given by the same expression as above, but with the boundary condition $z = g$ on $\partial\Omega \times [0,T]$.

3. Laser tomography: Stationary case

Here the governing equation is

$$\begin{aligned} \theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) \ &- \ b(x) \int_{S_{n-1}} \eta(\theta\cdot\theta')u(x,\theta')d\theta' = q(x-s) \text{ in } \Omega \times S^{n-1}, \\ u = 0 \qquad &\text{on } \ (\partial\Omega \times S^{n-1})_- \ , \end{aligned} \tag{3.10}$$

where

$$(\partial\Omega \times S^{n-1})_\pm = \{(x,\theta) \in \partial\Omega \times S^{n-1} : \nu(x)\cdot\theta \ \substack{\geq \\ \leq} \ 0\}$$

with $\nu(x)$ the exterior normal to $\partial\Omega$ at $x \in \partial\Omega$. The source term q is delta-like, and the sources s are on $\partial\Omega$. η is assumed to be known. We want to recover a, b from knowing u on $\partial\Omega$ for sources $s_1, \dots, s_p$. Denoting by u_j the solution of (3.10) for $s = s_j$ we have

$$\begin{aligned} \theta \cdot \nabla u_j(x,\theta) + a(x)u_j(x,\theta) &- b(x) \int\limits_{S^{n-1}} \eta(\theta\cdot\theta')u_j(x,\theta')d\theta' = q(x-s_j) \text{ in } \Omega \times S^{n-1}, \\ u_j = 0 & \qquad \text{on } (\partial\Omega \times S^{n-1})_- . \end{aligned} \tag{3.11}$$

The information on u on $\partial\Omega$ which we want to make use of is

$$\int\limits_{S^{n-1}} \nu(x) \cdot \theta u_j(x,\theta) d\theta = g_j(x) , \quad x \in \partial\Omega . \tag{3.12}$$

Obviously, (3.11)-(3.12) is of the type of (1.1)-(1.2) with $f = (a, b)$ and

$$\begin{aligned} U_j &= \{u \in L_2(\Omega \times S^{n-1}) : u = 0 \text{ on } (\partial\Omega \times S^{n-1})_- \} , \\ F &= L_2(\Omega) \times L_2(\Omega) , \qquad Z_j = L_2(\partial\Omega) . \end{aligned}$$

The operator $R_j : L_2(\Omega) \times L_2(\Omega) \to L_2(\partial\Omega)$ is given by

$$R_j\begin{pmatrix} a \\ b \end{pmatrix}(x) = \int\limits_{S^{n-1}} \nu(x) \cdot \theta u_j(x,\theta)d\theta - g_j(x)$$

where u_j is the solution to (3.11). The linearized operator is

$$R_j'\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \int\limits_{S^{n-1}} \nu(x) \cdot \theta w(x,\theta) d\theta$$

where w solves

$$\begin{aligned} & \theta \cdot \nabla w(x,\theta) + a(x) w(x,\theta) - b(x) \int\limits_{S^{n-1}} \eta(\theta\cdot\theta') w(x,\theta') d\theta \\ = \; & k(x) \int\limits_{S^{n-1}} \eta(\theta\cdot\theta') u_j(x,\theta')d\theta' - h(x)u_j(x,\theta) \text{ in } \Omega \times S^{n-1} , \\ & w = 0 \quad \text{on } (\partial\Omega \times S^{n-1})_- . \end{aligned} \tag{3.13}$$

Theorem 3.3 *The operator* $(R_j'\begin{pmatrix} a \\ b \end{pmatrix})^* : L_2(\partial\Omega) \to L_2(\Omega) \times L_2(\Omega)$ *is given by*

$$(R_j'\begin{pmatrix} a \\ b \end{pmatrix})^* g(x) = \int\limits_{S^{n-1}} \begin{pmatrix} -u_j(x,\theta) \\ \int\limits_{S^{n-1}} \eta(\theta\cdot\theta')u_j(x,\theta')d\theta' \end{pmatrix} z(x,\theta)d\theta$$

where z *is the solution of*

$$\begin{aligned} -\theta\cdot\nabla z(x,\theta) + a(x)z(x,\theta) \quad &- \quad b(x)\int\limits_{S^{n-1}} \eta(\theta\cdot\theta')z(x,\theta')d\theta' = 0 \ \ in \ \ \Omega\times S^{n-1} \ , \\ z = g \qquad & on \ \ (\partial\Omega\times S^{n-1})_+ \ . \end{aligned} \tag{3.14}$$

Proof: For any z, w we have

$$\begin{aligned} &\int\limits_{\Omega}\int\limits_{S^{n-1}} (\theta\cdot\nabla w(x,\theta) + a(x)w(x,\theta) \quad - \quad b(x)\int\limits_{S^{n-1}} \eta(\theta\cdot\theta')w(x,\theta')d\theta')z(x,\theta)dxd\theta \\ = &\int\limits_{\Omega}\int\limits_{S^{n-1}} (-\theta\cdot\nabla z(x,\theta) + a(x)z(x,\theta) \quad - \quad b(x)\int\limits_{S^{n-1}} \eta(\theta'\cdot\theta)z(x,\theta')d\theta')w(x,\theta)dxd\theta \\ &\qquad + \int\limits_{\partial\Omega}\int\limits_{S^{n-1}} z(x,\theta)w(x,\theta)\nu(x)\cdot\theta ds d\theta \ . \end{aligned}$$

For z, w from (3.13)-(3.14) this implies

$$\begin{aligned} &\int\limits_{\Omega}\int\limits_{S^{n-1}} \left(k(x)\int\limits_{S^{n-1}} \eta(\theta\cdot\theta')u_j(x,\theta')d\theta' - h(x)u_j(x,\theta)\right) z(x,\theta)dxd\theta \\ = &\int\limits_{\partial\Omega} g(x)R_j'\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} ds \ , \end{aligned}$$

hence the theorem.

□

The algorithm is as follows:

For $j = 1, \dots, p$

Solve

$$\theta \cdot \nabla u_j(x,\theta) + a_{j-1}(x) u_j(x,\theta) \quad - \quad b_{j-1}(x) \int\limits_{S^{n-1}} \eta(\theta \cdot \theta') u_j(x,\theta) d\theta' = q(x - s_j) \text{ in } \Omega \times S^{n-1} ,$$

$$u_j = 0 \qquad \text{on } (\partial\Omega \times S^{n-1})_-$$

and

$$-\theta \cdot \nabla z_j(x,\theta) + a_{j-1}(x) z_j(x,\theta) \quad - \quad b_{j-1}(x) \int\limits_{S^{n-1}} \eta(\theta \cdot \theta') z_j(x,\theta') d\theta' = 0 \text{ in } \Omega \times S^{n-1} ,$$

$$z_j = u_j - g_j \qquad \text{on } (\partial\Omega \times S^{n-1})_+ .$$

Put

$$\begin{aligned} a_j &= a_{j-1} - w \int\limits_{S^{n-1}} u_j(x,\theta) z_j(x,\theta) d\theta , \\ b_j &= b_{j-1} + w \int\limits_{S^{n-1}} \int\limits_{S^{n-1}} \eta(\theta \cdot \theta') u_j(x,\theta') z_j(x,\theta) d\theta' d\theta . \end{aligned}$$

4. SPECT

In SPECT one has to compute the source distribution f from

$$\theta_j \cdot \nabla u_j(x) + a(x) u_j(x) = f(x) \quad \text{in } \Omega \tag{3.15}$$

$$\begin{aligned} u_j &= 0 \quad \text{on } \partial\Omega_j^- , \\ u_j &= g_j \quad \text{on } \partial\Omega_j^+ \end{aligned} \tag{3.16}$$

where $\partial\Omega_j^{\pm} = \{x \in \partial\Omega : \nu(x) \cdot \theta_j \genfrac{}{}{0pt}{}{\geq}{\leq} 0\}$.

Here, Ω is a bounded domain in $\mathbb{R}^2$, and $\theta_1, \dots, \theta_p$ are direction vectors $\in S^1$. If a is known, then (3.15)-(3.16) reduces to a linear problem which can be solved by inverting the attenuated Radon transform. We consider the case in which a is unknown, too.

(3.15)-(3.16) is of the form (1.1)-(1.2) again, with (a, f) playing the role of f and

$$\begin{aligned} U_j &= \{u \in L_2(\Omega) : u = 0 \quad \text{on } \partial\Omega_j^-\} \\ F &= L_2(\Omega) \times L_2(\Omega)\,, \quad Z_j = L_2(\partial\Omega_j^+)\,. \end{aligned}$$

The operator $R_j(\begin{smallmatrix} a \\ f \end{smallmatrix}) : L_2(\Omega) \times L_2(\Omega) \to L_2(\partial\Omega_j^+)$ is given by

$$R_j(\begin{smallmatrix} a \\ f \end{smallmatrix})(x) = u_j(x) - g_j(x)\,, \quad x \in \partial\Omega_j^+$$

where u_j is the solution of (3.15). The linearized operator is

$$R'_j(\begin{smallmatrix} a \\ f \end{smallmatrix})(\begin{smallmatrix} h \\ k \end{smallmatrix}) = w_j|_{\partial\Omega_j^+}$$

where w_j solves

$$\theta_j \cdot \nabla w_j + a w_j = k - h u_j \quad \text{in } \Omega\,, \quad w = 0 \quad \text{on } \partial\Omega_j^-\,. \tag{3.17}$$

Theorem 3.4 *The operator* $(R'_j(\begin{smallmatrix} a \\ f \end{smallmatrix}))^* : L_2(\partial\Omega_j^+) \to L_2(\Omega) \times L_2(\Omega)$ *is given by*

$$(R'_j(\begin{smallmatrix} a \\ f \end{smallmatrix}))^* g = \begin{pmatrix} -u_j z_j \\ z_j \end{pmatrix}$$

where z_j *is the solution of*

$$-\theta_j \cdot \nabla z_j + a z_j = 0 \quad \text{in }\,, \quad z_j = g \;\; \nu(x) \cdot \theta_j \quad \text{on } \partial\Omega_j^+\,. \tag{3.18}$$

Proof: For any z, w we have

$$\int_\Omega (\theta_j \cdot \nabla w + a w) z\,dx = \int_\Omega (-\theta_j \cdot \nabla z + a z) w\,dx + \int_{\partial\Omega} \theta_j \cdot \nu(x) w z\,ds\,.$$

For z, w as in (3.13)-(3.17) this means

$$\int_\Omega (k - hu_j) z_j dx = \int_{\partial\Omega_j^+} R_j'\begin{pmatrix} a \\ f \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} ds \ ,$$

hence the theorem.

$\square$

The algorithm is:

For $j = 1, \ldots, p$

Solve

$$\theta_j \cdot \nabla u_j + a_{j-1} u_j = f_{j-1} \quad \text{in } \Omega \ , \quad u_j = 0 \quad \text{on } \partial\Omega_j^- \tag{3.19}$$

and

$$-\theta_j \cdot \nabla z_j + a_{j-1} z_j = 0 \quad \text{in } \Omega \ , \quad z_j = u_j - g_j \quad \text{on } \partial\Omega_j^+ \ . \tag{3.20}$$

Put

$$\begin{aligned} a_j &= a_{j-1} - \omega u_j z_j \ , \\ f_j &= f_{j-1} + \omega z_j \ . \end{aligned}$$

The forward problem (3.19) can be solved by inverting the attenuated Radon transform, and (3.20) is the attenuated backprojection. Thus the algorithm is very similar to the ART algorithm in computerized tomography and in fact reduces to this algorithm if a is known, e.g. $a = 0$.

5. Ultrasound tomography: Time harmonic case

Here we have the Helmholtz equation

$$\Delta u + k^2(1 - f)u = 0 \quad \text{in } \mathbb{R}^n$$

where $u = e^{ikx\cdot\theta} + v$ with the scattered wave v satisfying the radiation condition

$$\lim_{r\to\infty} \left(\frac{\partial v}{\partial r} - ikv \right) = 0 \ .$$

Here, f is a complex valued function which has to be recovered from knowing u for each direction $\theta \in S^{n-1}$ outside the support of f. k is a real parameter controlling the spatial resolution.

Let Ω be a ball containing $\mathrm{supp}(f)$ in its interior. We rewrite the radiation condition as a boundary condition on $\partial\Omega$, to wit

$$\frac{\partial v}{\partial \nu} = Bv \qquad \text{on } \partial\Omega$$

with a linear operator B on $L_2(\partial\Omega)$. Assume that the measurements are made for finitely many directions $\theta_1, \ldots, \theta_p$, and let $u_j = e^{ikx\cdot\theta_j} + v_j$ be the solution for direction θ_j. Then,

$$\begin{aligned} \Delta v_j + k^2(1-f)v_j &= k^2 f e^{ikx\cdot\theta_j} \quad \text{in } \Omega \\ \frac{\partial v_j}{\partial \nu} &= Bv_j \quad \text{on } \partial\Omega \end{aligned} \tag{3.21}$$

and f has to be determined from

$$v_j = g_j \quad \text{on } \partial\Omega\ . \tag{3.22}$$

Obviously, (3.21)-(3.22) is of the form (1.1)-(1.2), with

$$\begin{aligned} U_j &= \{v \in L_2(\Omega) : \frac{\partial v}{\partial \nu} = Bv \quad \text{on } \partial\Omega\}\ , \\ F &= L_2(\Omega)\ , \qquad Z_j = L_2(\partial\Omega)\ . \end{aligned}$$

The operator $R_j : L_2(\Omega) \to L_2(\partial\Omega)$ is

$$R_j f = v_j - g_j \quad \text{on } \partial\Omega$$

where v_j solves (3.21). The linearized operator is given by $R_j'(f)h = w_j$, where w_j is a solution of

$$\Delta w_j + k^2(1-f)w_j = k^2 h u_j \quad \text{in } \Omega \tag{3.23}$$

$$\frac{\partial w_j}{\partial \nu} = Bw_j \quad \text{on } \partial\Omega\ .$$

Theorem 3.5 *The operator* $(\mathbb{R}_j'(f))^* : L_2(\partial\Omega) \to L_2(\Omega)$ *is given by*

$$(R_j'(f))^* g = k^2 \overline{u}_j z$$

where z *is the solution of*

$$\Delta z + k^2(1-\overline{f})z = 0 \quad in \quad \Omega \ , \ \ \frac{\partial z}{\partial \nu} - B^* z = -g \quad on \ \ \partial\Omega \ . \tag{3.24}$$

Proof: For any z, w we have

$$\begin{aligned} &\int_\Omega \{(\Delta w + k^2(1-f)w)\overline{z} - w(\overline{\Delta z + k^2(1-\overline{f})z})\} dx \\ &\quad = \int_{\partial\Omega} \left(\frac{\partial w}{\partial \nu}\overline{z} - w\frac{\partial \overline{z}}{\partial \nu} \right) ds \ . \end{aligned}$$

For the functions z, w_j from (3.24)-(3.23) this yields

$$\begin{aligned} \int_\Omega k^2(e^{ikx\cdot\theta_j} + v_j)\overline{z} dx &= \int_{\partial\Omega} ((Bw)\overline{z} - w_j(\overline{B^* z - g})) ds \\ &= \int_{\partial\Omega} w_j \overline{g} ds \\ &= \int_{\partial\Omega} (R_j'(f)h)\overline{g} ds \end{aligned}$$

or

$$(h, k^2(e^{-ikx\cdot\theta_j} + \overline{v}_j)z) = (R_j'(f)h, g)_{L_2(\partial\Omega)} \ .$$

□

6. Ultrasound tomography: Time resolved case

Here we have the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u + q(x-s)\delta(t) \quad \text{in} \ \ \Omega \times [0, T] \tag{3.25}$$

with initial and boundary conditions

$$u = \frac{\partial u}{\partial t} = 0 \quad \text{on} \quad \Omega \times \{0\} \, , \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \;\; \partial\Omega \times [0,T] \tag{3.26}$$

with ν the exterior normal on $\partial\Omega$. The source term q is δ like and the sources s sit on $\partial\Omega$. The problem is to recover the sound speed c in Ω from the measurement of u on $\partial\Omega \times [0,T]$, or a part thereof, for sources $s_1, \ldots, s_p$. We assume $c^2 = 1 + f$ with $f = 0$ on $\partial\Omega$. Again, this problem is obviously a special case of (1.1), (1.2) with

$$Z_j = Z = L_2(\partial\Omega \times [0,T]) \, , \quad F = L_2(\Omega) \, .$$

The linearized operator $R_j'(f) : F \to Z$ is given by

$$R_j'(f)h = w_j \Big|_{\partial\Omega \times [0,T]} \, ,$$

where w_j is the solution of

$$\begin{aligned} \frac{\partial^2 w_j}{\partial t^2} &= (1+f)\Delta w_j + h\Delta u_j \qquad \text{in} \;\; \Omega \times [0,T] \, , \\ w_j &= \frac{\partial w_j}{\partial t} = 0 \quad \text{on} \;\; \Omega \times \{0\} \, , \quad \frac{\partial w_j}{\partial \nu} = 0 \quad \text{on} \;\; \partial\Omega \times [0,T] \end{aligned} \tag{3.27}$$

with u_j the solution to (3.25), (3.26) for $c^2 = 1 + f$ and $s = s_j$.

Theorem 3.6 *The operator $(R_j'(f))^* : L_2(\partial\Omega \times [0,T]) \to L_2(\Omega)$ is given by*

$$(R_j'(f))^* g = \int_0^T \Delta u_j \cdot z dt$$

where z is the solution to

$$\frac{\partial^2 z}{\partial t^2} = \Delta((1+f)z) \quad \textit{in} \quad \Omega \times [0,T] \tag{3.28}$$

$$z = \frac{\partial z}{\partial t} = 0 \quad \textit{on} \;\; \Omega \times \{T\} \, , \quad \frac{\partial z}{\partial \nu} = g \quad \textit{on} \;\; \partial\Omega \times [0,T] \, .$$

Proof: For arbitrary w, z we have

$$\int_\Omega \int_0^T \left(\frac{\partial^2 w}{\partial t^2} - (1+f)\Delta w\right) z dt dx = \int_\Omega \int_0^T \left(\frac{\partial^2 z}{\partial t^2} - \Delta((1+f)z)\right) w dt dx$$
$$- \int_{\partial\Omega} \int_0^T \left(\frac{\partial w}{\partial \nu} z - w \frac{\partial z}{\partial \nu}\right) dt dx + \left[\int_\Omega \left(\frac{\partial w}{\partial t} z - w \frac{\partial z}{\partial t}\right) dx\right]_0^T .$$

For $w = w_j$ from (3.27) and z as in (3.28) this reads

$$\int_\Omega h \int_0^T \Delta u_j \cdot z dt dx = \int_{\partial\Omega} \int_0^T R_j'(f) h \cdot g dt dx ,$$

hence

$$(R_j'(f))^* g = \int_0^T \Delta u_j \cdot z dt$$

$\square$

If u is measured only on a part $\partial\overline{\Omega} \times [0,T]$ of $\partial\Omega \times [0,T]$, then Z has to be replaced by $\overline{Z} = L_2(\partial\overline{\Omega} \times [0,T])$. With E the extension from $\partial\overline{\Omega} \times [0,T]$ to $\partial\Omega \times [0,T]$ by zero, the corresponding adjoint $(\overline{R}_j'(f))^* : \overline{Z} \to F$ is then given by $(R_j'(f))^* E$.

7. Laser tomography in the diffusion approximation: Time harmonic case.

Here the partial differential equation is

$$\begin{aligned}
\frac{\partial v_j}{\partial t}(x,t) &= \operatorname{div}(\sigma(x)\nabla v_j(x,t)) - a(x) v_j(x,t) + e^{i\omega t} q(x - s_j) \text{ in } \Omega \times \mathbb{R}^1 \\
v_j(x,t) &= 0 \ , \ \frac{\partial v_j}{\partial \nu}(x,t) = e^{i\omega t} g_j(x) \text{ on } \partial\Omega \times \mathbb{R}^1 .
\end{aligned}$$

The modulating frequency ω is fixed, and g_j is the measured data for the source $s_j \in \partial\Omega$.

Putting $v_j(x,t) = e^{i\omega t} u_j(x)$ we obtain

$$\begin{aligned} 0 &= \operatorname{div}(\sigma(x)\nabla u_j(x)) - (a(x) + i\omega)u_j(x) + q(x - s_j) \text{ in } \Omega \, , \\ u_j &= 0 \, , \quad \frac{\partial u_j}{\partial \nu} = g_j \text{ on } \partial\Omega \, . \end{aligned} \tag{3.29}$$

The problem is to recover σ, a from the g_j. It is clear that (3.29) is of the form (1.1-2). We have

$$R_j \begin{pmatrix} \sigma \\ a \end{pmatrix} = \frac{\partial u_j}{\partial \nu} - g_j \, ,$$

R_j being considered as an operator from $L_2(\Omega) \times L_2(\Omega)$ into $L_2(\partial\Omega)$. We readily compute the operator $R_j' \begin{pmatrix} \sigma \\ a \end{pmatrix} : L_2(\Omega) \times L_2(\Omega) \to L_2(\partial\Omega)$, obtaining

$$R_j' \begin{pmatrix} \sigma \\ a \end{pmatrix} \begin{pmatrix} \eta \\ h \end{pmatrix} = \frac{\partial w_j}{\partial \nu}$$

where w_j solves

$$0 = \operatorname{div}(\eta \nabla u_j) + \operatorname{div}(\sigma \nabla w_j) - h u_j - (a + i\omega) w_j \text{ in } \Omega$$

$$w_j = 0 \quad \text{on } \partial\Omega \, .$$

Theorem 3.7 *The operator* $\left(R_j' \begin{pmatrix} \sigma \\ a \end{pmatrix} \right)^* = L_2(\partial\Omega) \to L_2(\Omega) \times L_2(\Omega)$ *is given by*

$$\left(R_j' \begin{pmatrix} \sigma \\ a \end{pmatrix} \right)^* g = \begin{pmatrix} \nabla \overline{u_j} \cdot \nabla z \\ \overline{u_j} z \end{pmatrix}$$

where z *is the solution of*

$$div(\sigma \nabla z) - (a - i\omega) z = 0 \quad in \;\; \Omega$$

$$z = g \quad on \;\; \partial\Omega$$

Proof: The proof is done by verification. We compute

$$\left(R_j'\begin{pmatrix}\sigma\\ a\end{pmatrix}\begin{pmatrix}\eta\\ h\end{pmatrix},g\right)=\int\limits_{\partial\Omega}\frac{\partial w_j}{\partial\nu}\,\overline{g}\,ds$$

$$=\int\limits_{\partial\Omega}w_j\frac{\partial\overline{g}}{\partial\nu}ds-\int\limits_{\Omega}\{w_j\mathrm{div}(\sigma\nabla\overline{z})-\ \mathrm{div}(\sigma\nabla w_j)\overline{z}\}\,dx$$

by way of Green's theorem. The integral over $\partial\Omega$ vanishes, and the integral over Ω can be rewritten as

$$\int\limits_{\Omega}\{w_j(\mathrm{div}(\sigma\nabla\overline{z})-(a+i\omega)\overline{z})-\ (\mathrm{div}(\sigma\nabla w_j)-(a+i\omega)w_j)\overline{z}\}\,dx$$

which, by the differential equations for z and w_j, can be written as

$$-\int\limits_{\Omega}(-\mathrm{div}(\eta\nabla\overline{u}_j)+h\overline{u}_j)\overline{z}dx\ .$$

Hence

$$\begin{aligned}\left(R_j'\begin{pmatrix}\sigma\\ a\end{pmatrix}\begin{pmatrix}\eta\\ h\end{pmatrix},g\right) &= \int\limits_{\Omega}(-\mathrm{div}(\eta\nabla u_j)+hu_j)\overline{z}dx\\ &= \int\limits_{\Omega}(\eta\nabla u_j\cdot\nabla\overline{z}+hu_j\overline{z})dx\\ &= \left(\begin{pmatrix}\eta\\ h\end{pmatrix},\begin{pmatrix}\nabla\overline{u}_j\cdot\nabla z\\ \overline{u}_j z\end{pmatrix}\right)\ .\end{aligned}$$

□

8. Laser tomography in the diffusion approximation: Time resolved case

In this case the governing equation reads

$$\begin{aligned}\frac{\partial u_j}{\partial t} &= \mathrm{div}(\sigma(x)\nabla u_j)-a(x)u_j+q_j \quad \text{in} \quad \Omega\times[0,T],\\ u_j(x,0) &= 0 \quad \text{in} \quad \Omega, \quad u_j(x,t)=0 \quad \text{on} \quad \partial\Omega\times[0,T].\end{aligned} \tag{3.30}$$

The source terms q_j are assumed to be δ-functions at $t = 0$ and $x = s_j \in \partial\Omega$. Now the invers problem is to determine both the diffusion coeffient $\sigma(x)$ and the absorption coefficient $a(x)$ from the given data

$$\frac{\partial u_j}{\partial \nu}(x,t) = g_j(x,t) \quad \text{on} \quad \partial\Omega \times [0,T].$$

As before this problem is of the form (1.1)-(1.2).

The operator $R_j : L_2(\Omega) \times L_2(\Omega) \to L_{2,\sigma}(\partial\Omega \times [0,T])$ is given by

$$R_j\begin{pmatrix} \sigma \\ a \end{pmatrix} = \frac{\partial u_j}{\partial \nu} - g_j \quad \text{on} \quad \partial\Omega \times [0,T]$$

where u_j is the solution of (3.30). According to (2.2) we calculate the linearized operator to be

$$R_j'\begin{pmatrix} \sigma \\ a \end{pmatrix}\begin{pmatrix} \eta \\ h \end{pmatrix} = \left.\frac{\partial w_j}{\partial \nu}\right|_{\partial\Omega \times [0,T]}$$

where w_j solves the equation

$$\begin{aligned} \frac{\partial w_j}{\partial t} - \operatorname{div}(\sigma(x)\nabla w_j) + a(x) w_j &= \operatorname{div}(\eta(x)\nabla u_j) - h(x) u_j \quad \text{in} \quad \Omega \times [0,T], \\ w_j(x,0) = 0 \quad \text{in} \quad \Omega, &\qquad w_j(x,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,T]. \end{aligned} \tag{3.31}$$

Theorem 3.8 *The operator* $(R_j'\begin{pmatrix} \sigma \\ a \end{pmatrix})^* : L_{2,\sigma}(\partial\Omega \times [0,T]) \to L_2(\Omega) \times L_2(\Omega)$ *is given by*

$$(R_j'\begin{pmatrix} \sigma \\ a \end{pmatrix})^* g(x) = \int_0^T \begin{pmatrix} (\nabla u_j \cdot \nabla z)(x,t) \\ (u_j z)(x,t) \end{pmatrix} dt.$$

where $z(x,t)$ *is the solution of*

$$\begin{aligned} \frac{\partial z}{\partial t} + div(\sigma(x)\nabla z) - a(x) z = 0 \qquad in \quad \Omega \times [0,T], \\ z(x,T) = 0 \quad for \quad x \in \Omega, \qquad z(x,t) = g \quad on \quad \partial\Omega \times [0,T]. \end{aligned} \tag{3.32}$$

Proof: For any z, w we have

$$\int_0^T \int_\Omega \left\{ (\frac{\partial w}{\partial t} - \operatorname{div}(\sigma\nabla w) + aw)z + (\frac{\partial z}{\partial t} + \operatorname{div}(\sigma\nabla z) - az)w \right\} dx\, dt$$

$$= \int_\Omega ((wz)(x,T) - (wz)(x,0))\, dx + \int_0^T \int_{\partial\Omega} \sigma(\frac{\partial z}{\partial\nu} w - z\frac{\partial w}{\partial\nu})\, ds\, dt.$$

For w_j, z from (3.31)-(3.32) this reduces to

$$\int_0^T \int_\Omega (\operatorname{div}(\eta\nabla u_j) - hu_j)z\, dx\, dt = -\int_0^T \int_{\partial\Omega} \sigma g R_j'(\begin{array}{c}\sigma\\a\end{array})(\begin{array}{c}\eta\\h\end{array})\, ds\, dt.$$

An integration by parts on the left hand side leads to

$$\int_\Omega \left\{ \eta \int_0^T (\nabla u_j \cdot \nabla z)\, dt + h\int_0^T (u_j z)\, dt \right\} dx = \left(g, R_j'(\begin{array}{c}\sigma\\a\end{array})(\begin{array}{c}\eta\\h\end{array}) \right)_{L_{2,\sigma}}.$$

This proofs the theorem. $\square$

4 Convergence

We now prove convergence of the iterative method for linear equations

$$R_j f = g_j \ , \qquad j = 1, \ldots, p \tag{4.1}$$

where $R_j : F \to Z_j$ are linear bounded operators. To avoid purely technical difficulties we assume F and Z_j to be of finite dimension. The iteration reads

$$\begin{array}{ll} \text{(i)} & f_0 = f^0 \ , \\ \text{(ii)} & \text{For } \ j = 1, \ldots, p \\ & \\ & f_j = f_{j-1} + \omega h_j \ , \quad h_j = -R_j^* \mathcal{C}_j^{-1}(g_j - R_j f_{j-1}) \ , \\ \text{(iii)} & f^1 = f_p \ . \end{array} \tag{4.2}$$

We note in passing that for dim $(Z_j) = 1$ this is just the Kaczmarz method.

Theorem 4.1 *Let $\mathcal{C}_j$ be positive definite and $\mathcal{C}_j \geq R_j R_j^*$. Then, (4.2) converges. If (4.1) is consistent and*

$$f^0 \in \sum_{j=1}^{p} range(R_j^*) \ , \tag{4.3}$$

then (4.2) converges to the solution of (4.1) with least norm.

Proof: For $\mathcal{C}_j = R_j R_j^*$ this is just Theorem 3.9 of [3]. Our proof is a slight extension of this result.

To begin with we write f_j in the form

$$f_j = f_0 + \sum_{k=1}^{j} R_k^* u_k \tag{4.4}$$

with certain vectors u_k. Inserting this into the recursion (4.2) yields

$$f_0 + \sum_{k=1}^{j} R_k^* u_k = f_0 + \sum_{k=1}^{j-1} R_k^* u_k + \omega R_j^* \mathcal{C}_j^{-1}(g_j - R_j f_0 - \sum_{k=1}^{j-1} R_j R_k^* u_k) \ .$$

We apply R_j. After obvious manipulations we obtain

$$R_j R_j^* u_j = \omega R_j R_j^* \mathcal{C}_j^{-1}(g_j - R_j f_0 - \sum_{k=1}^{j-1} R_j R_k^* u_k) .$$

Hence,

$$\mathcal{C}_j u_j = \omega(g_j - R_j f_0 - \sum_{k=1}^{j-1} R_j R_k^* u_k) . \tag{4.5}$$

Now let A be the operator $(R_j R_k^*)_{j,k=1,\ldots,p}$ and R the map $(R_j)_{j=1,\ldots,p}$, i.e. $A = RR^*$. Let $A = L + L^* + D$ where D is the (block) diagonal and L the (block) lower left part of A. Let the vectors u, g consist of the components u_j, g_j, resp. and let $\mathcal{C}$ be the (block) diagonal $\mathcal{C}_1, \ldots, \mathcal{C}_p$. Then (4.5) reads

$$\mathcal{C}u = \omega(g - Rf_0 - Lu)$$

or

$$u = \omega(\mathcal{C} + \omega L)^{-1}(g - Rf_0) . \tag{4.6}$$

Using (4.4) with $j = p$ we see that

$$f^1 = f^0 + R^* u . \tag{4.7}$$

Now assume that $f^0 \in \sum_{j=1}^{p} \text{range}(R_j^*) = \text{range}(R_j^*)$. Then, $f^k = R^* u^k$ for all iterates, and (4.6)-(4.7) yield

$$R^* u^1 = R^* u^0 + \omega R^*(\mathcal{C} + \omega L)^{-1}(g - RR^* u^0)$$

or

$$u^1 = u^0 + \omega(\mathcal{C} + \omega L)^{-1}(g - RR^* u^0) .$$

Hence

$$u^1 = \mathcal{C}(\omega)u^0 + c(\omega) , \quad \mathcal{C}(\omega) = I - \omega(\mathcal{C} + \omega L)^{-1} RR^* . \tag{4.8}$$

We show that all the eigenvalues of $\mathcal{C}(\omega)$ are either 1 or less than 1 in absolute value if $\mathcal{C}_j \geq R_j R_j^*$, $j = 1, \ldots, p$. Let λ be such an eigenvalue and u the corresponding eigenvector, i.e.

$$(I - \omega(\mathcal{C} + \omega L)^{-1} RR^*)u = \lambda u$$

or

$$(\mathcal{C} + \omega L - \omega RR^*)u = \lambda(\mathcal{C} + \omega L)u \ .$$

Using $RR^* = D + L + L^*$ yields

$$(\mathcal{C} - \omega D - \omega L^*)u = \lambda(\mathcal{C} + \omega L)u \ . \tag{4.9}$$

Now let $(u, Du) = 1$ and $(Lu, u) = a = \alpha + i\beta$, $\gamma = (\mathcal{C}u, u)$ with α, β, γ real. (4.9) immediately yields

$$\gamma - \omega - \omega\overline{a} = \lambda(\gamma + \omega a)$$

hence

$$\lambda = \frac{\gamma - \omega - \omega\alpha + i\omega\beta}{\gamma + \omega\alpha + i\omega\beta} \ .$$

Since $RR^* \geq 0$ we have $\alpha \geq -1/2$, and since $\mathcal{C}_j \geq R_j R_j^*$ we have $\gamma \geq 1$. For $\alpha = -1/2$ we have $\lambda = 1$. For $\alpha > -1/2$ we have $|\gamma - \omega - \omega\alpha| < |\gamma + \omega\alpha|$ provided that $\omega < 2\gamma$. Thus $|\lambda| < 1$ for $\alpha > -1/2$ and $0 < \omega < 2$.

Returning to (4.8) we see that the sequence u^k converges except in the eigenspace of $\mathcal{C}(\omega)$ for the eigenvalue 1. But this eigenspace is identical to the null space of R^*. This does not prevent the $f^k = R^* u^k$ from converging.

□

References

[1] Bui, H.D.: Inverse Problems in the Mechanics of Materials. CRC Press 1994.

[2] Gutman, S. - Klibanov, M.: Regularized quasi-Newton method for inverse scattering problems. *Math. Comput. Modelling* **18**, 5-31 (1993). Oxford: Pergamon.

[3] Natterer, F.: The Mathematics of Computerized Tomography. Wiley - Teubner 1986.

[4] Natterer, F. - Wübbeling, F.: A propagation-backpropagation method for uzltrasound tomography. *Inverse Problems* **11**, 1225-1232 (1995). IOP Publishing Ltd.